Math 361, Section E1, Fall 1999
Quiz 1, September 10 **Name:** ____________________

1. 10 points Assume that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.5$, and $\mathbb{P}(A \cup B) = 0.6$.
   (a) 5 points What is $\mathbb{P}(A \cap B)$?
   (b) 5 points Are A and B independent?

Answers

1. (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.4 + 0.5 - 0.6 = 0.3.$
   (b) No; $\mathbb{P}(A \cap B) = 0.3 \neq 0.4 \times 0.5 = \mathbb{P}(A)\mathbb{P}(B).$

Math 361, Section E1, Fall 1999

Quiz 2, September 17 **Name:** ____________________

1. There are two boxes, A and B. Box A contains 1 black marble and 3 white marbles, and box B contains 2 black marbles and 4 white marbles. A box is selected at random and then a marble is randomly drawn from the chosen box.
   (a) [5 points] What is the probability that the marble is black?
   (b) [5 points] Given that the marble is white, what is the probability that it came from box A?

1. Define $A \stackrel{\text{def}}{=} \{\text{box A}\}$ and $W \stackrel{\text{def}}{=} \{\text{white marble}\}$.

   (a)

$$\mathbb{P}(W^c) = \mathbb{P}(W^c|A)\mathbb{P}(A) + \mathbb{P}(W^c|A^c)\mathbb{P}(A^c) = \frac{1}{4} \times \frac{1}{2} + \frac{2}{6} \times \frac{1}{2}.$$

   (b)

$$\mathbb{P}(A|W) = \frac{\mathbb{P}(W|A)\mathbb{P}(A)}{\mathbb{P}(W|A)\mathbb{P}(A) + \mathbb{P}(W|A^c)\mathbb{P}(A^c)} = \frac{\frac{3}{4} \times \frac{1}{2}}{\frac{3}{4} \times \frac{1}{2} + \frac{4}{6} \times \frac{1}{2}}.$$

Math 361, Section E1, Fall 1999
Quiz 3, September 20 **Name:** ____________________

1. In a raffle with 100 tickets, there are three winning tickets. Suppose you buy 10 tickets at random. What is the probability that you will win exactly two prizes?

Answers

1.

$$\frac{\binom{3}{2}\binom{97}{8}}{\binom{100}{10}} = \frac{\binom{10}{2}\binom{90}{1}}{\binom{100}{3}}.$$

Math 361, Section E1, Fall 1999
Quiz 4, October 1 **Name:** ____________________

1. Suppose that you toss an unfair coin and win $X = \$12$ if it is heads and win $X = \$5$ if it is tails. Suppose that the probability of tossing a heads is $1/4$.

   (a) [5 points] What is the density of X?

   (b) [5 points] What is $\mathbb{E}[X^2]$?

Answers

1. (a)

$$f_X(j) = \begin{cases} \frac{1}{4} & \text{if } j = 12 \\ \frac{3}{4} & \text{if } j = 5 \\ 0 & \text{else} \end{cases}$$

(b)

$$\mathbb{E}[X^2] = \frac{1}{2}(12)^2 + \frac{3}{4}(5)^2.$$

Math 361, Section E1, Fall 1999
Quiz 5, October 6 **Name:** ______________________

1. Suppose we flip a sequence of independent and identically-distributed coins, where $\mathbb{P}\{H\} = 0.2$ for each coin. Let X be the position of the first heads.
   (a) What is the distribution (i.e., density) of X?
   (b) Evaluate $\varphi_X(\theta) = \mathbb{E}[e^{\theta X}]$
   (c) Using this last result, compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.

Answers

1. (a)

$$f_X(j) = \begin{cases} (0.8)^{j-1}(0.2) & \text{if } j \in \{1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

(b)

$$\varphi_X(\theta) = \begin{cases} \frac{0.2e^\theta}{1-0.8e^\theta} & \text{if } \theta < -\ln 0.8 \\ \infty & \text{else} \end{cases}$$

(c) $\mathbb{E}[X] = 5$ and $\mathbb{E}[X^2] = 45$.

Math 361, Section E1, Fall 1999
Quiz 6, October 11 **Name:** ____________________

1. Suppose that we have two independent random variables X and Y. Fix a parameter $p \in (0,1)$ and suppose that X has density

$$f_X(j) = \begin{cases} (1-p)^j p & \text{for } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

and suppose that Y is uniform on $\{0,1,2\}$; i.e.,

$$f_Y(j) = \begin{cases} \frac{1}{3} & \text{for } j \in \{0,1,2\} \\ 0 & \text{else} \end{cases}$$

Define $Z = X + Y$. Find the density of Z via the following three steps.

(a) Find $f_Z(0)$.

(b) Find $f_Z(1)$.

(c) Find $f_Z(j)$ for $j \in \{2, 3 \dots\}$.

Answers

1. (a) $f_Z(0) = \frac{p}{3}$.

   (b) $f_Z(1) = \frac{1}{3}\{p + p(1-p)\}$.

   (c) $f_Z(j) = \frac{1}{3}p(1-p)^{j-2}\{1 + (1-p) + (1-p)^2\}$.

Math 361, Section E1, Fall 1999

Quiz 7, October 22 **Name:** ______________________

1. Suppose that X is a random variable which is uniform on $[0,1]$; i.e., it has density

$$f_X(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{else.} \end{cases}$$

Suppose that $Y = X^2$.

(a) Find the cumulative distribution function F_Y for Y.

(b) If Y has a density, compute it. Otherwise, state that it does not have a density.

1. (a)

$$F_Y(t) = \begin{cases} 1 & \text{if } t \ge 1 \\ \sqrt{t} & \text{if } 0 \le t < 1 \\ 0 & \text{if } t < 0 \end{cases}$$

(b)

$$f_Y(t) = \begin{cases} \frac{1}{2\sqrt{t}} & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Fall 1999
Quiz 8, October 29 **Name:** ____________________

1. Suppose that X is a random variable which has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{2} & \text{if } 0 \le t < 1/2 \\ \frac{t}{3} + \frac{1}{3} & \text{if } 1/2 \le t < 1 \\ 1 & t \ge 1. \end{cases}$$

   (a) Compute $\mathbb{P}\{X < 1\}$.

   (b) Compute $\mathbb{P}\{1/2 \le X < 1\}$.

Answers

1. (a) $2/3$
   (b) $2/3 - 1/4 = 5/12$

Math 361, Section E1, Fall 1999
Quiz 9, November 8 **Name:** ____________________

1. Suppose that X is an exponential random variable with parameter λ; i.e.,

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Define a new random variable $Y = \max\{X, 10\}$. Find the cumulative distribution function of Y.

1.

$$F_Y(t) = \mathbb{P}\{Y \le t\} = \begin{cases} 0 & \text{if } t < 10 \\ \mathbb{P}\{Y \le t\} & \text{if } t \ge 10 \end{cases} = \begin{cases} 0 & \text{if } t < 10 \\ \int_{s=-\infty}^{t} f_Y(s)ds & \text{if } t \ge 10 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 10 \\ \int_{s=0}^{t} \lambda e^{-\lambda s} ds & \text{if } t \ge 10 \end{cases} = \begin{cases} 0 & \text{if } t < 10 \\ 1 - e^{-\lambda t} & \text{if } t \ge 10 \end{cases}$$

Math 361, Section E1, Fall, 1999
Quiz 10, November 15 **Name:** ____________________

1. 5 points Remember that if X is a Gaussian random variable with mean μ and variance σ^2, then

$$\mathbb{E}[\exp[i\theta X]] = \exp\left[i\mu\theta - \frac{\sigma^2}{2}\theta^2\right].$$

Suppose that Y and Z are independent random variables with respective means 2 and 4 and respective variances 25 and 36. Define $U = Y + Z$.

(a) Compute $\mathbb{E}[\exp[i\theta U]]$.

(b) Identify the distribution (i.e., density if appropriate) of U.

Answers

1. (a)

$$\mathbb{E}[\exp[i\theta U]] = \exp\left[6i\theta - \frac{61}{2}\theta^2\right]$$

(b)

$$f_U(t) = \frac{1}{\sqrt{122\pi}} \exp\left[-\frac{(t-6)^2}{122}\right] \qquad t \in \mathbb{R}$$

Math 361, Section E1, Fall, 1999
Quiz 11, November 19 **Name:** ____________________

1. [5 points] Suppose that X and Y are random variables with joint characteristic function

$$\mathbb{E}\left[\exp[i\theta X + i\psi Y]\right] = \exp\left[2i\theta + 5i\psi - 5\theta^2/2 - 9\psi^2/2 - 3\psi\theta\right].$$

   (a) What is the mean of X?

   (b) What is the variance of X?

   (c) What is the covariance $\sigma_{X,Y}$ between X and Y?

1. Set $\varphi_{X,Y}(\theta,\psi) \stackrel{\text{def}}{=} \mathbb{E}[\exp[i\theta X + i\psi Y]]$. Could just use fact that (X,Y) must be jointly Gaussian.
   (a) Note that
   $$\frac{\partial \varphi_{X,Y}}{\partial \theta}(\theta,\psi) = (2i - 5\theta - 3\psi)\varphi_{X,Y}(\theta,\psi).$$
   Thus $\mathbb{E}[iX] = \frac{\partial \varphi_{X,Y}}{\partial \theta}(0,0) = 2i$, so $\mathbb{E}[X] = 2$.
   (b) We have that
   $$\frac{\partial^2 \varphi_{X,Y}}{\partial \theta^2}(\theta,\psi) = \{(2i - 5\theta - 3\psi)^2 - 5\}\varphi_{X,Y}(\theta,\psi).$$
   Thus $-\mathbb{E}[X^2] = \frac{\partial^2 \varphi_{X,Y}}{\partial \theta^2}(0,0) = -4 - 5 = -9$, so $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 9 - 4 = 5$.
   (c) We have that
   $$\frac{\partial \varphi_{X,Y}}{\partial \psi}(\theta,\psi) = \{(5i - 9\psi - 3\theta)\varphi_{X,Y}(\theta,\psi)$$
   $$\frac{\partial^2 \varphi_{X,Y}}{\partial \theta \partial \psi}(\theta,\psi) = \{(2i - 5\theta - 3\psi)(5i - 9\psi - 3\theta) - 3\}\varphi_{X,Y}(\theta,\psi).$$
   From the first, we get that $\mathbb{E}[iY] = \frac{\partial \varphi_{X,Y}}{\partial \psi}(0,0) = 5i$, so $\mathbb{E}[Y] = 5$. From the second we get that $-\mathbb{E}[XY] = \frac{\partial^2 \varphi_{X,Y}}{\partial \theta^2}(0,0) = -10 - 3$, so $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 12 - 2 \cdot 5 = 3$.

Math 361, Section E1, Fall, 1999
Quiz 12, November 29 **Name:** ____________________

1. 5 points Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 1 & \text{if } 0 \le s \le 1 \text{ and } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

Compute $\mathbb{P}\{XY \le 1/2\}$

Answers

1.

$$
\begin{aligned}
\mathbb{P}\left\{XY \le \frac{1}{2}\right\} &= 1 - \mathbb{P}\left\{XY > \frac{1}{2}\right\} = 1 - \int_{s=1/2}^{1}\int_{t=1/(2s)}^{1} dt\,ds \\
&= 1 - \int_{s=1/2}^{1}\left\{1 - \frac{1}{2s}\right\} ds = 1 - \frac{1}{2} - \frac{1}{2}\ln s\Big|_{s=1/2}^{1} \\
&= \frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \frac{1}{2} + \frac{1}{2}\ln 2
\end{aligned}
$$

Math 361, Section E1, Fall 1999
Exam 1, September 22

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [30 points] Suppose that

$$\mathbb{P}(A) = 0.7 \qquad \mathbb{P}(B) = 0.5 \qquad \text{and} \qquad \mathbb{P}(A \cap B^c) = 0.4.$$

   (a) [5 points] What is $\mathbb{P}(A \cap B)$?

   (b) [5 points] What is $\mathbb{P}(A|B)$?

   (c) [5 points] Are A and B independent (and why)?

   (d) [5 points] What is $\mathbb{P}(B \cap A^c)$?

   (e) [5 points] What is $\mathbb{P}(A \cup B)$?

   (f) [5 points] What is $\mathbb{P}(A^c \cup B^c)$?

2. [30 points] Consider a lottery with 100 tickets, where there are 3 winning tickets. Suppose that you buy 10 tickets.

   (a) [15 points] What is the probability that you win at least one prize?

   (b) [15 points] Given that you win at least one prize, what is the probability that you win exactly two prizes?

3. [15 points] Suppose that the probability of hitting a target is 1/8. Five (independent) shots are fired at the target.

   (a) [10 points] What is the probability that the target is hit on the first and last shots but all other shots miss?

   (b) [5 points] What is the probability that the target is hit exactly twice?

4. [25 points] Suppose that 1% of people in the U.S. have a certain disease. There is an imperfect test for the disease. If you have the disease, you will test positive 80% of the time, but if you don't have the disease, you will also test positive 10% of the time.

   (a) [15 points] Suppose you test positive for the disease; what is the probability that you actually have the disease?

   (b) [10 points] What is the probability that you have the disease but do not test positive?

1. (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c) = 0.7 - 0.5 = 0.2$
   (b) $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{0.2}{0.5} = 0.4$.
   (c) $\mathbb{P}(A|B) = 0.4 \neq 0.7 = \mathbb{P}(A)$; A and B are *not* independent
   (d) $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.5 - 0.2 = 0.3$.
   (e) $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c) = 0.5 + 0.4 = 0.9$.
   (f) $\mathbb{P}(A^c \cup B^c) = 1 - \mathbb{P}(A \cap B) = 1 - 0.2 = 0.8$.
2. X = number of prizes.
   (a)
$$\mathbb{P}\{X \geq 1\} = 1 - \mathbb{P}\{X = 0\} = 1 - \frac{\binom{97}{10}}{\binom{100}{10}}$$
   (b)
$$\mathbb{P}\{X = 2 | X \geq 1\} = \frac{\mathbb{P}\{X = 2, X \geq 1\}}{\mathbb{P}\{X \geq 1\}} = \frac{\mathbb{P}\{X = 2\}}{\mathbb{P}\{X \geq 1\}} = \frac{\binom{97}{8}\binom{3}{2}}{1 - \binom{97}{10}/\binom{100}{10}}$$
3. (a) $\left(\frac{1}{8}\right)^2 \left(\frac{7}{8}\right)^3$.
   (b) $\binom{5}{2} \left(\frac{1}{8}\right)^2 \left(\frac{7}{8}\right)^3$.
4. $D = \{\text{you have the disease}\}$, $P = \{\text{you test positive for the disease}\}$.
   (a)
$$\mathbb{P}\{D|P\} = \frac{\mathbb{P}(D \cap P)}{\mathbb{P}(D)} = \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P|D)\mathbb{P}(D) + \mathbb{P}(P|D^c)\mathbb{P}(D^c)}$$
$$= \frac{(0.8)(0.01)}{(0.8)(0.01) + (0.1)(0.99)}.$$
   (b)
$$\mathbb{P}(P^c|D) = 1 - \mathbb{P}(P|D) = 1 - 0.8 = 0.2.$$

Math 361, Section E1, Fall 1999
Exam 2, November 1

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. 25 points Suppose that X is a geometric random variable with parameter p; i.e.,

$$p_X(j) \stackrel{\text{def}}{=} \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

Define a new random variable $Y \stackrel{\text{def}}{=} \max\{X, 10\}$. Find the density of Y.

2. 30 points Suppose that X is an exponential random variable with parameter λ; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Define a new random variable $Y \stackrel{\text{def}}{=} \max\{X, 10\}$.

(a) 25 points Find the cumulative distribution function of Y.

(b) 5 points Does Y have a density? Why or why not? If so, find it.

(Hint: This is the continuous analogue of question 1).

3. 20 points Suppose that X and Y are independent random variables which are identically distributed with a geometric distribution with parameter p; i.e.,

$$p_X(j) = p_Y(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

Define a new random variable $Z = X + Y$. Find the density of Z. (Hint; you might try, for example, to first find $\mathbb{P}\{Z = 2\}$ and then generalize).

4. 15 points Suppose that X is a random variable which has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{2} & \text{if } 0 \leq t < 1/2 \\ \frac{t}{3} + \frac{1}{3} & \text{if } 1/2 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$

(a) 10 points Compute $\mathbb{P}\{X \geq 1\}$.

(b) 10 points Compute $\mathbb{P}\{1/2 \leq X < 1 \text{ or } X < 1/4\}$.

5. 10 points Suppose that X is a random variable which has moment generating function

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\exp[\theta X]] = \begin{cases} \left(\frac{2}{2-\theta}\right)^n & \text{if } \theta < 2 \\ \infty & \text{else} \end{cases}$$

Compute $\mathbb{E}[X]$.

Answers

1.

$$f_Y(j) = \mathbb{P}\{\max\{X,10\} = j\} = \begin{cases} \mathbb{P}\{X \le 10\} & \text{if } j = 10 \\ \mathbb{P}\{X = j\} & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \sum_{j=0}^{10} p(1-p)^j & \text{if } j = 10 \\ p(1-p)^j & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases} = \begin{cases} 1-(1-p)^{11} & \text{if } j = 10 \\ p(1-p)^j & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases}$$

2. (a)

$$F_Y(t) = \mathbb{P}\{Y \le t\} = \begin{cases} 0 & \text{if } t < 10 \\ \mathbb{P}\{Y \le t\} & \text{if } t \ge 10 \end{cases} = \begin{cases} 0 & \text{if } t < 10 \\ \int_{s=-\infty}^t f_Y(s)ds & \text{if } t \ge 10 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 10 \\ \int_{s=0}^t \lambda e^{-\lambda s} ds & \text{if } t \ge 10 \end{cases} = \begin{cases} 0 & \text{if } t < 10 \\ 1 - e^{-\lambda t} & \text{if } t \ge 10 \end{cases}$$

(b) No; F_Y has a jump at $t = 10$

3.

$$f_Z(j) = \sum_{k=-\infty}^{\infty} p_X(k)p_Y(j-k) = \sum_{k=-\infty}^{\infty} \chi_{\{k\ge 0\}}\chi_{\{j-k\ge 0\}}(1-p)^k p(1-p)^{j-k}p$$

$$= p^2(1-p)^j \sum_{k=-\infty}^{\infty} \chi_{\{0\le k\le j\}} = \begin{cases} (j+1)p^2(1-p)^j & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

4. (a) $\mathbb{P}\{X \ge 1\} = 1 - \mathbb{P}\{X < 1\} = 1 - F_X(1-) = 1 - \frac{2}{3} = \frac{1}{3}$.

(b)

$$\mathbb{P}\left\{\frac{1}{2} \le X < 1\right\} + \mathbb{P}\left\{X < \frac{1}{4}\right\} = F_X(1-) - F_X(\frac{1}{2}-) + F_X(\frac{1}{4}-)$$

$$= \frac{2}{3} - \frac{1}{4} + \frac{1}{8} = \frac{16-6+3}{24} = \frac{13}{24}.$$

5.

$$\dot{\varphi}_X(\theta) = n\left(\frac{2}{2-\theta}\right)^{n-1} \frac{2}{2-\theta)^2}$$

for $\theta < 2$. Thus

$$\mathbb{E}[X] = \dot{\varphi}_X(0) = \frac{n}{2}.$$

Math 361, Section E1, Fall 1999
Exam 3, December 6

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] Suppose that X and Y are independent Gaussian random variables with mean zero and variance one. Define $Z \stackrel{\text{def}}{=} 3X + Y$.
   (a)
   (b) [10 points] Compute either the moment generating function or characteristic function of the pair X and Z.
   (c) [10 points] Compute the variance $\sigma_{Z,Z}$ of Z and the covariance $\sigma_{X,Z}$ of X and Z.

2. [30 points] Suppose that X is a Gaussian random variable with mean zero and variance 3. Define $Z = X^2$.
   (a) [15 points] Find the cumulative distribution function of Z.
   (b) [15 points] Does Z have a density? Why or why not? If so, find it.

3. [30 points] Suppose that the random variable X has uniform distribution on $(0, 1)$. Given that $X = t$ (where $0 < t < 1$), the random variable Y is exponential with parameter $1 + t$; i.e.,

$$f_{Y|X}(s|t) = \begin{cases} (1+t)e^{-(1+t)s} & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$

   (a) [10 points] Compute $\mathbb{E}[Y \,|X = t]$ for $0 < t < 1$.
   (b) [5 points] Compute $\mathbb{E}[Y]$.
   (c) [15 points] Compute the joint density $f_{X,Y}$

4. [20 points] Suppose that X and Y have joint density function

$$f_{X,Y}(s,t) = \begin{cases} c/s^3 & \text{if } s > 1 \text{ and } 0 < t < s \\ 0 & \text{else.} \end{cases}$$

   (a) [5 points] Find c
   (b) [15 points] Find the marginal density of X; i.e., find f_X

Answers

1. (a)

$$\begin{aligned}\varphi_{X,Z}(\theta,\psi) &= \mathbb{E}\left[\exp\left[i\theta X + i\psi Z\right]\right] = \mathbb{E}\left[\exp\left[i\theta X + i\psi(3X+Y\right]\right] \\ &= \mathbb{E}\left[\exp\left[i(\theta+3\psi)X\right]\exp\left[i\psi Y\right]\right] \\ &= \exp\left[-\frac{1}{2}(\theta+3\psi)^2 - \frac{1}{2}\psi^2\right] = \exp\left[-\frac{1}{2}\theta^2 - \frac{10}{2}\psi^2 - 3\theta\psi\right]\end{aligned}$$

(b) $\mathbb{E}[X] = \mathbb{E}[Z] = 0$, so

$$\mathbb{E}[(X-0)(Z-0)] = \mathbb{E}[XZ] = \mathbb{E}[X(3X+Y)] = 3\mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[Y] = 3.$$

2. (a)

$$\begin{aligned}F_Z(t) = \mathbb{P}\{X^2 \le t\} &= \begin{cases}0 & \text{if } t<0 \\ \mathbb{P}\{|X| \le \sqrt{t}\} & \text{if } t \ge 0\end{cases} \\ &= \begin{cases}0 & \text{if } t<0 \\ \int_{s=-\sqrt{t}}^{\sqrt{t}} f_X(s)ds & \text{if } t \ge 0\end{cases} = \begin{cases}0 & \text{if } t<0 \\ \int_{s=-\sqrt{t}}^{\sqrt{t}} \frac{\exp[-s^2/6]}{\sqrt{6\pi}}ds & \text{if } t \ge 0\end{cases} \\ &= \begin{cases}0 & \text{if } t<0 \\ \int_{s=0}^{t} \frac{\exp[-s/6]}{\sqrt{6\pi s}}ds & \text{if } t \ge 0\end{cases}\end{aligned}$$

(b) Yes; F_Z is differentiable;

$$f_Z(t) = \begin{cases}0 & \text{if } t<0 \\ \frac{\exp[-t/6]}{\sqrt{6\pi t}} & \text{if } t \ge 0\end{cases}$$

3. (a)

$$\mathbb{E}[Y|X=t] = \int_{s\in\mathbb{R}} s f_{Y|X}(s|t)ds = \int_{s=0}^{\infty} s(1+t)e^{-(1+t)s}ds = \frac{1}{1+t}$$

(b)

$$\mathbb{E}[Y] = \int_{t=0}^{1} \mathbb{E}[Y|X=t]dt = \int_{t=0}^{1} \frac{1}{1+t}dt = \ln\frac{1}{2}.$$

(c)

$$f_{X,Y}(s,t) = f_{Y|X}(t|s)f_X(s) = \begin{cases}(1+s)e^{-(1+s)t} & \text{if } t \ge 0 \text{ and } s \in [0,1] \\ 0 & \text{else}\end{cases}$$

4. (a)

$$1 = \int_{\mathbb{R}^2} f_{X,Y}(s,t)ds\,dt = c\int_{s=1}^{\infty}\int_{t=0}^{s} \frac{1}{s^3}dt\,ds = c\int_{s=1}^{\infty}\frac{1}{s^2}ds = 2c;$$

$c = \frac{1}{2}$.

(b)

$$f_X(s) = \int_{t\in\mathbb{R}} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=0}^{s} \frac{1}{2s^3} ds & \text{if } s \ge 1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{2s^2} & \text{if } s \ge 1 \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Fall 1999
Final, December 13

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [15 points] Suppose that X and Y are independent Gaussian random variables with mean zero and variance one. Define $Z \stackrel{\text{def}}{=} 3X + Y$.
   (a) [10 points] Compute either the moment generating function or characteristic function of the pair X and Z.
   (b) [5 points] Compute the variance $\sigma_{Z,Z}$ of Z and the covariance $\sigma_{X,Z}$ of X and Z.

2. [25 points] Suppose that X is a Gaussian random variable with mean zero and variance 3. Define $Z = X^2$.
   (a) [15 points] Find the cumulative distribution function of Z.
   (b) [10 points] Does Z have a density? Why or why not? If so, find it.

3. [20 points] Suppose that the random variable X has uniform distribution on $(0, 1)$. Given that $X = t$ (where $0 < t < 1$), the random variable Y is exponential with parameter $1 + t$; i.e.,

$$f_{Y|X}(s|t) = \begin{cases} (1+t)e^{-(1+t)s} & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$

   (a) [10 points] Compute $\mathbb{E}[Y|X = t]$ for $0 < t < 1$.
   (b) [5 points] Compute $\mathbb{E}[Y]$.
   (c) [5 points] Compute the joint density $f_{X,Y}$

4. [15 points] Suppose that X and Y have joint density function

$$f_{X,Y}(s,t) = \begin{cases} c/s^3 & \text{if } s > 1 \text{ and } 0 < t < s \\ 0 & \text{else.} \end{cases}$$

   (a) [5 points] Find c
   (b) [10 points] Find the marginal density of X; i.e., find f_X

5. [10 points] Suppose that X and Y are independent geometric random variables with common parameter $p \in (0, 1)$; i.e.,

$$p_X(j) = p_Y(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1 \ldots\} \\ 0 & \text{else} \end{cases}$$

   Define $Z \stackrel{\text{def}}{=} \min\{X, Y\}$.

(a) 5 points Compute $\mathbb{P}\{Z = 3\}$.

(b) 5 points Compute the density of Z.

6. 20 points Suppose that X and Y are independent exponential random variables with common parameter $\lambda > 0$; i.e.,

$$f_X(t) = f_Y(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} \min\{X, Y\}$.

(a) 5 points Compute $\mathbb{P}\{Z > 3\}$.

(b) 5 points Compute $\mathbb{P}\{Z \le 3\}$.

(c) 5 points Compute the cumulative distribution function F_Z.

(d) 5 points Compute the density of Z if it exists; if the density does not exist, explain why.

7. 10 points Suppose that X and Y are independent continuous random variables. Suppose that X is uniform on $[0, 1]$ and Y is uniform on $[0, 2]$; i.e.,

$$f_X(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

and

$$f_Y(t) = \begin{cases} \frac{1}{2} & \text{if } 0 \le t \le 2 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$. Compute the density of Z.

8. 10 points Suppose that

$$\mathbb{P}(A) = 0.6 \qquad \mathbb{P}(B) = 0.5 \qquad \text{and} \qquad \mathbb{P}(A \cap B^c) = 0.4.$$

(a) 5 points What is $\mathbb{P}(B \cap A^c)$?

(b) 5 points What is $\mathbb{P}(B \cup A^c)$?

9. 15 points Consider a lottery with 100 tickets, where there are 7 winning tickets. Suppose that you buy 20 tickets.

(a) 5 points What is the probability that you win at least two prizes?

(b) 5 points What is the probability that you win at least two prizes but less than 6 prizes?

(c) $\boxed{\text{5 points}}$ Given that you win at least two prizes, what is the probability that you win less than 6 prizes?

10. $\boxed{\text{10 points}}$ Suppose that X is a random variable which has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{2} & \text{if } 0 \le t < 1/2 \\ \frac{t}{3} + \frac{1}{3} & \text{if } 1/2 \le t < 1 \\ 1 & t \ge 1. \end{cases}$$

(a) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X > 1\}$.

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X = 1/2\}$.

Answers

1. (a)

$$\begin{aligned}\varphi_{X,Z}(\theta,\psi) &= \mathbb{E}\left[\exp\left[i\theta X + i\psi Z\right]\right] = \mathbb{E}\left[\exp\left[i\theta X + i\psi(3X+Y\right]\right] \\ &= \mathbb{E}\left[\exp\left[i(\theta+3\psi)X\right]\exp\left[i\psi Y\right]\right] \\ &= \exp\left[-\frac{1}{2}(\theta+3\psi)^2 - \frac{1}{2}\psi^2\right] = \exp\left[-\frac{1}{2}\theta^2 - \frac{10}{2}\psi^2 - 3\theta\psi\right]\end{aligned}$$

(b) $\mathbb{E}[X] = \mathbb{E}[Z] = 0$, so

$$\mathbb{E}[(X-0)(Z-0)] = \mathbb{E}[XZ] = \mathbb{E}[X(3X+Y)] = 3\mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[Y] = 3.$$

2. (a)

$$\begin{aligned}F_Z(t) = \mathbb{P}\{X^2 \le t\} &= \begin{cases} 0 & \text{if } t<0 \\ \mathbb{P}\{|X| \le \sqrt{t}\} & \text{if } t \ge 0\end{cases} \\ &= \begin{cases} 0 & \text{if } t<0 \\ \int_{s=-\sqrt{t}}^{\sqrt{t}} f_X(s)ds & \text{if } t \ge 0\end{cases} = \begin{cases} 0 & \text{if } t<0 \\ \int_{s=-\sqrt{t}}^{\sqrt{t}} \frac{\exp[-s^2/6]}{\sqrt{6\pi}}ds & \text{if } t \ge 0\end{cases} \\ &= \begin{cases} 0 & \text{if } t<0 \\ \int_{s=0}^{t} \frac{\exp[-s/6]}{\sqrt{6\pi s}}ds & \text{if } t \ge 0\end{cases}\end{aligned}$$

(b) Yes; F_Z is differentiable;

$$f_Z(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{\exp[-t/6]}{\sqrt{6\pi t}} & \text{if } t \ge 0\end{cases}$$

3. (a)

$$\mathbb{E}[Y|X=t] = \int_{s\in\mathbb{R}} s f_{Y|X}(s|t)ds = \int_{s=0}^{\infty} s(1+t)e^{-(1+t)s}ds = \frac{1}{1+t}$$

(b)

$$\mathbb{E}[Y] = \int_{t=0}^{1} \mathbb{E}[Y|X=t]dt = \int_{t=0}^{1} \frac{1}{1+t}dt = \ln\frac{1}{2}.$$

(c)

$$f_{X,Y}(s,t) = f_{Y|X}(t|s)f_X(s) = \begin{cases} (1+s)e^{-(1+s)t} & \text{if } t \ge 0 \text{ and } s \in [0,1] \\ 0 & \text{else}\end{cases}$$

4. (a)

$$1 = \int_{\mathbb{R}^2} f_{X,Y}(s,t)ds\,dt = c\int_{s=1}^{\infty}\int_{t=0}^{s} \frac{1}{s^3}dt\,ds = c\int_{s=1}^{\infty} \frac{1}{s^2}ds = 2c;$$

$c = \frac{1}{2}$.

(b)

$$f_X(s) = \int_{t\in\mathbb{R}} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=0}^s \frac{1}{2s^3}ds & \text{if } s \ge 1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{2s^2} & \text{if } s \ge 1 \\ 0 & \text{else} \end{cases}$$

5. First, note that for all $n \in \{0, 1 \dots\}$,

$$\mathbb{P}\{Z \ge n\} = \mathbb{P}\{X \ge n\}\mathbb{P}\{Y \ge n\} = \left(\sum_{j=n}^\infty p(1-p)^j\right)^2 = (1-p)^{2n}.$$

(a)

$$\mathbb{P}\{Z = 3\} = \mathbb{P}\{Z \ge 3\} = \mathbb{P}\{Z \ge 4\} = (1-p)^6 - (1-p)^8 = (1-p)^6\{1-(1-p)^2\}.$$

(b) For $n \in \{0, 1 \dots\}$,

$$\begin{aligned} p_Z(n)\} = \mathbb{P}\{Z \ge n\} = \mathbb{P}\{Z \ge n+1\} = (1-p)^{2n} - (1-p)^{2(n+1)} \\ = (1-p)^{2n}\{1-(1-p)^2\}. \end{aligned}$$

We also have that $p_Z(n) = 0$ otherwise.

6. (a)

$$\mathbb{P}\{Z > 3\} = \mathbb{P}\{X > 3\}\mathbb{P}\{Y > 3\} = \left(\int_{t=3}^\infty \lambda e^{-\lambda t}dt\right)^2 = e^{-6\lambda}.$$

(b) $\mathbb{P}\{Z \le 3\} = 1 - e^{-6\lambda}$.

(c)

$$F_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-2\lambda t} & \text{if } t \ge 0 \end{cases}$$

(d) F_Z is continuous and has density

$$f_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2\lambda e^{-2\lambda t} & \text{if } t \ge 0 \end{cases}$$

7.

$$\begin{aligned} f_Z(t) &= \int_{s=-\infty}^\infty f_X(s)f_Y(t-s)ds = \frac{1}{2}\int_{s=-\infty}^\infty \chi_{[0,1]}(s)\chi_{[0,2]}(t-s)ds \\ &= \frac{1}{2}\int_{s=-\infty}^\infty \chi_{[0,1]}(s)\chi_{[t-s,t]}(s)ds = \frac{1}{2}\int_{s=-\infty}^\infty \chi_{[0,1]\cap[t-s,t]}(s)ds \\ &= \begin{cases} \frac{1}{2}\{1-(t-2)\} & \text{if } 0 \le t-2 \le 1 \\ \frac{1}{2} & \text{if } t-2 < 0 \text{ and } t \ge 1 \\ \frac{1}{2}t & \text{if } 0 \le t < 1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{2}\{3-t\} & \text{if } 2 \le t \le 3 \\ \frac{1}{2} & \text{if } 1 \le t < 2 \\ \frac{1}{2}t & \text{if } 0 \le t < 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

8. (a) First, compute that $\mathbb{P}(A\cap B)=\mathbb{P}(A)-\mathbb{P}(A\cap B^c)=0.6-0.4=0.2$. Then compute that $\mathbb{P}(B\cap A^c)=\mathbb{P}(B)-\mathbb{P}(A\cap B)=0.5-0.2=0.3$.

   (b) $\mathbb{P}(B\cup A^c)=1-\mathbb{P}(A\cap B^c)=1-0.4=0.6$.

9. Define $q \stackrel{\text{def}}{=} 1/\binom{100}{20}$. Let X = number of prizes you win.

   (a) $\mathbb{P}\{X\ge 2\}=1-\mathbb{P}\{X=0\}-\mathbb{P}\{X=1\}=1-q\left\{\binom{93}{20}+\binom{93}{19}\cdot 7\right\}$.

   (b) $\mathbb{P}\{2\le X\le 6\}=1-\mathbb{P}\{X=0\}-\mathbb{P}\{X=1\}-\mathbb{P}\{X=7\}=1-q\left\{\binom{93}{20}+\binom{93}{19}\cdot 7+1\right\}$.

   (c)

$$\mathbb{P}\{X\le 6|X\ge 2\}=\frac{\mathbb{P}\{2\le X\le 6\}}{\mathbb{P}\{X\ge 2\}}=\frac{1-q\left\{\binom{93}{20}+\binom{93}{19}\cdot 7+1\right\}}{1-q\left\{\binom{93}{20}+\binom{93}{19}\cdot 7\right\}}$$

10. (a) $\mathbb{P}\{X>1\}=1-F_X(1)=1-1=0$.

   (b) $\mathbb{P}\left\{X=\frac{1}{2}\right\}=F_X(1/2)-F_X(1/2-)=\frac{1}{6}+\frac{1}{3}-\frac{1}{4}=\frac{2+4-3}{12}=\frac{3}{12}=\frac{1}{4}$.