Math 361, Section F1, Fall 1998
Quiz 1, August 31 **Name:** ____________________

1. 10 points Assume that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.5$, and $\mathbb{P}(A \cup B) = 0.6$. What is $\mathbb{P}(A \setminus B)$?

Answers

1. $\mathbb{P}(A \setminus B) = \mathbb{P}(A \cup B) - \mathbb{P}(B) = 0.6 - 0.5 = 0.1.$

Math 361, Section F1, Fall 1998

Quiz 2, September 2 **Name:** ______________________

1. 10 points Assume that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.5$, and $\mathbb{P}(A \cap B) = 0.2$. What is $\mathbb{P}(A|B^c)$?

Answers

1.
$$\mathbb{P}(A|B^c) = \frac{\mathbb{P}(A \setminus B)}{\mathbb{P}(B^c)} = \frac{\mathbb{P}(A) - \mathbb{P}(A \cap B)}{1 - \mathbb{P}(B)} = \frac{0.4 - 0.2}{1 - 0.5} = \frac{0.2}{0.5} = 0.4.$$

Math 361, Section F1, Fall 1998
Quiz 3, September 14 **Name:** ______________________

1. [10 points] Evaluate $\binom{5}{2}$.

Answers

1. $\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 1} = 10$.

Math 361, Section F1, Fall 1998
Quiz 4, September 16 **Name:** ____________________

1. [10 points] A woman has 6 keys, exactly one of which fits a certain lock. She tries one at a time, at each time randomly choosing from one of the remaining keys. Find the probability that the 4th key she tries is the correct one.

Answers

1.

$$\frac{5 \cdot 4 \cdot 3 \cdot 1}{(6)_4} = \frac{1}{6}.$$

Math 361, Section F1, Fall 1998
Quiz 5, September 18 **Name:** ____________________

1. 10 points A box of 100 washers contains 25 defective ones. What is the probability that 3 washers, selected at random (without replacement) from the box are all good?

ANSWERS

1.

$$\frac{\binom{75}{3}}{\binom{100}{3}}.$$

Math 361, Section F1, Fall 1998

Quiz 6, September 7 **Name:** ____________________

1. [10 points] Let X be a geometric random variable with parameter p. Define $Y = \max\{X, 10\}$. Find the probability density function for Y.

1.

$$f_Y(j)' = \begin{cases} \mathbb{P}\{X \le 10\} & \text{if } j = 10 \\ \mathbb{P}\{X = j\} & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases} = \begin{cases} \sum_{k=1}^{10} p(1-p)^{k-1} & \text{if } j = 10 \\ p(1-p)^{j-1} & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 - (1-p)^{10} & \text{if } j = 10 \\ p(1-p)^{j-1} & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 1998
Quiz 7, October 19 **Name:** ____________________

1. Let X be a geometric random variable with parameter p and let Y have density

$$p_Y(j) = \begin{cases} \frac{1}{2} & \text{if } j = -1 \\ \frac{1}{2} & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$

Assume that X and Y are independent. Define $Z = X + Y$. Find the probability density function for Z.

1.

$$\begin{aligned}\mathbb{P}\{Z=j\} = \mathbb{P}\{X=j-1\}\mathbb{P}\{Y=1\} + \mathbb{P}\{X=j-1\}\mathbb{P}\{Y=1\} \\ = \frac{1}{2}\left\{\mathbb{P}\{X=j-1\} + \mathbb{P}\{X=j+1\}\right\}.\end{aligned}$$

We have that

$$\mathbb{P}\{X=j-1\} = \begin{cases} p(1-p)^{j-2} & \text{if } j \in \{2,3\dots\} \\ 0 & \text{else} \end{cases}$$

$$\mathbb{P}\{X=j+1\} = \begin{cases} p^j & \text{if } j \in \{0,1\dots\} \\ 0 & \text{else} \end{cases}$$

Thus

$$\mathbb{P}\{Z=j\} = \begin{cases} \frac{p}{2}\left\{(1-p)^{j-2} + (1-p)^j\right\} & \text{if } j \in \{2,3\dots\} \\ \frac{p}{2}(1-p)^j & \text{if } j \in \{0,1\} \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 1998
Quiz 8, November 1 **Name:** ______________________

1. 10 points Let X be an exponential random variable with parameter λ; i.e.,

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Compute $\mathbb{P}\{0.5 \leq X \leq 10\}$.

Answers

1.

$$\int_{t=0.5}^{10} f_X(t)dt = \int_{t=0.5}^{10} \lambda e^{-\lambda t}dt = e^{-0.5\lambda} - e^{-10\lambda}.$$

Math 361, Section F1, Fall 1998
Quiz 9, November 6 Name: ____________________

1. Let X be a uniform random variable in $(0,1)$; i.e.,

$$f_X(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else.} \end{cases}$$

   Define

$$Y = X^\beta$$

   where $\beta > 0$. Compute F_Y and f_Y.

1.
$$F_Y(t) = \mathbb{P}\{X^\beta \le t\} = \begin{cases} 0 & \text{if } t < 0 \\ \mathbb{P}\{X \le t^{1/\beta}\} & \text{if } t \ge 0 \end{cases} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/\beta} & \text{if } 0 \le t < 10 \\ 1 & \text{if } t \ge 1 \end{cases}$$

Hence Y has a density, and

$$f_Y(t) = \begin{cases} \beta^{-1} t^{1/\beta - 1} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

 Name: ______________________

1. Let X and Y be random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} \frac{1}{6}(t-s) & \text{if } 0 \le s \le t \le 1 \\ 0 & \text{else} \end{cases}$$

Compute $F_{X,Y}(1/2,1/2) = \mathbb{P}\{X \le 1/2,\, Y \le 1/2\}$.

1.

$$\begin{aligned}
\mathbb{P}\left\{X \le \frac{1}{2}, Y \le \frac{1}{2}\right\} &= \int_{s=-\infty}^{1/2}\int_{t=-\infty}^{1/2} f_{X,Y}(s,t)dt\,ds = \int_{s=0}^{1/2}\int_{t=s}^{1/2}\frac{1}{6}(t-s)dt\,ds \\
&= \int_{s=0}^{1/2}\frac{1}{6}\int_{t=0}^{1/2-s} tdt\,ds = \frac{1}{12}\int_{s=0}^{1/2}(1/2-s)^2 ds \\
&= \frac{1}{12}\int_{s=0}^{1/2} s^2 ds = \frac{1}{36}\left(\frac{1}{2}\right)^3.
\end{aligned}$$

Math 361, Section F1, Fall 1998
Quiz 11, November 20 **Name:** ______________________

1. 10 points Let X be a random variable with moment generating function

$$\phi_X(\theta) = \begin{cases} \frac{e^{\theta/2}-e^{-\theta/2}}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0. \end{cases}$$

What is the moment generating function of the random variable $Z = 2X$?

Answers

1.

$$\phi_Z(\theta) = \mathbb{E}\left[\exp\left[\theta(2X)\right]\right] = \mathbb{E}\left[\exp\left[(2\theta)X\right]\right] = \begin{cases} \frac{e^{\theta}-e^{-\theta/2}}{2\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0. \end{cases}$$

Math 361, Section F1, Fall 1998
Quiz 12, December 9 **Name:** ____________________

1. [10 points] Let A and B be sets such that

$$\mathbb{P}(A) = 0.2, \qquad \mathbb{P}(B) = 0.1 \quad \text{and} \quad \mathbb{P}(A \cap B) = 0.05.$$

Compute $\mathbb{P}(A \cap B^c)$.

Answers

1. $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.2 - 0.05 = 0.15.$

Math 361, Section F1, Fall 1998
Exam 1, September 23

1. [25 points] A standard six-sided fair die is rolled twice. Define

$$A \stackrel{\text{def}}{=} \{\text{face 2 appeared exactly once}\}$$
$$B \stackrel{\text{def}}{=} \{\text{face 2 appeared at least once}\},$$

   (a) [15 points] If it is known that face 2 (i.e., 2 dots) appeared at least once, what is the probability that it appeared exactly once?

   (b) [10 points] Are A and B independent?

2. [15 points] Suppose that the probability of hitting a target is 1/8. Five (independent) shots are fired at the target.

   (a) [10 points] What is the probability that the target is hit on the first and second shots but all other shots miss?

   (b) [5 points] What is the probability that the target is hit exactly twice?

3. [20 points] Suppose that we have 10 boxes. Balls are placed at random one at a time into the boxes until, for the first time, some box has two balls. Find the probability that this occurs with the 3rd ball.

4. [20 points] Suppose that

$$P(A) = 0.5 \qquad P(B) = 0.6 \qquad P(A \cap B) = 0.2.$$

   (a) [5 points] Are A and B independent?

   (b) [5 points] Calculate $P(A \cap B^c)$.

   (c) [5 points] Calculate $P(A \cup B^c)$.

   (d) [5 points] Calculate $P(A \cup B)$.

5. [20 points] If you hold 4 tickets for a lottery for which n tickets were sold, and 6 prizes are to be given, what is the probability that you will win exactly 2 prizes?

Answers

1. (a) Note that $A \subset B$, and that $B^c = \{\text{face 2 did not appear}\}$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{1 - \mathbb{P}(B^c)} = \frac{\{5+5\}(1/6)^2}{1 - 25(1/6)^2} = \frac{10}{11}.$$

   (b) No, since

$$\mathbb{P}(A) = 10(1/6)^2 \neq \frac{1}{11} = \mathbb{P}(A|B).$$

2. (a) $(1/8)^2(7/8)^3$.

   (b) $\binom{5}{2}(1/8)^2(7/8)^3$.

3. $\frac{10 \cdot 9 \cdot 2}{10^3}$.

4. (a) No, since $\mathbb{P}(A \cap B) = 0.2 \neq 0.5 \cdot 0.6 = \mathbb{P}(A)\mathbb{P}(B)$.

   (b) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.5 - 0.2 = 0.3$.

   (c) $\mathbb{P}(A \cup B^c) = 1 - \mathbb{P}(B \setminus A) = 1 - \{\mathbb{P}(A) - \mathbb{P}(A \cap B)\} = 1 - 0.6 + 0.2 = 0.6$.

   (d) $\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B) = 0.3 + 0.6 = 0.9$.

5.

$$\frac{\binom{6}{2}\binom{n-6}{2}}{\binom{n}{4}}.$$

Math 361, Section F1, Fall 1998
Exam 2, October 26

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] Let X be a random variable with moment generating function

$$\varphi_X(\theta) = \mathbb{E}[e^{\theta X}] = e^{\lambda(e^\theta - 1)}.$$

Compute

(a) [10 points] $\mathbb{E}[X]$

(b) [5 points] $\mathbb{E}[X^2]$

(c) [5 points] The variance of X.

2. [15 points] Suppose that X is a geometric random variable with parameter p.

(a) [10 points] Compute $\mathbb{P}\{X > L\}$ for all nonnegative integers L.

(b) [5 points] Compute the conditional probability $\mathbb{P}\{X > L + m | X > L\}$ for all nonnegative integers L and m.

3. [10 points] Suppose that X is a geometric random variable with parameter p. Set

$$Y = |2 - X|.$$

Compute the density of Y.

4. [45 points] Let X and Y be two random variables having the joint density given by the following table:

		Y			
		-1	0	2	6
X	-2	3/27	1/27	1/27	3/27
	1	6/27	0	3/27	3/27
	3	0	0	3/27	4/27

In other words, $\mathbb{P}\{X = 1, Y = -1\} = 6/27$.

(a) [5 points] Are X and Y independent?

(b) [5 points] Compute $\mathbb{P}\{X < 0 \text{ and } Y < 0\}$.

(c) [5 points] Compute the density of X.

(d) [5 points] Compute the density of Y.

(e) [5 points] Compute $\mathbb{E}[X]$.

(f) [5 points] Compute $\mathbb{E}[Y]$.

(g) 5 points Find $\mathbb{E}[X^2]$.

(h) 5 points Compute $\mathbb{E}[XY]$.

(i) 5 points Compute $\mathbb{E}[X+Y]$.

5. 10 points Let X be a geometric random variable with parameter p and let Y have density

$$p_Y(j) = \begin{cases} \frac{1}{2} & \text{if } j = -1 \\ \frac{1}{2} & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$

Assume that X and Y are independent. Define $Z = X+Y$. Find the probability density function for Z.

Answers

1. Note that $\dot{\varphi}_X(\theta) = \lambda e^\theta \varphi_X(\theta)$ and that hence $\ddot{\varphi}_X(\theta) = (\lambda e^\theta + \lambda^2 e^{2\theta})\varphi_X(\theta)$. Also note that $\varphi_X(0) = 1$.
   (a) $\mathbb{E}[X] = \dot{\varphi}_X(0) = \lambda$.
   (b) $\mathbb{E}[X^2] = \ddot{\varphi}_X(0) = \lambda + \lambda^2$.
   (c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$.
2. (a)
$$\mathbb{P}\{X > L\} = \sum_{k=L+1}^{\infty} p(1-p)^{k-1} = (1-p)^L.$$
   (b)
$$\begin{aligned}\mathbb{P}\{X > M+L | X > M\} &= \frac{\mathbb{P}\{X > M+L, X > L\}}{\mathbb{P}\{X > L\}} \\ &= \frac{\mathbb{P}\{X > M+L\}}{\mathbb{P}\{X > L\}} = \frac{(1-p)^{M+L}}{(1-p)^L} = (1-p)^M.\end{aligned}$$
3.
$$f_X(j) = \begin{cases} p(1-p)^{k-1} & \text{if } k \in \{1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$
   Thus, $f_Y(0) = \mathbb{P}\{X = 2\} = p(1-p)$, and if $j \in \{1, 2\}$,
$$f_Y(j) = \mathbb{P}\{X = 2-j\} + \mathbb{P}\{X = j-2\} = p\left\{(1-p)^{1-j} + (1-p)^{j+1}\right\}.$$
   Finally, if $j \in \{3, 4 \dots\}$,
$$f_Y(j) = \mathbb{P}\{X = j-2\} = p(1-p)^{j-3}.$$
4. (a) No, since
$$\mathbb{P}\{X = -2, Y = 0\} = 0 \neq \frac{12}{27} \cdot \frac{1}{27} = \mathbb{P}\{X = 1\}\mathbb{P}\{X = -2\}.$$
   (b)
$$\mathbb{P}\{X \geq 0, Y < 0\} = \mathbb{P}\{X = 1, Y = -1\} = \frac{6}{27}.$$
   (c)
$$f_X(j) = \begin{cases} \frac{8}{27} & \text{if } j = -2 \\ \frac{12}{27} & \text{if } j = 1 \\ \frac{7}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$

(d)

$$f_Y(j) = \begin{cases} \frac{9}{27} & \text{if } j = -1 \\ \frac{1}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 2 \\ \frac{10}{27} & \text{if } j = 6 \\ 0 & \text{else} \end{cases}$$

(e)

$$\mathbb{E}[X] = \frac{8(-2) + 12(1) + 7(3)}{27} = \frac{8}{27}.$$

(f)

$$\mathbb{E}[Y] = \frac{9(-1) + 1(0) + 7(2) + 10(6)}{27} = \frac{65}{27}.$$

(g)

$$\mathbb{E}[X^2] = \frac{8(4) + 12(1) + 7(9)}{27} = \frac{107}{27}.$$

(h)

$$\mathbb{E}[XY] = \frac{3(2) + 6(-1) + 1(-4) + 3(2) + 3(6) + 3(-12) + 3(6) + 4(18)}{27} = \frac{98}{27}.$$

(i) $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{72}{27}$.

5.

$$\begin{aligned} \mathbb{P}\{Z = j\} &= \mathbb{P}\{X = j-1\}\mathbb{P}\{Y = 1\} + \mathbb{P}\{X = j-1\}\mathbb{P}\{Y = 1\} \\ &= \frac{1}{2}\left\{\mathbb{P}\{X = j-1\} + \mathbb{P}\{X = j+1\}\right\}. \end{aligned}$$

We have that

$$\mathbb{P}\{X = j-1\} = \begin{cases} p(1-p)^{j-2} & \text{if } j \in \{2, 3\dots\} \\ 0 & \text{else} \end{cases}$$

$$\mathbb{P}\{X = j+1\} = \begin{cases} p^j & \text{if } j \in \{0, 1\dots\} \\ 0 & \text{else} \end{cases}$$

Thus

$$\mathbb{P}\{Z = j\} = \begin{cases} \frac{p}{2}\left\{(1-p)^{j-2} + (1-p)^j\right\} & \text{if } j \in \{2, 3\dots\} \\ \frac{p}{2}(1-p)^j & \text{if } j \in \{0, 1\} \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 1998
Exam 2 (Makeup), November 1

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] Let X be a random variable with moment generating function

$$\varphi_X(\theta) = \mathbb{E}[e^{\theta X}] = e^{\lambda(e^\theta - 1)}.$$

Compute

(a) [10 points] $\mathbb{E}[X]$

(b) [5 points] $\mathbb{E}[X^2]$

(c) [5 points] The variance of X.

2. [15 points] Suppose that X is a geometric random variable with parameter p.

(a) [10 points] Compute $\mathbb{P}\{X > L\}$ for all nonnegative integers L.

(b) [5 points] Compute the conditional probability $\mathbb{P}\{X > L + m | X > L\}$ for all nonnegative integers L and m.

3. [10 points] Suppose that X is a geometric random variable with parameter p. Set

$$Y = |1 - 2X|.$$

Compute the density of Y.

4. [45 points] Let X and Y be two random variables having the joint density given by the following table:

		Y			
		-1	0	2	6
X	-2	3/27	1/27	1/27	3/27
	1	6/27	0	3/27	3/27
	3	0	0	3/27	4/27

In other words, $\mathbb{P}\{X = 1, Y = -1\} = 6/27$.

(a) [5 points] Are X and Y independent?

(b) [5 points] Compute $\mathbb{P}\{X < 0 \text{ and } Y < 0\}$.

(c) [5 points] Compute the density of X.

(d) [5 points] Compute the density of Y.

(e) [5 points] Compute $\mathbb{E}[X]$.

(f) [5 points] Compute $\mathbb{E}[Y]$.

(g) 5 points Find $\mathbb{E}[X^2]$.

(h) 5 points Compute $\mathbb{E}[XY]$.

(i) 5 points Compute $\mathbb{E}[X + 2Y]$.

5. 10 points Let X be a geometric random variable with parameter p and let Y have density

$$p_Y(j) = \begin{cases} \frac{1}{2} & \text{if } j = 0 \\ \frac{1}{2} & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$

Assume that X and Y are independent. Define $Z = X + Y$. Find the probability density function for Z.

Answers

1. Note that $\dot{\varphi}_X(\theta) = \lambda e^\theta \varphi_X(\theta)$ and that hence $\ddot{\varphi}_X(\theta) = (\lambda e^\theta + \lambda^2 e^{2\theta})\varphi_X(\theta)$. Also note that $\varphi_X(0) = 1$.
   (a) $\mathbb{E}[X] = \dot{\varphi}_X(0) = \lambda$.
   (b) $\mathbb{E}[X^2] = \ddot{\varphi}_X(0) = \lambda + \lambda^2$.
   (c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda$.

2. (a)
$$\mathbb{P}\{X > L\} = \sum_{k=L+1}^{\infty} p(1-p)^{k-1} = (1-p)^L.$$
   (b)
$$\begin{aligned}\mathbb{P}\{X > M+L | X > M\} &= \frac{\mathbb{P}\{X > M+L, X > L\}}{\mathbb{P}\{X > L\}} \\ &= \frac{\mathbb{P}\{X > M+L\}}{\mathbb{P}\{X > L\}} = \frac{(1-p)^{M+L}}{(1-p)^L} = (1-p)^M.\end{aligned}$$

3. $f_Y(1) = \mathbb{P}\{X = 0\} + \mathbb{P}\{X = 1\}$. If $j \in \{3, 5, 7 \dots\}$,
$$f_X(j) = \mathbb{P}\left\{X = \frac{j+1}{2}\right\} = p(1-p)^{(j-1)/2}.$$

4. (a) No, since
$$\mathbb{P}\{X = -2, Y = 0\} = 0 \neq \frac{12}{27} \cdot \frac{1}{27} = \mathbb{P}\{X = 1\}\mathbb{P}\{X = -2\}.$$
   (b)
$$\mathbb{P}\{X \geq 0, Y < 0\} = \mathbb{P}\{X = 1, Y = -1\} = \frac{6}{27}.$$
   (c)
$$f_X(j) = \begin{cases} \frac{8}{27} & \text{if } j = -2 \\ \frac{12}{27} & \text{if } j = 1 \\ \frac{7}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$
   (d)
$$f_Y(j) = \begin{cases} \frac{9}{27} & \text{if } j = -1 \\ \frac{1}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 2 \\ \frac{10}{27} & \text{if } j = 6 \\ 0 & \text{else} \end{cases}$$

(e)

$$\mathbb{E}[X] = \frac{8(-2) + 12(1) + 7(3)}{27} = \frac{8}{27}.$$

(f)

$$\mathbb{E}[Y] = \frac{9(-1) + 1(0) + 7(2) + 10(6)}{27} = \frac{65}{27}.$$

(g)

$$\mathbb{E}[X^2] = \frac{8(4) + 12(1) + 7(9)}{27} = \frac{107}{27}.$$

(h)

$$\mathbb{E}[XY] = \frac{3(2) + 6(-1) + 1(-4) + 3(2) + 3(6) + 3(-12) + 3(6) + 4(18)}{27} = \frac{98}{27}.$$

(i) $\mathbb{E}[X + 2Y] = \mathbb{E}[X] + 2\mathbb{E}[Y] = \frac{138}{27}$.

5.

$$\begin{aligned}\mathbb{P}\{Z = j\} = \mathbb{P}\{X = j\}\mathbb{P}\{Y = 0\} + \mathbb{P}\{X = j-1\}\mathbb{P}\{Y = 1\}\\ = \frac{1}{2}\left\{\mathbb{P}\{X = j-1\} + \mathbb{P}\{X = j-1\}\right\}.\end{aligned}$$

We have that

$$\mathbb{P}\{X = j-1\} = \begin{cases} p(1-p)^{j-2} & \text{if } j \in \{2, 3\dots\} \\ 0 & \text{else} \end{cases}$$

Thus

$$\mathbb{P}\{Z = j\} = \begin{cases} \frac{p}{2}\left\{(1-p)^{j-1} + (1-p)^{j-2}\right\} & \text{if } j \in \{2, 3\dots\} \\ \frac{p}{2} & \text{if } j = 1 \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 1998
Exam 3, December 2

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [30 points] Let X be a continuous random variable with distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2}t + \frac{1}{4} & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

   (a) [5 points] Compute $\mathbb{P}\{X = 1\}$.

   (b) [5 points] Compute $\mathbb{P}\{X > 0\}$.

   (c) [5 points] Compute $\mathbb{P}\{0 \leq X \leq 1\}$.

   (d) [15 points] Define $Y = X^2$. Compute the cumulative distribution function for Y; i.e., compute F_Y.

2. [10 points] Suppose that X is a random variable with moment generating function

$$\varphi_X(\theta) = e^{\theta^2 + 2\theta}.$$

   Define $Y = 3X + 7$. Compute the moment generating function for Y.

3. [60 points] Suppose that X and Y are two continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} e^{-t} & \text{if } 0 \leq s < t \\ 0 & \text{else} \end{cases}$$

   (a) [30 points] Find the cumulative distribution function for (X, Y); i.e., compute $F_{X,Y}$. The answer should come in 3 parts.

   (b) [15 points] Find the density of X (i.e., the marginal density f_X). The answer should have two parts.

   (c) [15 points] Find the density of Y (i.e., the marginal density f_Y). The answer should have two parts.

Answers

1. (a) $\mathbb{P}\{X=1\} = F_X(1) - F_X(1-) = 1 - \frac{3}{4} = \frac{1}{4}$.

   (b) $\mathbb{P}\{X>0\} = 1 - F_X(0) = 1 - \frac{1}{4} = \frac{3}{4}$.

   (c) $\mathbb{P}\{0 \le X \le 1\} = \mathbb{P}\{X \le 1\} - \mathbb{P}\{X < 0\} = F_X(1) - F_X(0-) = 1 - 0 = 1$.

   (d) $F_Y(t) = \mathbb{P}\{X^2 \le t\}$. If $t<0$, $F_Y(t)=0$. If $t \ge 0$,

$$F_Y(t) = \mathbb{P}\{-\sqrt{t} \le X \le \sqrt{t}\} = \mathbb{P}\{-\sqrt{t} \le X \le \sqrt{t}\} = F_X(\sqrt{t}) - F_X(-\sqrt{t}-)$$
$$= \begin{cases} \frac{\sqrt{t}}{2} + \frac{1}{4} & \text{if } 0 \le \sqrt{t} \le 1 \\ 1 & \text{if } \sqrt{t} \ge 1 \end{cases} = \begin{cases} \frac{\sqrt{t}}{2} + \frac{1}{4} & \text{if } 0 \le t \le 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

2.

$$\varphi_Y(\theta) = \mathbb{E}\left[\exp\left[\theta Y\right]\right] = \mathbb{E}\left[\exp\left[\theta(3X+7)\right]\right] = \mathbb{E}\left[\exp\left[(3\theta)X\right]\right] e^{7\theta}$$
$$= \exp\left[9\theta^2 + 6\theta + 7\theta\right] = \exp\left[9\theta^2 + 13\theta\right]$$

3. (a)

$$F_{X,Y}(s,t) = \int_{u=-\infty}^{s} \int_{v=-\infty}^{t} f_{X,Y}(u,v)\,dv\,du = \int_{u=-\infty}^{s} \int_{v=-\infty}^{t} e^{-v} \chi_{\{0 \le u < v\}}\,dv\,du.$$

   If $0 \le s < t$,

$$F_{X,Y}(s,t) = \int_{u=0}^{s} \int_{v=u}^{t} e^{-v}\,dv\,du = \int_{u=0}^{s} \{e^{-u} - e^{-t}\}\,du = 1 - e^{-s} - se^{-t}.$$

   and if $0 \le t < s$,

$$F_{X,Y}(s,t) = \int_{u=0}^{t} \int_{v=u}^{t} e^{-v}\,dv\,du = 1 - e^{-t} - te^{-t}.$$

   Otherwise, $F_{X,Y}(s,t) = 0$.

   (b)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)\,dt = \begin{cases} \int_{t=s}^{\infty} e^{-t}\,dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

   (c)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)\,ds = \begin{cases} \int_{s=0}^{t} e^{-t}\,ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} te^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Fall 1998
Final, December 19

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [15 points] Suppose that X is a random variable with moment generating function

$$\varphi_X(\theta) = \mathbb{E}[e^{\theta X}] = e^{\theta^2 + 2\theta}.$$

Define $Y = 2X + 6$. Compute the moment generating function for Y.

2. [60 points] Suppose that X and Y are two continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} e^{-t} & \text{if } 0 \le s < t \\ 0 & \text{else} \end{cases}$$

   (a) [30 points] Find the cumulative distribution function for (X, Y); i.e., compute $F_{X,Y}$. The answer should come in 3 parts.

   (b) [15 points] Find the density of X (i.e., the marginal density f_X). The answer should have two parts.

   (c) [15 points] Find the density of Y (i.e., the marginal density f_Y). The answer should have two parts.

3. [15 points] At a certain prestigious institution of higher learning, 60% of the students are female and 40% are male. It is known that 30% of the men have some sort of body piercing (i.e., ears, nose, etc.) and 20% of the women have some sort of body piercing.

   (a) [5 points] What is the probability that a randomly selected student will have a pierced body?

   (b) [5 points] If you randomly select a student who has some body piercing, what is the probability that the student will be male?

   (c) [5 points] Are manliness and body-piercing independent?

4. [20 points] Let X be a Gaussian random variable with mean zero and variance 1. Set $Y = e^X$. Find the density of Y. (Hint: you might first find an expression for the cumulative distribution function F_Y of Y.

5. [15 points] Consider a four-sided die, and assume that

$$\mathbb{P}\{1\} = \mathbb{P}\{2\} = \mathbb{P}\{3\} = \mathbb{P}\{4\} = 1/4.$$

Show that the two sets

$$A = \{1, 2\} \quad \text{and} \quad B = \{1, 3\}$$

are independent.

6. 25 points A box contains 3 white balls and 2 black balls. Two balls are drawn from the box without replacement.

   (a) 5 points Calculate the probability that the first ball is black.

   (b) 5 points Calculate the probability that the second ball is black.

   (c) 5 points Calculate the probability that both balls are black.

   (d) 5 points Calculate the probability that both balls are the same color.

   (e) 5 points Calculate the probability that the second ball is black given that the first ball is black.

1.

$$\varphi_Y(\theta) = \mathbb{E}[\exp[\theta Y]] = \mathbb{E}[\exp[\theta(2X+6)]] = \mathbb{E}[\exp[(2\theta)X]]\, e^{6\theta}$$
$$= \exp\left[4\theta^2 + 4\theta + 6\theta\right] = \exp\left[4\theta^2 + 10\theta\right]$$

2. (a)

$$F_{X,Y}(s,t) = \int_{u=-\infty}^{s} \int_{v=-\infty}^{t} f_{X,Y}(u,v)\,dv\,du = \int_{u=-\infty}^{s} \int_{v=-\infty}^{t} e^{-v} \chi_{\{0 \le u < v\}}\,dv\,du.$$

If $0 \le s < t$,

$$F_{X,Y}(s,t) = \int_{u=0}^{s} \int_{v=u}^{t} e^{-v}\,dv\,du = \int_{u=0}^{s} \{e^{-u} - e^{-t}\}du = 1 - e^{-s} - se^{-t}.$$

and if $0 \le t < s$,

$$F_{X,Y}(s,t) = \int_{u=0}^{t} \int_{v=u}^{t} e^{-v}\,dv\,du = 1 - e^{-t} - te^{-t}.$$

Otherwise, $F_{X,Y}(s,t) = 0$.

(b)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=s}^{\infty} e^{-t}dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

(c)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^{t} e^{-t}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} te^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

3. (a)

$$\mathbb{P}(P) = \mathbb{P}(P|M)\mathbb{P}(M) + \mathbb{P}(P|F)\mathbb{P}(F) = 0.3 \cdot 0.4 + 0.2 \cdot 0.6 = 0.12 + 0.12 = 0.24.$$

(b)

$$\mathbb{P}(M|P) = \frac{\mathbb{P}(M \cap P)}{\mathbb{P}(P)} = \frac{\mathbb{P}(P|M)\mathbb{P}(M)}{\mathbb{P}(P)} = \frac{0.3 \cdot 0.4}{0.24} = \frac{0.12}{0.24} = 0.5.$$

(c) No, since $\mathbb{P}(M) = 0.4 \neq 0.5 = \mathbb{P}(M|P)$.

4. We have that

$$F_Y(t) = \mathbb{P}\{e^X \le t\} = \begin{cases} 0 & \text{if } t \le 0 \\ \mathbb{P}\{X \le \ln t\} & \text{if } t > 0 \end{cases} = \begin{cases} 0 & \text{if } t \le 0 \\ F_X(\ln t) & \text{if } t > 0 \end{cases}$$

Note that $F_Y(0) = F_Y(0-)$. We can differentiate to see that Y has density given by

$$f_Y(t) = \begin{cases} \frac{1}{t} f_X(\ln t) & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{\sqrt{2\pi t^2}} \exp\left[-\frac{(\ln t)^2}{2}\right] & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

5. We have that $\mathbb{P}(A \cap B) = \mathbb{P}(\{1\}) = \frac{1}{4}$. On the other hand, $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$. Thus

$$\mathbb{P}(A \cap B) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(B),$$

so A and B are dependent.

6. (a) $\mathbb{P}\{C_1 = B\} = \frac{2}{5}$.
   (b) $\mathbb{P}\{C_2 = B\} = \frac{2}{5}$.
   (c) $\mathbb{P}\{C_1 = C_2 = B\} = 1/\binom{5}{2}$.
   (d) $\mathbb{P}\{C_1 = C_2\} = \left\{1 + \binom{3}{2}\right\} / \binom{5}{2}$.
   (e)

$$\mathbb{P}\{C_2 = B | C_1 = B\} = \frac{\mathbb{P}\{C_1 = C_2 = B\}}{\mathbb{P}\{C_1 = B\}} = \frac{1/\binom{5}{2}}{2/5} = \frac{1}{4}.$$