**Math 361, Section F1, Fall 1997**
**Quiz 1, September 8** **Name:** ____________________

1. [8 points] Suppose that we have a three-sided coin, with sides labelled heads (H), tails (T), and stomach (S). We toss the coin.

   (a) [2 points] What is the event space corresponding to this experiment?

   Assume that the $P\{H, S\} = 7/8$ and $P\{H\} = 1/4$.

   (b) [3 points] What is $P\{T\}$?

   (c) [3 points] What is $P\{S\}$?

2. [2 points] Evaluate

$$\int_0^x y^2 dy.$$

## Answers

1. (a) $\Omega = \{H, T, S\}$.

   (b) $\mathbb{P}\{T\} = \frac{1}{8}$.

   (c) $\mathbb{P}\{S\} = \frac{7}{8} - \frac{1}{4} = \frac{5}{8}$.

2. $$\int_0^x y^2 dy = \frac{x^3}{3}.$$

**Math 361, Section F1, Fall 1997**

**Quiz 2, September 26** **Name:** ____________________

1. Suppose that a box has $b$ black balls and $r$ red balls. We draw $n$ balls from the box. What is the probability that we selected either 2 or 3 red balls (assume that $b \geq 3$)?

## Answers

1.

$$\left\{ \binom{b}{2}\binom{r}{n-2} + \binom{b}{3}\binom{r}{n-3} \right\} \Big/ \binom{b+r}{n}$$

**Math 361, Section F1, Fall 1997**
**Quiz 3, October 15** **Name:** ____________________

1. [10 points] Suppose two fair and independent dice are tossed. Let $X$ be the larger of the two numbers observed from the two tosses.
   (a) [5 points] What is the probability density function of $X$?
   (b) [5 points] What is the cumulative probability distribution function of $X$?

1. (a) For any $j \in \{1,2,3,4,5,6\}$,

$$\mathbb{P}\{X \le j\} = \mathbb{P}\{D_1 \le j\}\mathbb{P}\{D_2 \le j\} = \frac{j^2}{36}.$$

Thus for $j \in \{1,2,3,4,5,6\}$,

$$f_X(j) = \frac{j^2 - (j-1)^2}{36} = \frac{2j+1}{36}.$$

(b)

$$F_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{j^2}{36} & \text{if } j \le t < j+1,\ j \in \{1,2\dots 6\} \\ 1 & \text{if } t \ge 6 \end{cases}$$

**Math 361, Section F1, Fall 1997**
**Quiz 4, October 24** **Name:** ______________________

1. Let $X$ be a geometric random variable with parameter $p$; i.e.,

$$p_X(j) = \begin{cases} p(1-p)^j & \text{if } j = 0,1,2,\dots \\ 0 & \text{else.} \end{cases}$$

Define

$$Y = \max\{X, 10\} = \begin{cases} X & \text{if } X \geq 10 \\ 10 & \text{else.} \end{cases}$$

Compute the probability density function for $Y$.

## Answers

1.

$$p_Y(j) = \begin{cases} \mathbb{P}\{X \le 10\} & \text{if } j = 10 \\ \mathbb{P}\{X = j\} & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases} = \begin{cases} \sum_{j=0}^{10} p(1-p)^j & \text{if } j = 10 \\ p(1-p)^j & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 - (1-p)^{11} & \text{if } j = 10 \\ p(1-p)^j & \text{if } j \in \{11, 12 \dots\} \\ 0 & \text{else} \end{cases}$$

**Math 361, Section F1, Fall 1997**
**Quiz 5, October 31** **Name:** ____________________

1. Let $X$ be a random variable with moment generating function

$$\varphi(\theta) = \frac{e^{\theta} + e^{2\theta}}{2}. \qquad \theta \in \mathbb{R}$$

(recall that $\varphi(\theta) = \mathbb{E}[e^{\theta X}]$).

(a) Compute $\mathbb{E}[X]$.

(b) Compute $\mathbb{E}[X^2]$

# Answers

1. Note that
$$\varphi'(\theta) = \frac{e^\theta + 2e^{2\theta}}{2} \qquad \text{and} \qquad \varphi''(\theta) = \frac{e^\theta + 4e^{2\theta}}{2}$$
for all $\theta \in \mathbb{R}$.
   (a) $\mathbb{E}[X] = \varphi'(0) = \frac{3}{2}$
   (b) $\mathbb{E}[X^2] = \varphi''(0) = \frac{5}{2}$.

**Math 361, Section F1, Fall 1997**
**Quiz 6, November 12** **Name:** ____________________

1. Let $X$ be a continuous random variable with density

$$f_X(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 3e^{-3t} & \text{if } t \ge 0. \end{cases}$$

Compute $\mathbb{P}\{-2 \le X \le 7 \,\text{or}\, X \ge 25\}$.

1.

$$\begin{aligned}\mathbb{P}\{-2 \le X \le 7 \,\text{or}\, X \ge 25\} &= \int_{t=-2}^{7} f_X(t)dt + \int_{t=25}^{\infty} f_X(t)dt \\ &= \int_{t=0}^{7} 3e^{-3t}dt + \int_{t=25}^{\infty} 3e^{-3t}dt = 1 - e^{-2t} + e^{-75}\end{aligned}$$

**Math 361, Section F1, Fall 1997**
**Quiz 7, December 1** **Name:** ____________________

1. [10 points] Let $X$ be a continuous random variable with density $f_X$. Let $Y = |X| + 1$.

   (a) [5 points] Find the cumulative distribution function for $Y$.

   (b) [5 points] If $Y$ has a density, find it. If $Y$ does not have a density, state so.

1. (a)

$$F_Y(t) = \mathbb{P}\{|X|+1 \le t\} = \mathbb{P}\{|X| \le t-1\} = \begin{cases} 0 & \text{if } t-1<0 \\ \mathbb{P}\{1-t \le X \le t-1\} & \text{if } t-1 \ge 0 \end{cases}$$
$$= \begin{cases} 0 & \text{if } t<1 \\ \int_{s=1-t}^{t-1} f_X(s)ds & \text{if } t \ge 1 \end{cases}$$

(b)

$$f_Y(t) = \begin{cases} f_X(t-1)+f_X(1-t) & \text{if } t>1 \\ 0 & \text{else} \end{cases}$$

# Math 361, Section F1, Fall 1997
# Exam 1, October 3

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] A lottery is held, and $n$ tickets are sold. You buy 3 tickets, and there are 5 winning tickets. Let

$$A \stackrel{\text{def}}{=} \{\text{you win at least one prize}\}$$
$$B \stackrel{\text{def}}{=} \{\text{you win exactly one prize}\}$$

   (a) [5 points] Compute $\mathbb{P}(A)$
   (b) [5 points] Compute $\mathbb{P}(B)$
   (c) [5 points] Compute $\mathbb{P}(A \cap B)$
   (d) [3 points] Compute $\mathbb{P}(A|B)$
   (e) [2 points] Are $A$ and $B$ independent?

2. [47 points] Suppose that 10 balls are distributed into 10 boxes. Let

$$A \stackrel{\text{def}}{=} \{\text{exactly one box is empty}\} \qquad B \stackrel{\text{def}}{=} \{\text{box 1 is empty}\}.$$

   (a) [15 points] What is $\mathbb{P}(A)$?
   (b) [10 points] What is $\mathbb{P}(B)$?
   (c) [12 points] What is $\mathbb{P}(A \cap B)$?
   (d) [5 points] What is $\mathbb{P}(A|B)$?
   (e) [5 points] What is $\mathbb{P}(B|A)$?

3. [20 points] Suppose you know that

$$\mathbb{P}(A) = 0.6 \qquad \text{and} \qquad \mathbb{P}(B) = 0.3$$

   and that $A$ and $B$ are independent.
   (a) [5 points] Compute $\mathbb{P}(A \cap B)$.
   (b) [5 points] Compute $\mathbb{P}(A \setminus B)$.
   (c) [5 points] Compute $\mathbb{P}(A \cup B)$.
   (d) [5 points] Compute $\mathbb{P}((A \cup B) \setminus (A \cap B))$.

4. [13 points] At a certain prestigious institution of higher learning, 60% of the students are female and 40% are male. It is know that 10% of the men have tattoos and 15% of the women have tattoos.

(a) 5 points What is the probability that a randomly selected student is both male and has a tattoo?

(b) 5 points If you randomly select a student who has a tattoo, what is the probability that the student will be male?

(c) 3 points Are masculinity and tattoo-ness independent?

## Answers

1. (a)

$$1 - \mathbb{P}\{\text{no prizes}\} = 1 - \frac{\binom{n-5}{3}}{\binom{n}{3}}.$$

   (b)

$$\mathbb{P}(B) = \frac{(n-5)\binom{5}{2}}{\binom{n}{3}}.$$

   (c) $B \subset A$, so

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) = \frac{(n-5)\binom{5}{2}}{\binom{n}{3}}.$$

   (d) $\mathbb{P}(A|B) = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$

   (e) no, since $\mathbb{P}(A|B) = 1 \neq \mathbb{P}(A)$.

2. (a) $\mathbb{P}(A) = \binom{10}{2}(10)_9/10^{10}$.

   (b) $\mathbb{P}(B) = 9^{10}/10^{10}$.

   (c) $\mathbb{P}(A \cap B) = \binom{10}{2}9!/10^{10}$.

   (d) $\mathbb{P}(A|B) = \binom{10}{2}9!/9^{10}$.

   (e) $\mathbb{P}(B|A) = 9!/(10)_9 = \frac{1}{10}$.

3. (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = (0.6)(0.3) = 0.18$.

   (b) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.6 - 0.18 = 0.42$.

   (c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.6 + 0.3 - 0.18 = 0.72$.

   (d) $\mathbb{P}((A \cup B) \setminus (A \cap B)) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = 0.72 - 0.18 = 0.54$.

4. (a) $\mathbb{P}(M \cap T) = \mathbb{P}(T|M)\mathbb{P}(M) = (0.1)(0.4) = 0.04$.

   (b)

$$\mathbb{P}(M|T) = \frac{\mathbb{P}(M \cap T)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|M)\mathbb{P}(M)}{\mathbb{P}(T|M)\mathbb{P}(M) + \mathbb{P}(T|F)\mathbb{P}(F)}$$
$$= \frac{(0.4)(0.1)}{(0.4)(0.1) + (0.15)(0.6)} = \frac{4}{13}.$$

   (c) no; $\mathbb{P}(M|T) = \frac{4}{13} \neq 0.4 = \mathbb{P}(M)$.

# Math 361, Section F1, Fall 1997
# Exam 2, November 3

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [34 points] Assume that we flip a coin ten times. Assume that the probability of heads is 0.25. Let $X$ be the number of heads in the 10 coin flips.
   (a) [10 points] What is the density of $X$?
   (b) [10 points] Compute the moment generating expression $\varphi(\theta) = \mathbb{E}[e^{\theta X}]$. I want the answer in a closed form.
   (c) [7 points] Compute $\mathbb{E}[X]$.
   (d) [7 points] Compute $\mathbb{E}[X^2]$.

2. [47 points] Suppose that $X$ is a random variable with the following density

$$p_X(j) = \begin{cases} 0.1 & \text{if } j = -3 \\ 0.2 & \text{if } j = -1 \\ 0.15 & \text{if } j = 0 \\ 0.2 & \text{if } j = 1 \\ 0.1 & \text{if } j = 2 \\ 0.15 & \text{if } j = 3 \\ 0.05 & \text{if } j = 5 \\ c & \text{if } j = 8 \\ 0 & \text{else} \end{cases}$$

   (a) [10 points] Note that we have not given $p_X(8)$. What must $p_X(8)$ be?
   (b) [5 points] Compute the $\mathbb{P}\{X \text{ is negative}\}$.
   (c) [5 points] Compute the $\mathbb{P}\{X \text{ is odd}\}$.
   (d) [5 points] Compute the $\mathbb{P}\{X \text{ is odd and negative}\}$.
   (e) [2 points] Compute the $\mathbb{P}\{X \text{ is odd} | X \text{ is negative}\}$.
   (f) [10 points] Compute $\mathbb{E}[X]$.
   (g) [10 points] Compute $\mathbb{E}[\cos(\pi X/2)]$.

3. [19 points] $X$ and $Y$ are independent random variables with densities

$$p_X(j) = \begin{cases} \binom{10}{j} p^j (1-p)^{10-j} & \text{if } j = 0, 1, \dots 10 \\ 0 & \text{else} \end{cases}$$

$$p_Y(j) = \begin{cases} q(1-q)^j & \text{if } j = 0, 1, \dots \\ 0 & \text{else} \end{cases}$$

   Define $Z = X + Y$.

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Z = 20\}$.

(b) $\boxed{\text{9 points}}$ Recall that the average of $X$ is $10p$ and the average of $Y$ is $(1 - q)/q$. Compute $\mathbb{E}[Z]$.

## Answers

1. (a)

$$p_X(j) = \begin{cases} \binom{10}{j}\left(\frac{1}{4}\right)^j\left(\frac{3}{4}\right)^{10-j} & \text{if } j \in \{0,1,2\dots 10\} \\ 0 & \text{else} \end{cases}$$

(b)

$$\begin{aligned}\varphi_X(\theta) \overset{\text{def}}{=} \mathbb{E}\left[e^{\theta X}\right] &= \sum_{j=0}^{10} e^{\theta j}\binom{10}{j}\left(\frac{1}{4}\right)^j\left(\frac{3}{4}\right)^{10-j} \\ &= \sum_{j=0}^{10}\binom{10}{j}\left(\frac{e^\theta}{4}\right)^j\left(\frac{3}{4}\right)^{10-j} = \left(\frac{e^\theta}{4}+\frac{3}{4}\right)^{10}.\end{aligned}$$

Note that $\varphi'_X(\theta) = \frac{10e^\theta}{4}\varphi_X(\theta) = \frac{5e^\theta/2}{\varphi}(\theta)$.

(c) $\mathbb{E}[X] = \varphi'_X(0) = \frac{5}{2}$.

(d) We compute that

$$\varphi''_X(\theta) = \frac{5e^\theta}{2}\varphi_X(\theta) + \frac{25e^{2\theta}}{4}\varphi_X(\theta);$$

thus $\mathbb{E}[X^2] = \varphi''_X(0) = \frac{5}{2} + \frac{25}{4} = \frac{35}{4}$.

2. (a) $p_X(8) = 1 - 0.1 - 0.2 - 0.15 - 0.2 - 0.1 - 0.15 - 0.05 = 1 - 0.95 = 0.05$.

(b) $\mathbb{P}\{X < 0\} = 0.1 + 0.2 = 0.3$.

(c) $\mathbb{P}\{X \text{ is odd}\} = 0.1 + 0.2 + 0.2 + 0.15 + 0.05 = 0.7$.

(d) $\mathbb{P}\{X \text{ is odd and negative}\} = 0.1 + 0.2 = 0.3$.

(e)

$$\mathbb{P}\{X \text{ is odd}|X < 0\} = \frac{0.3}{0.3} = 1.$$

(f)

$$\begin{aligned}\mathbb{E}[X] &= 0.1(-3) + 0.2(-1) + 0.2(1) + 0.1(2) + 0.15(3) + 0.05(5) + 0.05(8) \\ &= -0.3 - 0.2 + 0.2 + 0.2 + 0.45 + 0.25 + 0.4 = 1.\end{aligned}$$

(g) Recall that $\cos(\pi n/2) = 0$ if $n$ is odd, while $\cos(\pi n/2) = \pm 1$ if $n$ is even. Thus

$$\mathbb{E}[\cos(\pi X/2)] = 0.15(1) + 0.1(-1) + 0.05(1) = 0.15 - 0.1 + 0.15 = 0.1.$$

3. (a)

$$\begin{aligned}\mathbb{P}\{Z = 20\} &= \sum_{j=0}^{10} p_X(j)p_Y(10-j) = \sum_{j=0}^{10}\binom{10}{j}p^j(1-p)^{10-j}q(1-q)^{10-j} \\ &= q\sum_{j=0}^{10}\binom{10}{j}p^j\{(1-p)(1-q)\}^{10-j} = q\{p + (1-p)(1-q)\}^{10}\end{aligned}$$

(b) $\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = 10p + (1-q)/q$.

# Math 361, Section F1, Fall 1997
# Exam 2 (makeup), November 3

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [34 points] Assume that we flip a coin such that the probability of heads is 0.25. Let $X$ be the position of the first heads.
   (a) [10 points] What is the density of $X$?
   (b) [10 points] Compute the moment generating expression $\varphi(\theta) = \mathbb{E}[e^{\theta X}]$. I want the answer in a closed form.
   (c) [7 points] Compute $\mathbb{E}[X]$.
   (d) [7 points] Compute $\mathbb{E}[X^2]$.

2. [47 points] Suppose that $X$ is a random variable with the following density

$$p_X(j) = \begin{cases} 0.1 & \text{if } j = -3 \\ 0.2 & \text{if } j = -1 \\ 0.15 & \text{if } j = 0 \\ 0.2 & \text{if } j = 1 \\ 0.1 & \text{if } j = 2 \\ 0.15 & \text{if } j = 3 \\ 0.05 & \text{if } j = 5 \\ c & \text{if } j = 8 \\ 0 & \text{else} \end{cases}$$

   (a) [10 points] Note that we have not given $p_X(8)$. What must $p_X(8)$ be?
   (b) [5 points] Compute the $\mathbb{P}\{X > 0\}$.
   (c) [5 points] Compute the $\mathbb{P}\{X \text{ is odd}\}$.
   (d) [5 points] Compute the $\mathbb{P}\{X \text{ is odd and positive}\}$.
   (e) [2 points] Compute the $\mathbb{P}\{X \text{ is odd} | X > 0\}$.
   (f) [10 points] Compute $\mathbb{E}[X]$.
   (g) [10 points] Compute $\mathbb{E}[|X|]$.

3. [19 points] $X$ and $Y$ are independent random variables with densities

$$p_X(j) = \begin{cases} \binom{10}{j} p^j (1-p)^{10-j} & \text{if } j = 0, 1, \dots 10 \\ 0 & \text{else} \end{cases}$$

$$p_Y(j) = \begin{cases} q(1-q)^j & \text{if } j = 0, 1, \dots \\ 0 & \text{else} \end{cases}$$

   Define $Z = X + Y$.

(a) 10 points Compute $\mathbb{P}\{Z = 5\}$.

(b) 9 points Recall that the average of $X$ is $10p$ and the average of $Y$ is $(1 - q)/q$. Compute $\mathbb{E}[2Z + 3]$.

## Answers

1. (a)

$$p_X(j) = \begin{cases} \frac{1}{4}\left(\frac{3}{4}\right)^{j-1} & \text{if } j \in \{1,2\dots\} \\ 0 & \text{else} \end{cases}$$

(b)

$$\varphi_X(\theta) \overset{\text{def}}{=} \mathbb{E}\left[e^{\theta X}\right] = \sum_{j=1}^{\infty} e^{\theta j}\frac{1}{4}\left(\frac{3}{4}\right)^{j-1} = \frac{1}{4}e^{\theta}\sum_{j=0}^{\infty}\left(\frac{3}{4}e^{\theta j}\right)^{j}$$
$$= \begin{cases} \frac{e^\theta/4}{1-3e^\theta/4} & \text{if } e^\theta < 4/3 \\ \infty & \text{else} \end{cases}$$

Note that $\varphi_X'(\theta) = \varphi_X(\theta) + (3e^\theta/4)\varphi_X(\theta)/(1-3e^\theta/4) = \varphi_X(\theta)/(1-3e^\theta/4)$ if $e^\theta < 4/3$.

(c) $\mathbb{E}[X] = \varphi_X'(0) = 4$.

(d) We have that

$$\varphi_X''(\theta) = \frac{\varphi_X(\theta)}{\left(1-\frac{3}{4}e^\theta\right)^2} + \frac{3}{4}e^\theta\frac{\varphi_X(\theta)}{\left(1-\frac{3}{4}e^\theta\right)^2}.$$

Thus $\mathbb{E}[X^2] = \varphi_X''(0) = 16 + 12 = 28$.

2. (a) $p_X(8) = 1 - 0.1 - 0.2 - 0.15 - 0.2 - 0.1 - 0.15 - 0.05 = 1 - 0.95 = 0.05$.

(b) $\mathbb{P}\{X > 0\} = 0.2 + 0.1 + 0.15 + 0.05 + 0.05 = 0.55$.

(c) $\mathbb{P}\{X \text{ is odd}\} = 0.1 + 0.2 + 0.2 + 0.15 + 0.05 = 0.7$.

(d) $\mathbb{P}\{X \text{ is odd and positive}\} = 0.2 + 0.15 + 0.05 = 0.4$.

(e)

$$\mathbb{P}\{X \text{ is odd}|X > 0\} = \frac{0.4}{0.55} = \frac{4}{55}.$$

(f)

$$\mathbb{E}[X] = 0.1(-3) + 0.2(-1) + 0.2(1) + 0.1(2) + 0.15(3) + 0.05(5) + 0.05(8)$$
$$= -0.3 - 0.2 + 0.2 + 0.2 + 0.45 + 0.25 + 0.4 = 1.$$

(g)

$$\mathbb{E}[|X|] = 0.1(3) + 0.2(1) + 0.2(1) + 0.1(2) + 0.15(3) + 0.05(5) + 0.05(8)$$
$$= 0.3 + 0.2 + 0.2 + 0.2 + 0.45 + 0.25 + 0.4 = 2.$$

3. (a)

$$\mathbb{P}\{Z = 5\} = \sum_{j=0}^{5}\binom{10}{j}p^j(1-p)^{10-j}q(1-q)^{5-j}.$$

(b) $\mathbb{E}[2Z+3] = 2\mathbb{E}[Z] + 3 = 2\left(\mathbb{E}[X] + \mathbb{E}[Y]\right) + 3 = 2(10p + (1-q)/q) + 3$.

# Math 361, Section F1, Fall 1997
# Exam 3, December 3

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [30 points] . Assume that $X$ is a continuous random variable which is uniformly distributed on $[0,1]$. Define $Y = \alpha X + \beta$, where $\alpha$ and $\beta$ are constants. Note that I am not specifying $\alpha$ or $\beta$; I want the answer for *all* $\alpha$ and $\beta$.
   (a) [20 points] Find the cumulative distribution function for $Y$.
   (b) [10 points] If $Y$ has a density, find it. If $Y$ does not have a density, state so.

2. [60 points] Suppose that $X$ and $Y$ are two continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 6(t-s) & \text{if } 0 \le s < t \le 1 \\ 0 & \text{else} \end{cases}$$

   (a) [30 points] Find the cumulative distribution function for $(X,Y)$.
   (b) [15 points] Find the density of $X$ (i.e., the marginal density $f_X$).
   (c) [15 points] Find the density of $Y$ (i.e., the marginal density $f_Y$).

3. [10 points] Let $X$ be a continuous random variable with distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2}t + \frac{1}{4} & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

   (a) [5 points] Compute $\mathbb{P}\{X \le 1\}$.
   (b) [5 points] Compute $\mathbb{P}\{0 \le X < 1\}$.

1. (a) $X$ has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

If $\alpha > 0$, then

$$\begin{aligned} F_Y(t) &= \mathbb{P}\{\alpha X + \beta \le t\} = \mathbb{P}\left\{X \le \frac{t-\beta}{\alpha}\right\} = F_X\left(\frac{t-\beta}{\alpha}\right) \\ &= \begin{cases} 0 & \text{if } (t-\beta)/\alpha < 0 \\ (t-\beta)/\alpha & \text{if } 0 \le (t-\beta)/\alpha < 1 \\ 1 & \text{if } (t-\beta)/\alpha \ge 1 \end{cases} = \begin{cases} 0 & \text{if } t < \beta < 0 \\ (t-\beta)/\alpha & \text{if } \beta \le t < \alpha+\beta \\ 1 & \text{if } t \ge \alpha+\beta \end{cases} \end{aligned}$$

If $\alpha < 0$, then

$$\begin{aligned} F_Y(t) &= \mathbb{P}\{\alpha X + \beta \le t\} = \mathbb{P}\left\{X \ge \frac{t-\beta}{\alpha}\right\} = 1 - F_X\left(\frac{t-\beta}{\alpha}-\right) \\ &= \begin{cases} 1-0 & \text{if } (t-\beta)/\alpha \le 0 \\ 1-(t-\beta)/\alpha & \text{if } 0 < (t-\beta)/\alpha \le 1 \\ 1-1 & \text{if } (t-\beta)/\alpha > 1 \end{cases} = \begin{cases} 0 & \text{if } t < \alpha+\beta \\ 1-(t-\beta)/\alpha & \text{if } \alpha+\beta \le t < \beta \\ 1 & \text{if } t \ge \beta \end{cases} \end{aligned}$$

Finally, if $\alpha = 0$,

$$F_Y(t) = \mathbb{P}\{\beta \le t\} = \begin{cases} 0 & \text{if } t < \beta \\ 1 & \text{if } t \ge \beta \end{cases}$$

(b) If $\alpha > 0$, then

$$f_Y(t) = \begin{cases} \frac{1}{\alpha} & \text{if } \beta < t < \alpha+\beta \\ 0 & \text{else} \end{cases}$$

If $\alpha < 0$, then

$$f_Y(t) = \begin{cases} -\frac{1}{\alpha} & \text{if } \alpha+\beta < t < \beta \\ 0 & \text{else} \end{cases}$$

Finally, if $\alpha = 0$, $Y$ does not have a density.

2. We can write

$$f_{X,Y}(s,t) = 6(t-s)\chi_{\{0 \le s < t \le 1\}}$$

(a) For $0 \le s \le t \le 1$,

$$\begin{aligned} F_{X,Y}(s,t) = 6\int_{u=-\infty}^{s}\int_{v=-\infty}^{t}(v-u)\chi_{\{0\le u<v\le 1\}}dv\,du &= 6\int_{u=0}^{s}\int_{v=u}^{t}(v-u)dv\,du \\ &= 3\int_{u=0}^{s}(t-u)^2du = t^3-(t-s)^3. \end{aligned}$$

For $0 \le s < 1 \le t$, $F_{X,Y}(s,t) = F_{X,Y}(s,1) = 1-(1-s)^3$. For $0 \le t \le 1$ and $s \ge t$, $F_{X,Y}(s,t) = F_{X,Y}(t,t) = t^3$. Finally, if $t \le 0$ or $s \le 0$, $F_{X,Y}(s,t) = 0$ while if $t \ge 1$ and $s \ge 1$, $F_{X,Y}(s,t) = 1$.

(b) We first have that

$$F_X(s) = F_{X,Y}(s,\infty) = \begin{cases} 0 & \text{if } s < 0 \\ 1-(1-s)^3 & \text{if } 0 \le s < 1 \\ 1 & \text{if } s \ge 1 \end{cases}$$

Hence

$$f_X(s) = \begin{cases} 3(1-s)^2 & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases}$$

(c) We first have that

$$F_Y(t) = F_{X,Y}(\infty,t) = \begin{cases} 0 & \text{if } t < 0 \\ t^3 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

Hence

$$f_X(s) = \begin{cases} 3t^2 & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases}$$

3. (a) $\mathbb{P}\{X \le 1\} = F_X(1) = 1$.

(b) $\mathbb{P}\{0 \le X < 1\} = \mathbb{P}\{X < 1\} - \mathbb{P}\{X < 0\} = F_X(1-) - F_X(0-) = \frac{3}{4} - 0 = \frac{3}{4}$.

# Math 361, Section F1, Fall 1997
# Final, December 19

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [20 points] Suppose a factory has three machines, $A$, $B$, and $C$, which make 10%, 30%, and 70% of the total production, respectively. Of their output, machine $A$ produces 3% defective items, machine $B$ produces 5% defective items, and machine $C$ produces 1% defective items,
   (a) [10 points] Compute the probability that a defective part was produced by machine $B$.
   (b) [10 points] Compute the probability that a defective part was produced by either machine $A$ or machine $B$.

2. [35 points] Let $X$ and $Y$ be independent random variables having geometric densities with parameters $p$ and $q$ respectively. In other words,

$$p_X(j) = \begin{cases} p(1-p)^j & \text{if } j = 0, 1 \dots \\ 0 & \text{else} \end{cases}$$

$$p_Y(j) = \begin{cases} q(1-q)^j & \text{if } j = 0, 1 \dots \\ 0 & \text{else} \end{cases}$$

   Let $Z = X + Y$.
   (a) [15 points] Compute $\mathbb{P}\{X \geq Y\}$
   (b) [10 points] Compute the moment generating function for $X$.
   (c) [5 points] Compute the moment generating function for $Y$.
   (d) [5 points] Compute the moment generating function for $Z$.

3. [25 points] Let $\theta$ be an angle uniformly chosen between $-\pi/2$ and $\pi/2$. Let $X$ be the sine of $\theta$.
   (a) [15 points] Find the distribution of $X$.
   (b) [10 points] Find the density of $X$ if it exists; if it doesn't, state so.

4. [10 points] A box has 10 balls labelled $1, 2 \dots 10$. Suppose a random sample of size 3 is taken. Find the probability that balls 2 and 7 are among those in the sample.

5. [60 points] Suppose that $X$ and $Y$ are two continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 6(t-s) & \text{if } 0 \leq s < t \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) [30 points] Find the cumulative distribution function for $(X, Y)$.

(b) [15 points] Find the density of $X$ (i.e., the marginal density $f_X$).

(c) [15 points] Find the density of $Y$ (i.e., the marginal density $f_Y$).

## Answers

1. First note that

$$\begin{aligned}\mathbb{P}(A\cap D)&=\mathbb{P}(D|A)\mathbb{P}(A)=(0.03)(0.1)=0.003\\ \mathbb{P}(B\cap D)&=\mathbb{P}(D|B)\mathbb{P}(B)=(0.05)(0.3)=0.015\\ \mathbb{P}(C\cap D)&=\mathbb{P}(D|C)\mathbb{P}(C)=(0.01)(0.7)=0.007\end{aligned}$$

Thus,

$$\mathbb{P}(D)=\mathbb{P}(A\cap D)+\mathbb{P}(B\cap D)+\mathbb{P}(C\cap D)=0.003+0.015+0.007=0.025.$$

(a)

$$\mathbb{P}(B|D)=\frac{\mathbb{P}(B\cap D)}{\mathbb{P}(D)}=\frac{0.015}{0.025}=\frac{15}{25}=0.6.$$

(b)

$$\mathbb{P}(A\cup B|D)=\frac{\mathbb{P}(A\cap D)+\mathbb{P}(B\cap D)}{\mathbb{P}(D)}=\frac{0.003+0.015}{0.025}=\frac{18}{25}=0.72.$$

2. (a)

$$\begin{aligned}\mathbb{P}\{X\ge Y\}&=\sum_{j=0}^\infty \mathbb{P}\{X\ge j\}\mathbb{P}\{Y=j\}=\sum_{j=0}^\infty\left\{\sum_{k=j}^\infty p(1-p)^k\right\}q(1-q)^j\\ &=\sum_{j=0}^\infty (1-p)^j q(1-q)^j=\frac{q}{1-(1-p)(1-q)}.\end{aligned}$$

(b)

$$\mathbb{E}\left[e^{\theta X}\right]=\sum_{j=0}^\infty e^{\theta j}p(1-p)^j=\begin{cases}\frac{p}{1-e^\theta(1-p)} & \text{if } e^\theta<1/(1-p)\\ \infty & \text{else}\end{cases}$$

(c)

$$\mathbb{E}\left[e^{\theta Y}\right]=\sum_{j=0}^\infty e^{\theta j}q(1-q)^j=\begin{cases}\frac{q}{1-e^\theta(1-q)} & \text{if } e^\theta<1/(1-q)\\ \infty & \text{else}\end{cases}$$

(d)

$$\begin{aligned}\mathbb{E}\left[e^{\theta Z}\right]&=\mathbb{E}\left[e^{\theta X}\right]\mathbb{E}\left[e^{\theta Y}\right]\\ &=\begin{cases}\frac{pq}{(1-e^\theta(1-p))(1-e^\theta(1-q))} & \text{if } e^\theta<\min\{1/(1-q),1/(1-p)\}\\ \infty & \text{else}\end{cases}\end{aligned}$$

3. (a)

$$F_X(t) = \mathbb{P}\{\sin\theta \le t\} = \begin{cases} 0 & \text{if } t < -1 \\ \mathbb{P}\{\theta \le \arcsin t\} & \text{if } -1 < t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < -1 \\ \frac{\arcsin t + \pi/2}{\pi} & \text{if } -1 < t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(b)

$$f_X(t) = \begin{cases} \frac{1}{\pi\sqrt{1-t^2}} & \text{if } -1 < t < 1 \\ 0 & \text{else} \end{cases}$$

4.

$$\frac{7}{\binom{10}{3}}.$$

5. We can write

$$f_{X,Y}(s,t) = 6(t-s)\chi_{\{0\le s<t\le 1\}}$$

(a) For $0 \le s \le t \le 1$,

$$F_{X,Y}(s,t) = 6\int_{u=-\infty}^{s}\int_{v=-\infty}^{t}(v-u)\chi_{\{0\le u<v\le 1\}}dv\,du = 6\int_{u=0}^{s}\int_{v=u}^{t}(v-u)dv\,du$$
$$= 3\int_{u=0}^{s}(t-u)^2du = t^3-(t-s)^3.$$

For $0 \le s < 1 \le t$, $F_{X,Y}(s,t) = F_{X,Y}(s,1) = 1-(1-s)^3$. For $0 \le t \le 1$ and $s \ge t$, $F_{X,Y}(s,t) = F_{X,Y}(t,t) = t^3$. Finally, if $t \le 0$ or $s \le 0$, $F_{X,Y}(s,t) = 0$ while if $t \ge 1$ and $s \ge 1$, $F_{X,Y}(s,t) = 1$.

(b) We first have that

$$F_X(s) = F_{X,Y}(s,\infty) = \begin{cases} 0 & \text{if } s < 0 \\ 1-(1-s)^3 & \text{if } 0 \le s < 1 \\ 1 & \text{if } s \ge 1 \end{cases}$$

Hence

$$f_X(s) = \begin{cases} 3(1-s)^2 & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases}$$

(c) We first have that

$$F_Y(t) = F_{X,Y}(\infty,t) = \begin{cases} 0 & \text{if } t < 0 \\ t^3 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

Hence

$$f_X(s) = \begin{cases} 3t^2 & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases}$$

# Math 361, Section F1, Fall 1997
# Final (makeup), December 19

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [20 points] Suppose a factory has three machines, $A$, $B$, and $C$, which make 10%, 30%, and 70% of the total production, respectively. Of their output, machine $A$ produces 3% defective items, machine $B$ produces 5% defective items, and machine $C$ produces 1% defective items,
   (a) [10 points] Compute the probability that a defective part was produced by machine $B$.
   (b) [10 points] Compute the probability that a defective part was produced by either machine $A$ or machine $B$.

2. [35 points] Let $X$ and $Y$ be independent continuous uniform random variables on $[0,1]$. Let $Z = X + Y$.
   (a) [15 points] Compute $\mathbb{P}\{X \geq Y\}$
   (b) [10 points] Compute the moment generating function for $X$.
   (c) [5 points] Compute the moment generating function for $Y$.
   (d) [5 points] Compute the moment generating function for $Z$.

3. [25 points] Let $\theta$ be an angle uniformly chosen between $-\pi/2$ and $\pi/2$. Let $X$ be the sine of $\theta$.
   (a) [15 points] Find the distribution of $X$.
   (b) [10 points] Find the density of $X$ if it exists; if it doesn't, state so.

4. [10 points] A box has 10 balls labelled $1, 2 \ldots 10$. Suppose a random sample of size 3 is taken. Find the probability that balls 2 and 7 are among those in the sample.

5. [60 points] Assume that $X$ is a continuous random variable which is uniformly distributed on $[0,1]$. Define $Y = \alpha X + \beta$, where $\alpha$ and $\beta$ are constants. Note that I am not specifying $\alpha$ or $\beta$; I want the answer for *all* $\alpha$ and $\beta$.
   (a) [40 points] Find the cumulative distribution function for $Y$.
   (b) [20 points] If $Y$ has a density, find it. If $Y$ does not have a density, state so.

## Answers

1. First note that

$$
\begin{aligned}
\mathbb{P}(A\cap D)=\mathbb{P}(D|A)\mathbb{P}(A)=(0.03)(0.1)=0.003\\
\mathbb{P}(B\cap D)=\mathbb{P}(D|B)\mathbb{P}(B)=(0.05)(0.3)=0.015\\
\mathbb{P}(C\cap D)=\mathbb{P}(D|C)\mathbb{P}(C)=(0.01)(0.7)=0.007
\end{aligned}
$$

Thus,

$$\mathbb{P}(D)=\mathbb{P}(A\cap D)+\mathbb{P}(B\cap D)+\mathbb{P}(C\cap D)=0.003+0.015+0.007=0.025.$$

(a)

$$\mathbb{P}(B|D)=\frac{\mathbb{P}(B\cap D)}{\mathbb{P}(D)}=\frac{0.015}{0.025}=\frac{15}{25}=0.6.$$

(b)

$$\mathbb{P}(A\cup B|D)=\frac{\mathbb{P}(A\cap D)+\mathbb{P}(B\cap D)}{\mathbb{P}(D)}=\frac{0.003+0.015}{0.025}=\frac{18}{25}=0.72.$$

2. (a)

$$\mathbb{P}\{X\ge Y\}=\int_{s=-\infty}^{\infty}\int_{t=-\infty}^{s}f_{X,Y}(s,t)ds\,dt=\int_{s=0}^{1}\int_{t=0}^{s}dt\,ds=\frac{1}{2}.$$

(b)

$$\mathbb{E}\left[e^{\theta X}\right]=\int_{s=0}^{1}e^{\theta s}ds=\begin{cases}\frac{e^{\theta}-1}{\theta} & \text{if } \theta\ne 0\\ 1 & \text{if } \theta=0\end{cases}$$

(c)

$$\mathbb{E}\left[e^{\theta Y}\right]=\int_{s=0}^{1}e^{\theta s}ds=\begin{cases}\frac{e^{\theta}-1}{\theta} & \text{if } \theta\ne 0\\ 1 & \text{if } \theta=0\end{cases}$$

(d)

$$\mathbb{E}\left[e^{\theta Z}\right]=\mathbb{E}\left[e^{\theta X}\right]\mathbb{E}\left[e^{\theta Y}\right]=\begin{cases}\frac{\left(e^{\theta}-1\right)^2}{\theta^2} & \text{if } \theta\ne 0\\ 1 & \text{if } \theta=0\end{cases}$$

3. (a)

$$
\begin{aligned}
F_X(t)=\mathbb{P}\{\sin\theta\le t\}&=\begin{cases}0 & \text{if } t<-1\\ \mathbb{P}\{\theta\le \arcsin t\} & \text{if } -1<t<1\\ 1 & \text{if } t\ge 1\end{cases}\\
&=\begin{cases}0 & \text{if } t<-1\\ \frac{\arcsin t+\pi/2}{\pi} & \text{if } -1<t<1\\ 1 & \text{if } t\ge 1\end{cases}
\end{aligned}
$$

(b)

$$f_X(t) = \begin{cases} \frac{1}{\pi\sqrt{1-t^2}} & \text{if } -1 < t < 1 \\ 0 & \text{else} \end{cases}$$

4.

$$\frac{7}{\binom{10}{3}}.$$

5. (a) $X$ has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

If $\alpha > 0$, then

$$F_Y(t) = \mathbb{P}\{\alpha X + \beta \le t\} = \mathbb{P}\left\{X \le \frac{t-\beta}{\alpha}\right\} = F_X\left(\frac{t-\beta}{\alpha}\right)$$
$$= \begin{cases} 0 & \text{if } (t-\beta)/\alpha < 0 \\ (t-\beta)/\alpha & \text{if } 0 \le (t-\beta)/\alpha < 1 \\ 1 & \text{if } (t-\beta)/\alpha \ge 1 \end{cases} = \begin{cases} 0 & \text{if } t < \beta < 0 \\ (t-\beta)/\alpha & \text{if } \beta \le t < \alpha+\beta \\ 1 & \text{if } t \ge \alpha+\beta \end{cases}$$

If $\alpha < 0$, then

$$F_Y(t) = \mathbb{P}\{\alpha X + \beta \le t\} = \mathbb{P}\left\{X \ge \frac{t-\beta}{\alpha}\right\} = 1 - F_X\left(\frac{t-\beta}{\alpha}-\right)$$
$$= \begin{cases} 1-0 & \text{if } (t-\beta)/\alpha \le 0 \\ 1-(t-\beta)/\alpha & \text{if } 0 < (t-\beta)/\alpha \le 1 \\ 1-1 & \text{if } (t-\beta)/\alpha > 1 \end{cases} = \begin{cases} 0 & \text{if } t < \alpha+\beta \\ 1-(t-\beta)/\alpha & \text{if } \alpha+\beta \le t < \beta \\ 1 & \text{if } t \ge \beta \end{cases}$$

Finally, if $\alpha = 0$,

$$F_Y(t) = \mathbb{P}\{\beta \le t\} = \begin{cases} 0 & \text{if } t < \beta \\ 1 & \text{if } t \ge \beta \end{cases}$$

(b) If $\alpha > 0$, then

$$f_Y(t) = \begin{cases} \frac{1}{\alpha} & \text{if } \beta < t < \alpha+\beta \\ 0 & \text{else} \end{cases}$$

If $\alpha < 0$, then

$$f_Y(t) = \begin{cases} -\frac{1}{\alpha} & \text{if } \alpha+\beta < t < \beta \\ 0 & \text{else} \end{cases}$$

Finally, if $\alpha = 0$, $Y$ does not have a density.