Math 361, Section D1, Fall 1996
Quiz 1, September 30 **Name:** ____________________

1. 10 points The Morse code consists of a sequence of dots and dashes with repetitions permitted.
   (a) 5 points How many letters can be represented using exactly n symbols?
   (b) 5 points How many letters can be represented using n or fewer symbols?

Answers

1. (a) 2^n.

   (b) $\sum_{j=1}^n 2^j = 2^{n+1} - 2$.

Math 361, Section D1, Fall 1996
Quiz 2, October 11 **Name:** ____________________

1. 10 points Evaluate the following:

$$\int_{x=0}^{3}\int_{y=0}^{4} xy^2 dx\, dy.$$

Answers

1. 84.

Math 361, Section D1, Fall 1996
Quiz 3, October 21 **Name:** ____________________

1. [10 points] Let X have a binomial distribution with parameters n and p. Calculate $\mathbb{E}[t^X]$.

Answers

1.

$$\mathbb{E}[t^X] = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} t^j = \sum_{j=0}^{n} \binom{n}{j} (tp)^j (1-p)^{n-j} = (tp+1-p)^n.$$

Math 361, Section D1, Fall 1996
Quiz 4, October 28 **Name:** ____________________

1. [10 points] Let X be a random variable with moment generating function $\Phi(t) = e^{\lambda(t-1)}$ where $\lambda > 0$ is some parameter (recall that $\Phi(t) \stackrel{\text{def}}{=} \mathbb{E}[t^X]$).
   (a) [5 points] Calculate $\mathbb{E}[X]$.
   (b) [5 points] Calculate the variance of X.

1. Note that $\Phi'(t) = \lambda\Phi(t)$ and $\Phi''(t) = \lambda^2\Phi(t)$.
   (a) $\mathbb{E}[X] = \Phi'(1) = \lambda$.
   (b) Since $\Phi''(t) = \mathbb{E}[X^2 - X]$,

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2 - X] + \mathbb{E}[X] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Math 361, Section D1, Fall 1996
Quiz 5, November 14 **Name:** ______________________

1. 10 points Let X be a random variable with distribution function

$$F(x) = \begin{cases} \frac{1}{3}e^{x+2} & \text{if } x < -2 \\ \frac{1}{2} & \text{if } -2 \le x < 10 \\ 1 & \text{if } x \ge 10. \end{cases}$$

   (a) 5 points Calculate $\mathbb{P}\{X \ge -100\}$.

   (b) 5 points Calculate $\mathbb{P}\{X \ge -2\}$.

Answers

1. (a) $\mathbb{P}\{X \geq -100\} = 1 - F_X(-100-) = 1 - \frac{1}{3}e^{-98}$.

   (b) $\mathbb{P}\{X \geq -2\} = 1 - F_X(-2-) = 1 - \frac{1}{3}$.

Math 361, Section D1, Fall 1997
Quiz 6, November 22 **Name:** ____________________

1. 10 points Let X and Y be independent random variables. Let X be uniform on $[0,1]$, and let Y be exponential with parameter λ. Define $Z = X + Y$. Calculate $f_Z(5)$; i.e., the density of Z at 5.

Answers

1.

$$f_Z(5) = \int_{s=-\infty}^{\infty} \lambda e^{-\lambda s} \chi_{[0,\infty)}(s) \chi_{[0,1]}(5-s) ds = \int_{s=4}^{5} \lambda e^{-\lambda s} ds = e^{-4\lambda} - e^{-5\lambda}.$$

Math 361, Section D1, Fall 1996
Quiz 7, December 6 **Name:** ____________________

1. [10 points] Compute $1 + 1$.

Answers

1. 2.

Math 361, Section D1, Fall 1996
Exam 1, October 2

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [15 points] A domino is a rectangular block divided into two equal subrectangles. Each rectangle has a number on it; let these be x and y (not necessarily distinct). Since the block is symmetric, the domino (x, y) is the same as (y, x). How many different domino blocks can be made using n different numbers?

2. [20 points] A box has 10 balls labelled $1, 2, \dots 10$. Suppose a random sample of size 3 is selected. Find the probability that balls 1 and 6 are among the three selected balls.

3. [40 points] Suppose n balls are distributed into n boxes. Let

$$A \stackrel{\text{def}}{=} \{\text{one box is empty}\} \qquad \text{and} \qquad B \stackrel{\text{def}}{=} \{\text{box 1 is empty}\}.$$

   (a) [10 points] What is $\mathbb{P}(A)$?
   (b) [10 points] What is $\mathbb{P}(B)$?
   (c) [10 points] What is $\mathbb{P}(A \cap B)$?
   (d) [5 points] What is $\mathbb{P}(A|B)$?
   (e) [5 points] What is $\mathbb{P}(B|A)$?

4. [20 points] If you hold 4 tickets for a lottery for which n tickets were sold, and 6 prizes are to be given, what is the probability that you will win at least one prize? Note what is going on here. You buy your tickets and then the selection of the winning tickets occurs.

5. [5 points] Suppose a factory has two machines A and B that make 70% and 30% of the total production, respectively. Of their output, machine A produces 3% defective items, while machine B produces 5% defective items. Find the probability that a given defective part was produced by machine B.

1.
$$\binom{n}{2} + n.$$

2.
$$\frac{1}{\binom{8}{3}}.$$

3. (a)
$$\mathbb{P}(A) = \frac{\binom{n}{2}(n)_{n-1}}{n^n}.$$

(b)
$$\mathbb{P}(B) = \frac{(n-1)^n}{n^n}.$$

(c)
$$\mathbb{P}(A \cap B) = \frac{\binom{n}{2}(n-1)!}{n^n}.$$

(d)
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\binom{n}{2}(n-1)!}{(n-1)^n}.$$

(e)
$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{(n-1)!}{(n)_{n-1}} = \frac{1}{n}.$$

4.
$$1 - \mathbb{P}\{\text{no prizes}\} = 1 - \frac{\binom{n-4}{6}}{\binom{n}{4}}.$$

5.
$$\mathbb{P}(B|D) = \frac{\mathbb{P}(D \cap B)}{\mathbb{P}(D)} = \frac{\mathbb{P}(D \cap B)}{\mathbb{P}(D \cap A) + \mathbb{P}(D \cap B)} = \frac{\mathbb{P}(D|B)\mathbb{P}(B)}{\mathbb{P}(D|A)\mathbb{P}(A) + \mathbb{P}(D|B)\mathbb{P}(B)}$$
$$= \frac{(0.05)(0.3)}{(0.03)(0.7) + (0.05)(0.3)} = \frac{15}{12 + 15} = \frac{15}{36} = \frac{5}{12}.$$

Math 361, Section D1, Fall 1996
Exam 2, October 30

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [35 points] Suppose that X is a random variable with density

$$p_X(j) = \begin{cases} 0.1 & \text{if } j = -3 \\ 0.2 & \text{if } j = -1 \\ 0.15 & \text{if } j = 0 \\ 0.2 & \text{if } j = 1 \\ 0.1 & \text{if } j = 2 \\ 0.15 & \text{if } j = 3 \\ 0.05 & \text{if } j = 5 \\ 0.05 & \text{if } j = 8 \end{cases}$$

   (a) [5 points] Compute $\mathbb{P}\{X \ge 0\}$.

   (b) [5 points] Compute $\mathbb{P}\{X \text{ is even}\}$.

   (c) [5 points] Compute $\mathbb{P}\{1 \le X \le 8\}$.

   (d) [10 points] Compute $\mathbb{P}\{X = -3 | X \le 0\}$.

   (e) [10 points] Compute $\mathbb{P}\{X \ge 3 | X > 0\}$.

2. [10 points] Suppose that X and Y are two independent random variables with

$$\mathbb{E}[X^4] = 5, \qquad \mathbb{E}[Y^2] = 10, \qquad \mathbb{E}[X^2] = 2, \qquad \text{and} \qquad \mathbb{E}[Y] = 3.$$

   Compute the variance of $Z \stackrel{\text{def}}{=} X^2 Y$.

3. [55 points] Let X and Y be independent geometric random variables with parameters p_1 and p_2 respectively.

   (a) [8 points] Compute $\mathbb{P}\{X \ge Y\}$.

   (b) [8 points] Compute $\mathbb{P}\{X = Y\}$.

   (c) [8 points] Compute the density of $Z_1 = \min(X, Y)$.

   (d) [8 points] Compute the density of $Z_2 = X + Y$.

   (e) [8 points] Compute the mean of Z_1.

   (f) [8 points] Compute the mean of Z_2.

   (g) [7 points] Compute the moment generating function of Z_2.

Answers

1. (a) $\mathbb{P}\{X \geq 0\} = 0.15 + 0.2 + 0.1 + 0.15 + 0.05 + 0.05 = 0.7$.

   (b) $\mathbb{P}\{X \text{ is even}\} = 0.15 + 0.1 + 0.05 = 0.3$.

   (c) $\mathbb{P}\{1 \leq X \leq 8\} = 0.2 + 0.1 + 0.15 + 0.05 + 0.05 = 0.55$.

   (d)
$$\mathbb{P}\{X = -3 | X \leq 0\} = \frac{\mathbb{P}\{X = -3\}}{\mathbb{P}\{X \leq 0\}} = \frac{0.1}{0.15 + 0.2 + 0.1} = \frac{0.1}{0.45} = \frac{2}{9}.$$

   (e)
$$\mathbb{P}\{X \geq 3 | X > 0\} = \frac{\mathbb{P}\{X \geq 3\}}{\mathbb{P}\{X > 0\}} = \frac{0.15 + 0.05 + 0.05}{0.2 + 0.1 + 0.15 + 0.05 + 0.05} = \frac{0.25}{0.55} = \frac{5}{11}.$$

2.
$$\begin{aligned}\mathbb{E}\left[Z^2\right] - \mathbb{E}[Z]^2 &= \mathbb{E}\left[(X^2Y)^2\right] - \mathbb{E}\left[X^2Y\right]^2 \\ &= \mathbb{E}\left[X^4\right]\mathbb{E}\left[Y^2\right] - \mathbb{E}\left[X^2\right]^2\mathbb{E}[Y]^2 = 5 \cdot 10 - 4 \cdot 0 = 50 - 36 = 14.\end{aligned}$$

3. (a)
$$\begin{aligned}\mathbb{P}\{X \geq Y\} &= \sum_{k=1}^{\infty}\mathbb{P}\{X \geq Y = k\} = \sum_{k=1}^{\infty}\left\{\sum_{j=k}^{\infty}\mathbb{P}\{X = j\}\right\}\mathbb{P}\{Y = k\} \\ &= p_1p_2\sum_{k=1}^{\infty}\left\{\sum_{j=k}^{\infty}(1-p_1)^{j-1}\right\}(1-p_2)^{k-1} \\ &= p_1p_2\sum_{k=1}^{\infty}\left\{\sum_{j=k-1}^{\infty}(1-p_1)^{j}\right\}(1-p_2)^{k-1} = p_2\sum_{k=1}^{\infty}(1-p_1)^{k-1}(1-p_2)^{k-1} \\ &= p_2\sum_{k=0}^{\infty}\{(1-p_1)(1-p_2)\}^k = \frac{p_2}{1-(1-p_1)(1-p_2)}.\end{aligned}$$

   (b)
$$\begin{aligned}\mathbb{P}\{X = Y\} &= \sum_{k=1}^{\infty}\mathbb{P}\{X = Y = k\} = \sum_{k=1}^{\infty}\mathbb{P}\{X = k\}\mathbb{P}\{Y = k\} \\ &= p_1p_2\sum_{k=1}^{\infty}(1-p_1)^{k-1}(1-p_2)^{k-1} = p_1p_2\sum_{k=0}^{\infty}\{(1-p_1)(1-p_2)\}^k \\ &= \frac{p_1p_2}{1-(1-p_1)(1-p_2)}\end{aligned}$$

(c) For $k \in \{1, 2 \dots\}$,

$$\mathbb{P}\{Z_1 \ge k\} = \mathbb{P}\{X \ge k\}\mathbb{P}\{Y \ge k\} = p_1 p_2 \left\{\sum_{j=k}^{\infty}(1-p_1)^{j-1}\right\}\left\{\sum_{j=k}^{\infty}(1-p_2)^{j-1}\right\}$$
$$= p_1 p_2 \left\{\sum_{j=k-1}^{\infty}(1-p_1)^{j}\right\}\left\{\sum_{j=k-1}^{\infty}(1-p_2)^{j}\right\} = \{(1-p_1)(1-p_2)\}^{k-1}$$

Thus, for $k \in \{1, 2 \dots\}$,

$$p_{Z_1}(k) = \mathbb{P}\{Z_1 \ge k\} - \mathbb{P}\{Z_1 \ge k+1\} = \{(1-p_1)(1-p_2)\}^{k-1} - \{(1-p_1)(1-p_2)\}^{k}$$
$$= \{(1-p_1)(1-p_2)\}^{k-1}\{1-(1-p_1)(1-p_2)\}.$$

For all other k, $p_{Z_1}(k) = 0$.

(d) For $j \in \{2, 3 \dots\}$,

$$p_{Z_2}(k) = \sum_{j=-\infty}^{\infty}\mathbb{P}\{X = j\}\mathbb{P}\{Y = k-j\} = p_1 p_2 \sum_{j=1}^{k-1}(1-p_1)^{j-1}(1-p_2)^{k-j-1}$$
$$= p_1 p_2 (1-p_2)^{k-2}\sum_{j=1}^{k-1}\left(\frac{1-p_1}{1-p_2}\right)^{j-1} = p_1 p_2 (1-p_2)^{k-2}\sum_{j=0}^{k-2}\left(\frac{1-p_1}{1-p_2}\right)^{j}$$
$$= p_1 p_2 (1-p_2)^{k-2}\left\{\frac{1-\left(\frac{1-p_1}{1-p_2}\right)^{k-1}}{1-\left(\frac{1-p_1}{1-p_2}\right)}\right\} = p_1 p_2 \left\{\frac{(1-p_2)^{k-1}-(1-p_2)^{k-1}}{(1-p_2)-(1-p_1)}\right\}$$
$$= \frac{p_1 p_2}{p_1 - p_2}\left\{(1-p_2)^{k-1} - (1-p_1)^{k-1}\right\}$$

if $p_1 \ne p_2$; if $p_1 = p_2 = q$, then

$$p_{Z_2}(k) = q^2 \sum_{j=1}^{k-1}(1-q)^{j-1}(1-q)^{k-j-1} = q^2(1-q)^{k-2}\sum_{j=1}^{k-1} 1 = q^2(k-1)(1-q)^{k-2}.$$

If W is geometric with parameter q, then

$$\varphi_W(\theta) \overset{\text{def}}{=} \mathbb{E}\left[e^{\theta W}\right] = q\sum_{j=1}^\infty e^{\theta j}(1-q)^{j-1} = qe^\theta \sum_{j=1}^\infty \left(e^\theta(1-q)\right)^{j-1}$$
$$= qe^\theta \sum_{j=0}^\infty \left(e^\theta(1-q)\right)^{j} = \begin{cases} \frac{qe^\theta}{1-e^\theta(1-q)} & \text{if } e^\theta < 1/(1-q) \\ \infty & \text{else.}\end{cases}$$

Thus, if $e^\theta < 1/(1-q)$,

$$\varphi'_W(\theta) = \varphi_W(\theta) + \varphi_W(\theta)\frac{e^\theta(1-q)}{1-e^\theta(1-q)} = \frac{\varphi_W(\theta)}{1-e^\theta(1-q)}.$$

Thus $\mathbb{E}[W] = \varphi'_W(0) = \frac{1}{q}$.

(e)

$$\mathbb{E}[Z_1] = \frac{1}{1-(1-p_1)(1-p_2)}.$$

(f)

$$\mathbb{E}[Z_2] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] = \frac{1}{p_1} + \frac{1}{p_2}.$$

(g)

$$\varphi_{Z_2}(\theta) = \varphi_X(\theta)\varphi_Y(\theta)$$
$$= \begin{cases} \frac{p_1p_2e^{2\theta}}{(1-e^\theta(1-p_1))(1-e^\theta(1-p_2))} & \text{if } e^\theta < \min\{1/(1-p_1), 1/(1-p_2)\} \\ \infty & \text{else.}\end{cases}$$

Math 361, Section D1, Fall 1996
Exam 3, December 4

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] Suppose that X is a random variable having density f_X. Define two new random variables
$$Y = X + 3 \qquad \text{and} \qquad Z = 4X.$$
   (a) [10 points] Compute the density of Y.
   (b) [10 points] Compute the density of Z.

2. [30 points] Let X and Y be independent and uniformly distributed on $[0,1]$. We are interested in the density of $Z = |X - Y|$.
   (a) [15 points] Compute the density of $\tilde{Z} = X - Y$.
   (b) [15 points] Compute the density of $\tilde{Z} = |Z|$.

3. [20 points] Here we are interested in finding the density of the sum of two Gaussian random variables. Let X and Y be independent Gaussian random variables. Suppose that X has mean μ_1 and variance σ_1^2. Suppose that Y has mean μ_2 and variance σ_2^2. Set $Z = X + Y$.
   (a) [15 points] Compute the moment generating function $\Phi_Z(t) = \mathbb{E}[e^{tZ}]$ of Z. Remember that X and Y are independent.
   (b) [5 points] Assuming that moment generating functions are unique (i.e., that one can completely characterize a density by its moment generating function), what is the density of Z?

4. [30 points] Suppose that X is a random variable with distribution function
$$F(x) = \begin{cases} \frac{1}{3}e^x & \text{if } x < 0 \\ \frac{x+1}{2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$
   (a) [5 points] Compute $\mathbb{P}\{X < 0\}$.
   (b) [5 points] Compute $\mathbb{P}\{X < 1\}$.
   (c) [5 points] Compute $\mathbb{P}\{X \le 0\}$.
   (d) [5 points] Compute $\mathbb{P}\{X \le 1\}$.
   (e) [5 points] Compute $\mathbb{P}\{X > 0.5\}$.
   (f) [5 points] Compute $\mathbb{P}\{0 \le X < 0.5\}$.

ANSWERS

1. (a) Since $F_Y(t) = \mathbb{P}\{X+3 \le t\} = F_X(t-3)$ for all $t \in \mathbb{R}$, $f_Y(t) = f_X(t-3)$ for all $t \in \mathbb{R}$.

   (b) Since $F_Z(t) = \mathbb{P}\{4X \le t\} = F_X(t/4)$ for all $t \in \mathbb{R}$, $f_Z(t) = \frac{1}{4} f_X(t/4)$ for all $t \in \mathbb{R}$.

2. (a) Since $F_{-Y}(t) = \mathbb{P}\{-Y \le t\} = \mathbb{P}\{Y \ge -t\} = 1 - F_Y(-t^-) = 1 - F_Y(-t)$, $f_{-Y}(t) = f_Y(-t)$; thus by convolution,

$$f_{\tilde{Z}}(t) = \int_{s=-\infty}^{\infty} f_X(s) f_{-Y}(t-s)ds = \int_{s=-\infty}^{\infty} f_X(s) f_Y(s-t)ds$$
$$= \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[0,1]}(s-t)ds = \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[t,t+1]}(s)ds$$
$$= \int_{s=-\infty}^{\infty} \chi_{[0,1]\cap[t,t+1]}(s)ds = \begin{cases} 1+t & \text{if } 0 \le t+1 \le 1 \\ 1-t & \text{if } 0 < t \le 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 1+t & \text{if } -1 \le t \le 0 \\ 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

   (b) Note that $F_Z(t) = 0$ if $t < 0$. Since

$$F_Z(t) = \mathbb{P}\{-t \le X \le t\} = F_X(t) - F_X(-t^-) = F_X(t) - F_X(-t),$$
$$f_Z(t) = f_X(t) + f_X(-t) = \begin{cases} 2(1-t) & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

3. Recall that W is a Gaussian random variable with mean μ and variance σ if and only if

$$\mathbb{E}[e^{tX}] = \exp\left[\frac{\sigma^2 t^2}{2} + \mu t\right]$$

   for all $t \in \mathbb{R}$.

   (a)

$$\Phi_Z(t) = \mathbb{E}\left[e^{tZ}\right] = \mathbb{E}\left[e^{t(X+Y)}\right] = \mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[e^{tY}\right]$$
$$= \exp\left[\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right] \exp\left[\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right] = \exp\left[\frac{(\sigma_1^2+\sigma_2^2)t^2}{2} + (\mu_1+\mu_2)t\right]$$

   (b) Z is Gaussian with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

4. (a) $F_X(0^-) = \frac{1}{3}$.

   (b) $F_X(1^-) = 1$.

   (c) $F_X(0) = \frac{1}{2}$.

   (d) $F_X(1) = 1$.

   (e) $1 - F_X(0.5) = 1 - \frac{3}{4} = \frac{1}{4}$.

   (f) $F_X(0.5^-) - F_X(0^-) = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$.

Math 361, Section D1, Fall 1996
Final, December 19

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [15 points] Cards are dealt from an ordinary deck of playing cards one at a time until the first king appears. Find the probability that this occurs with the 10th card dealt.

2. [15 points] Four cards are drawn from a deck. What is the probability that 2 are red and 2 are black?

3. [40 points] Suppose a box has 12 balls labelled $1, 2 \dots 12$. A random sample of size 5 is selected without replacement. Let Y denote the largest of the numbers drawn and let Z denote the smallest of the numbers drawn.
   (a) [10 points] Compute $\mathbb{P}\{Y \le y\}$.
   (b) [10 points] Compute $\mathbb{P}\{Z \ge z\}$.
   (c) [10 points] Compute $\mathbb{P}\{Y = y\}$.
   (d) [10 points] Compute $\mathbb{P}\{Y < Z\}$.

4. [10 points] Let X be an geometric random variable with parameter p. Compute $\mathbb{E}[e^{tX}]$.

5. [25 points] Let X and Y be two random variables having the joint density given by the following table:

		Y			
		-1	0	2	6
	-2	1/9	1/27	1/27	1/9
X	1	2/9	0	1/9	1/9
	3	0	0	1/9	4/27

   (a) [2 points] Compute $\mathbb{P}\{XY \text{ is odd}\}$.
   (b) [2 points] Compute $\mathbb{P}\{X \ge 0 \text{ and } Y < 0\}$.
   (c) [5 points] Compute the density of X.
   (d) [5 points] Compute the density of Y.
   (e) [5 points] Compute $\mathbb{E}[X]$.
   (f) [3 points] Compute $\mathbb{E}[Y]$.
   (g) [3 points] Find $\mathbb{E}[X^2]$.

6. [30 points] Let X and Y be independent and uniformly distributed on $[0, 1]$. We are interested in the density of $Z = |X - Y|$.

(a) $\boxed{\text{15 points}}$ Compute the density of $\tilde{Z} = X - Y$.

(b) $\boxed{\text{15 points}}$ Compute the density of $Z = |\tilde{Z}|$.

7. $\boxed{\text{15 points}}$ Let X have a Gaussian distribution with mean 0 and variance 1. Find the density of $Y = e^X$.

Answers

1.
$$\frac{(48)_9 \cdot 4}{(52)_{10}}.$$

2.
$$\frac{\binom{26}{2}\binom{26}{2}}{\binom{52}{4}}.$$

3. (a) Note that $5 \le Y \le 12$. For all integers y,

$$\mathbb{P}\{Y \le y\} = \begin{cases} 0 & \text{if } y < 5 \\ \frac{\binom{y}{5}}{\binom{12}{5}} & \text{if } y \in \{5, 6, \dots 12\} \\ 1 & \text{if } y > 12 \end{cases}$$

(b) Note that $1 \le Z \le 8$. For all integers z,

$$\mathbb{P}\{Z \ge z\} = \begin{cases} \frac{\binom{13-z}{5}}{\binom{12}{5}} & \text{if } 1 \le z \le 8 \\ 1 & \text{if } z < 1 \\ 0 & \text{if } z > 8 \end{cases}$$

Thus

$$\mathbb{P}\{Z \le z\} - 1 - \mathbb{P}\{Z \ge z+1\} = \begin{cases} 1 - \frac{\binom{12-z}{5}}{\binom{12}{5}} & \text{if } 1 \le z+1 \le 8 \\ 1-1 & \text{if } z+1 < 1 \\ 1-0 & \text{if } z+1 > 8 \end{cases}$$
$$= \begin{cases} 0 & \text{if } z < 0 \\ 1 - \frac{\binom{12-z}{5}}{\binom{12}{5}} & \text{if } 0 \le z \le 7 \\ 1 & \text{if } z > 7 \end{cases}$$

(c)

$$\mathbb{P}\{Y = y\} = \mathbb{P}\{Y \le y\} - \mathbb{P}\{Y \le y-1\} = \begin{cases} \frac{\binom{y}{5} - \binom{y-1}{5}}{\binom{12}{5}} & \text{if } y \in \{6, 7 \dots 12\} \\ \frac{1}{\binom{12}{5}} & \text{if } y = 5 \\ 0 & \text{else} \end{cases}$$
$$= \begin{cases} \frac{\binom{y-1}{4}}{\binom{12}{5}} & \text{if } y \in \{5, 6 \dots 12\} \\ 0 & \text{else} \end{cases}$$

(check that $\binom{y}{5} - \binom{y-1}{5} = \binom{y-1}{4}$).

4.

$$\mathbb{E}\left[e^{tX}\right] = p\sum_{j=1}^{\infty} e^{tj}(1-p)^{j-1} = pe^t \sum_{j=1}^{\infty}\left(e^{2t}(1-p)\right)^{j-1}$$
$$= pe^t \sum_{j=0}^{\infty}\left(e^{2t}(1-p)\right)^{j} = \begin{cases} \frac{pe^t}{1-e^t(1-p)} & \text{if } e^t < 1/(1-p) \\ \infty & \text{else} \end{cases}$$

5. (a) 2/9+0=2/9.
   (b) 2/9+0=2/9.
   (c)

$$p_X(j) = \begin{cases} \frac{8}{27} & \text{if } j=-2 \\ \frac{12}{27} & \text{if } j=1 \\ \frac{7}{27} & \text{if } j=3 \\ 0 & \text{else} \end{cases}$$

   (d)

$$p_Y(j) = \begin{cases} \frac{9}{27} & \text{if } j=-1 \\ \frac{1}{27} & \text{if } j=0 \\ \frac{7}{27} & \text{if } j=2 \\ \frac{10}{27} & \text{if } j=6 \\ 0 & \text{else} \end{cases}$$

   (e)

$$\mathbb{E}[X] = \frac{1}{27}\{(-2)8+(1)12+3(7)\} = \frac{1}{27}\{-16+12+21\} = \frac{17}{27}.$$

   (f)

$$\mathbb{E}[Y] = \frac{1}{27}\{(-1)9+(0)1+(2)7+(6)10\} = \frac{-9+14+60}{27} = \frac{65}{27}.$$

   (g)

$$\mathbb{E}[X^2] = \frac{1}{27}\{(-2)^2 8+(1)^2 12+(3)^7 7\} = \frac{32+12+63}{27} = \frac{107}{27}.$$

6. Since $F_{-Y}(t) = \mathbb{P}\{-Y \le t\} = \mathbb{P}\{Y \ge -t\} = 1 - F_Y(-t^-) = 1 - F_Y(-t)$, we have

that $f_{-Y}(t) = f_Y(-t)$; thus by convolution,

$$\begin{aligned} f_{\tilde{Z}}(t) &= \int_{s=-\infty}^{\infty} f_X(s) f_{-Y}(t-s)\,ds = \int_{s=-\infty}^{\infty} f_X(s) f_Y(s-t)\,ds \\ &= \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[0,1]}(s-t)\,ds = \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[t,t+1]}(s)\,ds \\ &= \int_{s=-\infty}^{\infty} \chi_{[0,1]\cap[t,t+1]}(s)\,ds = \begin{cases} 1+t & \text{if } 0 \le t+1 \le 1 \\ 1-t & \text{if } 0 < t \le 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 1+t & \text{if } -1 \le t \le 0 \\ 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Note that $F_Z(t) = 0$ if $t < 0$. Since

$$F_Z(t) = \mathbb{P}\{-t \le X \le t\} = F_X(t) - F_X(-t^-) = F_X(t) - F_X(-t),$$

$$f_Z(t) = f_X(t) + f_X(-t) = \begin{cases} 2(1-t) & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

(a) 7. First, compute that

$$\begin{aligned} F_Y(t) = \mathbb{P}\left\{e^X \le t\right\} &= \begin{cases} 0 & \text{if } t \le 0 \\ \mathbb{P}\{X < \ln t\} & \text{if } t > 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } t \le 0 \\ F_X(\ln t) & \text{if } t > 0 \end{cases} = \begin{cases} 0 & \text{if } t \le 0 \\ \int_{s=-\infty}^{\ln t} \frac{\exp\left[-\frac{z^2}{2}\right]}{\sqrt{2\pi}}dz & \text{if } t > 0 \end{cases} \end{aligned}$$

Then Y has density

$$f_Y(t) = \begin{cases} \frac{\exp\left[-\frac{(\ln t)^2}{2}\right]}{t\sqrt{2\pi}}dz & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

Math 361, Section D1, Fall 1996
Final (Makeup), December 19

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [15 points] If 12 balls are distributed at random into 4 boxes, what is the probability that box 1 has exactly 7 balls?

2. [15 points] Four cards are drawn from a deck. What is the probability that exactly one is a king?

3. [40 points] Suppose a box has 12 balls labelled $1, 2 \dots 12$. A random sample of size 5 is selected without replacement. Let Y denote the largest of the numbers drawn and let Z denote the smallest of the numbers drawn.
   (a) [10 points] Compute $\mathbb{P}\{Y \geq j\}$ for all integers j.
   (b) [10 points] Compute $\mathbb{P}\{Z \leq j\}$ for all integers j.
   (c) [10 points] Compute $\mathbb{P}\{Y = j\}$ for all integers j.
   (d) [10 points] Compute $\mathbb{P}\{Y \geq Z\}$ for all integers j.

4. [10 points] Let X be an geometric random variable with parameter p. Compute $\mathbb{E}[e^{2tX}]$.

5. [25 points] Let X and Y be two random variables having the joint density given by the following table:

		Y: -1	0	2	6
X	-2	1/9	1/27	1/27	1/9
	1	2/9	0	1/9	1/9
	3	0	0	1/9	4/27

   (a) [2 points] Compute $\mathbb{P}\{XY \text{ is odd}\}$.
   (b) [2 points] Compute $\mathbb{P}\{X \geq 0 \text{ and } Y < 0\}$.
   (c) [5 points] Compute the density of X.
   (d) [5 points] Compute the density of Y.
   (e) [5 points] Compute $\mathbb{E}[2X]$.
   (f) [3 points] Compute $\mathbb{E}[Y]$.
   (g) [3 points] Find $\mathbb{E}[Y^2]$.

6. [30 points] Let X and Y be independent and uniformly distributed on $[0, 1]$. We are interested in the density of $Z = |X - Y|$.

(a) [15 points] Compute the density of $\tilde{Z} = X - Y$.

(b) [15 points] Compute the density of $Z = |\tilde{Z}|$.

7. [15 points] Let X have a Gaussian distribution with mean 0 and variance 1. Find the density of $Y = \ln |X|$.

Answers

1.
$$\frac{\binom{12}{7}3^5}{4^{12}}.$$

2.
$$\frac{4\binom{48}{3}}{\binom{52}{4}}.$$

3. Note that $5 \le Y \le 12$, and that

$$\mathbb{P}\{Y \le j\} = \frac{\binom{j}{5}}{\binom{20}{5}}$$

for all integers $j \in \{5, 6 \dots 12\}$.

(a) We have that

$$\mathbb{P}\{Y \ge j\} = 1 - \mathbb{P}\{Y \le j-1\} = \begin{cases} 1 - \frac{\binom{j-1}{5}}{\binom{20}{5}} & \text{if } 5 \le j-1 \le 12 \\ 1 - 0 & \text{if } j-1 < 4 \\ 1 - 1 & \text{if } j - 1 > 12 \end{cases}$$
$$= \begin{cases} 1 - \frac{\binom{j-1}{5}}{\binom{20}{5}} & \text{if } 6 \le j \le 13 \\ 1 & \text{if } j < 5 \\ 0 & \text{if } j > 13 \end{cases}$$

(b) Note that $1 \le Z \le 8$, and that

$$\mathbb{P}\{Z \ge j\} = \frac{\binom{13-j}{5}}{\binom{20}{5}}$$

for all integers $j \in \{1, 2 \dots 8\}$. Thus

$$\mathbb{P}\{Z \le j\} = 1 - \mathbb{P}\{Z \ge j+1\} = \begin{cases} 1 - 1 & \text{if } j+1 < 1 \\ 1 - \frac{\binom{13-j}{5}}{\binom{20}{5}} & \text{if } 1 \le j+1 \le 8 \\ 1 - 0 & \text{if } j+1 > \end{cases}$$
$$= \begin{cases} 0 & \text{if } j < 0 \\ 1 - \frac{\binom{13-j}{5}}{\binom{20}{5}} & \text{if } 0 \le j \le 7 \\ 1 & \text{if } j > 7. \end{cases}$$

(c) From the beginning comments, we have that

$$\mathbb{P}\{Y=j\} = \mathbb{P}\{Y \le j\} - \mathbb{P}\{Y \le j-1\} = \frac{\binom{j}{5} - \binom{j-1}{5}}{\binom{20}{5}} = \frac{\binom{j-1}{4}}{\binom{20}{5}}$$

for all integers $j \in \{6, 7 \dots 12\}$ (check that $\binom{j}{5} - \binom{j-1}{5} = \binom{j-1}{4}$). Thus

$$\mathbb{P}\{Y=j\} = \begin{cases} \frac{\binom{j-1}{4}}{\binom{20}{5}} & \text{if } 5 \le j \le 12 \\ 0 & \text{else} \end{cases}$$

for all integers j

(d) 1

4.

$$\mathbb{E}\left[e^{2tX}\right] = p\sum_{j=1}^{\infty} e^{2tj}(1-p)^{j-1} = pe^{2t}\sum_{j=1}^{\infty}\left(e^{2t}(1-p)\right)^{j-1}$$
$$= pe^{2t}\sum_{j=0}^{\infty}\left(e^{2t}(1-p)\right)^{j} = \begin{cases} \frac{pe^{2t}}{1-e^{2t}(1-p)} & \text{if } e^{2t} < 1/(1-p) \\ \infty & \text{else} \end{cases}$$

5. (a) 2/9+0=2/9.

(b) 2/9+0=2/9.

(c)

$$p_X(j) = \begin{cases} \frac{8}{27} & \text{if } j=-2 \\ \frac{12}{27} & \text{if } j=1 \\ \frac{7}{27} & \text{if } j=3 \\ 0 & \text{else} \end{cases}$$

(d)

$$p_Y(j) = \begin{cases} \frac{9}{27} & \text{if } j=-1 \\ \frac{1}{27} & \text{if } j=0 \\ \frac{7}{27} & \text{if } j=2 \\ \frac{10}{27} & \text{if } j=6 \\ 0 & \text{else} \end{cases}$$

(e)

$$\mathbb{E}[2X] = 2\mathbb{E}[X] = \frac{2}{27}\{(-2)8 + (1)12 + 3(7)\} = \frac{2}{27}\{-16+12+21\} = \frac{2\cdot 17}{27} = \frac{34}{27}.$$

(f)

$$\mathbb{E}[Y] = \frac{1}{27}\{(-1)9 + (0)1 + (2)7 + (6)10\} = \frac{-9+14+60}{27} = \frac{65}{27}.$$

(g)

$$\mathbb{E}[Y^2] = \frac{1}{27}\{(-1)^2 9 + (0)^2 1 + (2)^2 7 + (6)^2 10\} = \frac{9 + 28 + 360}{27} = \frac{397}{27}.$$

6. Since $F_{-Y}(t) = \mathbb{P}\{-Y \le t\} = \mathbb{P}\{Y \ge -t\} = 1 - F_Y(-t^-) = 1 - F_Y(-t)$, $f_{-Y}(t) = f_Y(-t)$; thus by convolution,

$$\begin{aligned} f_{\tilde{Z}}(t) &= \int_{s=-\infty}^{\infty} f_X(s) f_{-Y}(t-s)ds = \int_{s=-\infty}^{\infty} f_X(s) f_Y(s-t)ds \\ &= \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[0,1]}(s-t)ds = \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s)\chi_{[t,t+1]}(s)ds \\ &= \int_{s=-\infty}^{\infty} \chi_{[0,1]\cap[t,t+1]}(s)ds = \begin{cases} 1+t & \text{if } 0 \le t+1 \le 1 \\ 1-t & \text{if } 0 < t \le 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 1+t & \text{if } -1 \le t \le 0 \\ 1-t & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Note that $F_Z(t) = 0$ if $t < 0$. Since

$$F_Z(t) = \mathbb{P}\{-t \le X \le t\} = F_X(t) - F_X(-t^-) = F_X(t) - F_X(-t),$$

$$f_Z(t) = f_X(t) + f_X(-t) = \begin{cases} 2(1-t) & \text{if } 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

7. (a) For $t \in \mathbb{R}$,

$$F_Y(t) = \mathbb{P}\{\ln|X| \le t\} = \mathbb{P}\{|X| \le e^t\} = \int_{s=-e^t}^{e^t} f_X(s)ds;$$

thus

$$f_Y(t) = e^t f_X(e^t) + e^{-t} f_X(e^{-t}) = \frac{1}{\sqrt{2\pi}}\left\{\exp\left[t - \frac{e^{2t}}{2}\right] + \exp\left[-t - \frac{e^{-2t}}{2}\right]\right\}$$