Math 361, Section X1, Fall 2002
Quiz 1, September 13 **Name:** ____________________

1. [10 points] Suppose that we have 15 women and 10 men in a room. We want to make a committee of 7. What is the probability that there is at least 2 men in the committee?

Answers

1.

$$1-\frac{\binom{15}{7}}{\binom{25}{7}}-\frac{\binom{15}{6}\binom{10}{1}}{\binom{25}{7}}$$

Math 361, Section X1, Fall 2002
Quiz 2, September 18 **Name:** ______________________

1. 10 points A survey found that 30% of Chicago residents are male. The survey also found that 20% of men smoke and 15% of women smoke. Suppose a smoker is randomly picked. What is the probability that the smoker is a man?

Answers

1. Define $M \stackrel{\text{def}}{=} \{\text{male}\}$ and $S \stackrel{\text{def}}{=} \{\text{smoker}\}$. Then

$$\mathbb{P}(M|S) = \frac{\mathbb{P}(S|M)\mathbb{P}(M)}{\mathbb{P}(S|M)\mathbb{P}(M) + \mathbb{P}(S|M^c)\mathbb{P}(M^c)} = \frac{0.2 \times 0.3}{0.2 \times 0.3 + 0.15 \times 0.7}.$$

Math 361, Section X1, Fall 2002
Quiz 3, September 25 **Name:** ____________________

1. 10 points Suppose that we have 15 boxes, labelled 1 through 15. Distribute 22 balls into the boxes. What is the probability that there are exactly 4 balls in the first 7 boxes?

Answers

1.

$$\frac{\binom{22}{4}7^4 8^{18}}{15^{22}}$$

Math 361, Section X1, Fall 2002
Quiz 4, September 30 **Name:** ____________________

1. 10 points Suppose that we toss a fair six-sided die 4 times. Define

$$A \stackrel{\text{def}}{=} \{\text{face 2 appeared at least once}\}$$
$$B \stackrel{\text{def}}{=} \{\text{face 2 appeared exactly twice}\}$$

   (a) 5 points What is $\mathbb{P}(B|A)$?

   (b) 5 points Are the sets A and B independent?

Answers

1. Since $B \subset A$, $\mathbb{P}(B \cap A) = \mathbb{P}(B)$, so

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}.$$

We can compute that

$$\mathbb{P}(B) = \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$$

$$\mathbb{P}(A) = 1 - \mathbb{P}\{\text{face 2 doesn't appear at all}\} = 1 - \left(\frac{5}{6}\right)^4$$

(a)

$$\mathbb{P}(B|A) = \frac{\binom{4}{2}5^2}{6^4 - 5^4}$$

(b) Since $\mathbb{P}(A) < 1$, $\mathbb{P}(B|A) > \mathbb{P}(B)$, so $\mathbb{P}(B|A) \neq \mathbb{P}(A)$. Thus they are dependent.

Math 361, Section X1, Fall 2002
Quiz 5, October 9 **Name:** ____________________

1. [10 points] Let X be uniform on $\{1, 2, \dots 9, 10\}$; i.e., X has density

$$f_X(j) \overset{\text{def}}{=} \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2 \dots 9, 10\} \\ 0 & \text{else} \end{cases}$$

Define $Y \overset{\text{def}}{=} \max\{X, 4\}$. Compute the density of Y.

Answers

1.

$$f_Y(j) = \begin{cases} \frac{1}{10} & \text{if } j \in \{5, 6 \dots 10\} \\ \frac{4}{10} & \text{if } j = 4 \\ 0 & \text{else} \end{cases}$$

Math 361, Section X1, Fall 2002
Quiz 6, October 11 **Name:** ______________________

1. $\boxed{\text{10 points}}$ Let X be uniform on $\{1, 2, \dots 9, 10\}$; i.e., X has density

$$f_X(j) \overset{\text{def}}{=} \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2 \dots 9, 10\} \\ 0 & \text{else} \end{cases}$$

Define $Y \overset{\text{def}}{=} \min\{X, 4\}$. Compute the density of Y.

Answers

1.

$$f_Y(j) = \begin{cases} \frac{1}{10} & \text{if } j \in \{1,2,3\} \\ \frac{7}{10} & \text{if } j = 4 \\ 0 & \text{else} \end{cases}$$

Math 361, Section X1, Fall 2002
Quiz 7, October 28 **Name:** ______________________

1. 10 points Let X and Y be independent random variables with distributions

$$f_X(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{11, 12 \dots\} \\ 1-(1-p)^{11} & \text{if } j = 10 \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$. Compute $f_Z(14)$.

Answers

1.

$$f_Z(j) = 4p^2(1-p)^{14} + p(1-(1-p)^{11})(1-p)^4.$$

Math 361, Section X1, Fall 2002
Quiz 8, October 30 **Name:** ____________________

1. 10 points Suppose that X is a random variable with density

$$f_X(j) = \begin{cases} \frac{1}{10} & \text{if } j = -3 \\ \frac{1}{10} & \text{if } j = -1 \\ \frac{2}{10} & \text{if } j = 0 \\ \frac{3}{10} & \text{if } j = 2 \\ \frac{3}{10} & \text{if } j = 4 \\ 0 & \text{else.} \end{cases}$$

   (a) 4 points Compute $\mathbb{E}[X]$

   (b) 4 points Compute $\mathbb{E}[X^2]$

   (c) 2 points Compute $\mathbb{E}[2X]$

Answers

1. (a) $\frac{14}{10}$
   (b) $\frac{70}{10}$
   (c) $\frac{28}{10}$

Math 361, Section X1, Fall 2002
Quiz 9, November 8 **Name:** ______________________

1. [10 points] Suppose that X is a random variable with moment generating function

$$\mathbb{E}[e^{\theta X}] = \begin{cases} \frac{2}{2-\theta} & \text{if } \theta < 2 \\ \infty & \text{else.} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} X + 7$. What is the moment generating function of Y? In other words, compute $\mathbb{E}[e^{\theta Y}]$ for all $\theta \in \mathbb{R}$.

Answers

1.

$$\mathbb{E}[e^{\theta Y}] = \begin{cases} \frac{2e^{7\theta}}{2-\theta} & \text{if } \theta < 2 \\ \infty & \text{else.} \end{cases}$$

Math 361, Section X1, Fall 2002
Quiz 10, November 15 **Name:** ____________________

1. 10 points Suppose that X is a random variable with moment generating function

$$\mathbb{E}[e^{\theta X}] = \exp\left[\frac{9}{2}\theta^2 + 5\theta\right]$$

   (a) 5 points Define $Y \stackrel{\text{def}}{=} X/10$. What is the moment generating function of Y?

   (b) 5 points Let $Y_n \stackrel{\text{def}}{=} X/n$, and let φ_{Y_n} be the moment generating function of Y_n. Compute $\lim_{n\nearrow\infty} \varphi_{Y_n}(\theta)$ for each $\theta \in \mathbb{R}$.

Answers

1. (a)

$$\mathbb{E}[e^{\theta Y}] = \exp\left[\frac{9}{200}\theta^2 + \frac{5}{10}\theta\right]$$

(b)

$$\lim_{n\nearrow\infty} \varphi_{Y_n}(\theta) = \lim_{n\nearrow\infty} \exp\left[\frac{9}{2n^2}\theta^2 + \frac{5}{n}\theta\right] = e^0 = 1$$

for all $\theta \in \mathbb{R}$.

Math 361, Section X1, Fall 2002
Quiz 11, November 20 **Name:** ____________________

1. [10 points] Suppose that X is a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < -5 \\ \frac{1}{3} & \text{if } -5 \le t < 2 \\ 1 - e^{-30t} & \text{if } t \ge 2 \end{cases}$$

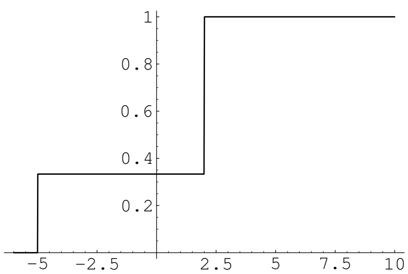
   (a) [2 points] Graph F_X.

   (b) [4 points] Compute $\mathbb{P}\{X = 2\}$

   (c) [4 points] Compute $\mathbb{P}\{-5 < X \le 10\}$

Answers

1. (a) See figure.



(b) $\mathbb{P}\{X = 2\} = F_X(2) - F_X(2-) = 1 - e^{-60} - \frac{1}{3} = \frac{2}{3} - e^{-60}$.

(c) $\mathbb{P}\{-5 < X \le 10\} = F_X(10) - F_X(-5) = 1 - e^{-300} - \frac{1}{3} = \frac{2}{3} - e^{-300}$.

 Name: ____________________

1. 10 points Fix $\lambda > 0$. Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} \lambda^2 e^{-\lambda(s+t)} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Compute $\mathbb{P}\{X + Y \leq 5\}$.

Answers

1.

$$\mathbb{P}\{X + Y \le 5\} = \int_{s=0}^{5} \int_{t=0}^{5-s} \lambda^2 e^{-\lambda(s+t)} ds\, dt = 1 - e^{-5\lambda} - 5\lambda e^{-5\lambda}.$$

Math 361, Section X1, Fall 2002
Exam 1, October 2

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [25 points] Consider a box containing 6 white balls and 8 black balls. Pick the balls from the box one by one.
   (a) [10 points] What is the probability that the fourth ball is white and the fifth ball is black?
   (b) [10 points] What is the probability that the fourth ball is white?
   (c) [5 points] What is the probability that the fifth ball is black given that the fourth ball is white?

   Hint: you may want to "reserve" balls.

2. [15 points] An instructor gives her class 15 problems, telling them that the exam will consist of a random selection of 7 of them. By the time of the exam, a student knows how to do 12 out of the 15.
   (a) [8 points] What is the probability that the student knows how to do all of the questions on the exam?
   (b) [7 points] What is the probability that the student knows how to do at least 6 of the questions on the exam?

   Hint: compare this problem to a lottery problem.

3. [15 points] There are 7 restaurants in a town. Suppose three students are hungry, and each student randomly chooses a restaurant.
   (a) [7 points] What is the probability that no two students go to the same restaurant?
   (b) [8 points] What is the probability that exactly two students go to the same restaurant?

4. [30 points] Suppose that each child born to Jim and Jane is equally likely to be a boy or girl, independently of the gender of all other children. Jim and Jane have 5 children. Compute the following probabilities:
   (a) [10 points] All children are of the same sex.
   (b) [5 points] Exactly 3 are boys
   (c) [10 points] youngest and oldest are girls
   (d) [5 points] At least one boy.

5. [15 points] Suppose that

$$\mathbb{P}(A \cap C|B) = 0.4 \quad \mathbb{P}(B) = 0.1 \quad \mathbb{P}(A \cap B \setminus C) = 0.02 \quad \text{and} \quad \mathbb{P}(A \setminus B) = 0.2$$

   Compute

(a) $\boxed{\text{5 points}}$ $\mathbb{P}(A \cap B)$

(b) $\boxed{\text{5 points}}$ $\mathbb{P}(A)$

(c) $\boxed{\text{5 points}}$ $\mathbb{P}(A \cup B)$

Answers

1. (a) Reserve a black ball and a white ball.

$$\frac{8 \times 6 \times (12)_3}{(14)_5}$$

(b) Reserve a white ball.

$$\frac{6 \times (13)_3}{(14)_4}$$

(c) Note that

$$\frac{6 \times (13)_3}{(14)_4} = \frac{6 \times (13)_4}{(14)_5}.$$

Answer is

$$\frac{8 \times 6 \times (12)_3}{6 \times (13)_4} = \frac{8}{13}.$$

2. Define $q \stackrel{\text{def}}{=} 1/\binom{15}{7}$.
   (a) $\binom{12}{7}q$.
   (b) $\left(\binom{12}{7} + \binom{12}{6}\binom{3}{1}\right) q$.
3. Define $q \stackrel{\text{def}}{=} 1/7^3$.
   (a) $\binom{7}{3}3!q$.
   (b) $\binom{3}{2}\binom{7}{2}2!q$.
4. Set $q \stackrel{\text{def}}{=} (1/2)^5$.
   (a) $2q$
   (b) $\binom{5}{3}q$
   (c) $2^3 q$
   (d) $1 - q$.
5. (a)

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap B \setminus C) \\ &= \mathbb{P}(A \cap C|B)\mathbb{P}(B) + \mathbb{P}(A \cap B \setminus C) = 0.4 \times 0.1 + 0.02 = 0.06.\end{aligned}$$

   (b) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B) = 0.06 + 0.2 = 0.26$.
   (c) $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \setminus B) = 0.1 + 0.2 = 0.3$.

Math 361, Section X1, Fall 2002
Exam 2, November 1

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [30 points] Suppose that X and Y are independent random variables with densities

$$f_X(j) \overset{\text{def}}{=} \begin{cases} (j+1)p^2(1-p)^j & \text{if } j \in \{0,1,2\dots\} \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) \overset{\text{def}}{=} \begin{cases} p(1-p)^j & \text{if } j \in \{0,1,2\dots\} \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$.

(a) [15 points] Compute the density of Z.

(b) [15 points] Compute $f_{X|Z}(j|k)$ when $0 \le j \le k$.

Hint: Remember that $\sum_{j=0}^{N} j = N(N+1)/2$.

2. [40 points] Let X be a random variable with density

$$f_X(j) \overset{\text{def}}{=} \begin{cases} p(1-p)^j & \text{if } j \in \{0,1,2\dots\} \\ 0 & \text{else} \end{cases}$$

Define the transformation

$$\varphi(u) \overset{\text{def}}{=} \begin{cases} 10-u & \text{if } u \le 10 \\ u-5 & \text{if } u > 10 \end{cases}$$

Define $Y \overset{\text{def}}{=} \varphi(X)$.

(a) [10 points] Compute $\mathbb{P}\{Y=6\}$

(b) [10 points] Compute $\mathbb{P}\{Y=4\}$

(c) [10 points] Compute $\mathbb{P}\{Y=22\}$

(d) [10 points] Compute the density of Y.

Hint: Note that φ is discontinuous and that $\varphi(10)=0$. You may want to *carefully* graph it.

3. [20 points] Let X and Y be random variables with joint density

$$f_{X,Y}(j,k) \overset{\text{def}}{=} \begin{cases} \frac{p(1-p)^j}{j+1} & \text{if } 0 \le k \le j \\ 0 & \text{else} \end{cases}$$

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X = 5\}$.

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y = 3 | X = 5\}$.

4. $\boxed{\text{10 points}}$ Suppose that X is a random variable with

$$\mathbb{P}\{X = -2\} = \frac{1}{4}, \qquad \mathbb{P}\{X = 0\} = \frac{1}{4}, \qquad \text{and} \qquad \mathbb{P}\{X = 1\} = \frac{2}{4}.$$

Compute $\mathbb{E}[X^3]$.

Answers

1. (a)

$$f_Z(j) \overset{\text{def}}{=} \begin{cases} \binom{j+2}{2} p^2(1-p)^j & \text{if } j \in \{1,2\dots\} \\ 0 & \text{else} \end{cases}$$

(note that $\binom{j+2}{2} = (j+1)(j+2)/2$; the given representation is a more standard representation for negative binomials).

(b)

$$f_{X|Z}(j|k) = \frac{j+1}{(k+1)(k+2)/2} = \frac{\binom{j+1}{1}}{\binom{k+2}{2}}$$

if $0 \le j \le k$.

2. (a) $f_Y(6) = f_X(4) + f_X(11) = p\{(1-p)^4 + (1-p)^{11}\}$.

(b) $f_Y(4) = f_X(6) = p(1-p)^6$.

(c) $f_Y(22) = f_X(27) = p(1-p)^{27}$.

(d)

$$f_Y(j) = \begin{cases} p(1-p)^{10-j} & \text{if } j \in \{0,1,2,3,4,5\} \\ p\{(1-p)^{10-j} + (1-p)^{j+5}\} & \text{if } j \in \{6,7,8,9,10\} \\ p(1-p)^{j+5} & \text{if } j \in \{11,12\dots\} \\ 0 & \text{else} \end{cases}$$

3. (a) $p(1-p)^5$

(b) $\frac{1}{6}$

4.

$$(-2)^3\frac{1}{4} + 0^3\frac{1}{4} + 1^3\frac{2}{4} = \frac{-8+2}{4} = -\frac{6}{4}.$$

Math 361, Section X1, Fall 2002
Exam 3, December 6

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [35 points] Let X be a continuous random variable with density

$$f_X(t) = \begin{cases} 1 & \text{if } t \in (0,1] \\ 0 & \text{else} \end{cases}$$

(i.e., X is $U(0,1)$). Define the transformation

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 4 & \text{if } u \in (0, \frac{1}{10}] \\ -2 & \text{if } u \in (\frac{1}{10}, \frac{4}{10}] \\ 3 & \text{if } u \in (\frac{4}{10}, 1] \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

   (a) [10 points] Compute F_Y, the cumulative distribution function of Y.

   (b) [5 points] Is Y continuous, or is it discrete (Hint: in this case, it is one or the other).

   (c) [10 points] Compute the density (i.e., the discrete density if Y is discrete, and the continuous density if Y is continuous).

   (d) [10 points] Compute $\mathbb{E}[Y]$.

2. [31 points] Suppose that X and Y are independent continuous random variables with densities

$$f_X(s) \stackrel{\text{def}}{=} \begin{cases} \lambda e^{-\lambda s} & \text{if } s \ge 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \nu e^{-\nu t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

where λ and ν are positive parameters. Define $Z \stackrel{\text{def}}{=} \min\{X, Y\}$.

   (a) [10 points] Compute $\mathbb{P}\{X \ge Y\}$.

   (b) [10 points] Compute $\mathbb{P}\{Z > 3\}$.

   (c) [3 points] Compute $\mathbb{P}\{Z \le 3\}$.

   (d) [8 points] Compute F_Z, the cumulative distribution function of Z.

3. [34 points] Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} 6t & \text{if } s \ge 0,\ t \ge 0, \text{ and } s+t \le 1 \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$.

(a) $\boxed{\text{10 points}}$ Compute f_X, the first marginal of $f_{X,Y}$.

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\left\{Z \le \frac{2}{3}\right\}$.

(c) $\boxed{\text{7 points}}$ Compute F_Z, the cumulative distribution function of Z. Hint: F_Z should be continuous.

(d) $\boxed{\text{7 points}}$ Compute f_Z, the density of Z.

1. (a)
$$F_Y(t) = \begin{cases} 0 & \text{if } t < -2 \\ \frac{3}{10} & \text{if } -2 \le t < 3 \\ \frac{9}{10} & \text{if } 3 \le t < 4 \\ 1 & \text{if } t \ge 4 \end{cases}$$
   (b) Y is discrete.
   (c)
$$f_Y(j) = \begin{cases} \frac{3}{10} & \text{if } j = -2 \\ \frac{6}{10} & \text{if } j = 3 \\ \frac{1}{10} & \text{if } j = 4 \end{cases}$$
   (d)
$$\mathbb{E}[Y] = (-3)\frac{1}{10} + 3\frac{6}{10} + 4\frac{1}{10} = 1.6.$$
2. (a)
$$\mathbb{P}\{X \ge Y\} = \int_{t=0}^{\infty} \int_{s=t}^{\infty} \lambda e^{-\lambda s} \nu e^{-\nu t} dt\, ds = \int_{t=0}^{\infty} \nu e^{-\lambda s - \nu s} dt = \frac{\nu}{\lambda + \nu}.$$
   (b)
$$\mathbb{P}\{Z > 3\} = \mathbb{P}\{X > 3\}\mathbb{P}\{Y > 3\} = \left(\int_{s=3}^{\infty} \lambda e^{-\lambda s} ds\right)\left(\int_{t=3}^{\infty} \nu e^{-\nu t} dt\right) = e^{-3(\lambda+\nu)}.$$
   (c)
$$\mathbb{P}\{Z \le 3\} = 1 - e^{-3(\lambda+\nu)}.$$
   (d)
$$F_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-(\lambda+\nu)t} & \text{if } t \ge 0 \end{cases}$$
3. (a)
$$f_X(s) = \begin{cases} 3(1-s)^2 & \text{if } s \in [0,1] \\ 0 & \text{else} \end{cases}$$
   (b)
$$\mathbb{P}\left\{Z \le \frac{2}{3}\right\} = \int_{s=0}^{2/3} \int_{t=0}^{2/3-s} (6t) dt\, ds = 3\int_{s=0}^{2/3} \left(\frac{2}{3} - s\right)^2 ds = \left(\frac{2}{3}\right)^3.$$
   (c)
$$F_Z(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^3 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(d)

$$f_Z(t) = \begin{cases} 3t^2 & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section X1, Fall 2002
Final, December 16

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [30 points] Suppose that there are 10 married couples in a room. Suppose for specificity that one of the married couples is Don and Donna. Randomly pick four people to sit at a table.
   (a) [10 points] What is the probability that you pick Don and Donna?
   (b) [10 points] What is the probability that you pick Don and Donna and another married couple?
   (c) [10 points] What is the probability that you pick two married couples?

2. [20 points] Suppose that X and Y have joint density

		Y = 0	Y = 1	Y = 2	Y = 3
X	-1	3/27	1/27	1/27	3/27
	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

   In other words, $f_{X,Y}(0,1) = 0$.
   (a) [5 points] Compute f_X.
   (b) [10 points] Compute $\mathbb{E}[X^2 Y]$.

   Now define $Z \stackrel{\text{def}}{=} X + Y$.
   (c) [5 points] Compute $\mathbb{P}\{Z = 1\}$.

3. [40 points] Fix $\lambda > 0$. Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} \lambda e^{-\lambda(t-s)} & \text{if } t \geq s \text{ and } 0 \leq s \leq 1 \\ 0 & \text{else} \end{cases}$$

   (a) [10 points] Compute $f_Y(4)$
   (b) [10 points] Compute $f_Y(\frac{1}{3})$
   (c) [10 points] Compute f_Y.
   (d) [10 points] Compute $f_{X|Y}(s|4)$ for $0 < s < 1$.

4. $\boxed{\text{50 points}}$ Let X be a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - e^{-3t} & \text{if } 1 \le t < 10 \\ 1 & \text{if } t \ge 10 \end{cases}$$

(a) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X < 10\}$.

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X = 1\}$.

Define now

$$\tilde{F}(t) \stackrel{\text{def}}{=} \mathbb{P}\{X \le t | X \in (1, 10)\}$$

for all $t \in \mathbb{R}$ (this is the cumulative distribution of X given that $X \in (1, 10)$).

(c) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \in (1, 10)\}$

(d) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \le 5 \text{ and } X \in (1, 10)\}$.

(e) $\boxed{\text{3 points}}$ Compute $\tilde{F}(5)$.

(f) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \le 12 \text{ and } X \in (1, 10)\}$.

(g) $\boxed{\text{3 points}}$ Compute $\tilde{F}(12)$.

(h) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \le -2 \text{ and } X \in (1, 10)\}$.

(i) $\boxed{\text{3 points}}$ Compute $\tilde{F}(-2)$.

(j) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \le 1 \text{ and } X \in (1, 10)\}$.

(k) $\boxed{\text{3 points}}$ Compute $\tilde{F}(1)$.

(l) $\boxed{\text{3 points}}$ Compute $\tilde{F}(t)$ for all t.

5. $\boxed{\text{10 points}}$ Let X and Y be independent Gaussian random variables with mean 0 and variance 1; i.e.,

$$f_X(t) = f_Y(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \qquad t \in \mathbb{R}$$

Define $Z_1 \stackrel{\text{def}}{=} 7X + 3$ and $Z_2 \stackrel{\text{def}}{=} 5X + 4Y$.

(a) $\boxed{\text{5 points}}$ Compute the moment generating function of Z_1.

(b) $\boxed{\text{5 points}}$ Compute the moment generating function of Z_2.

Answers

1. (a) $\frac{\binom{18}{2}}{\binom{20}{4}}$.

   (b) $\frac{9}{\binom{20}{4}}$.

   (c) $\frac{(10)_2}{\binom{20}{4}}$.

2. (a)

$$f_X(j) = \begin{cases} \frac{8}{27} & \text{if } j = -1 \\ \frac{12}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 3 \end{cases}$$

   (b)

$$\mathbb{E}[X^2 Y] = (-1)^2(1)\frac{1}{27} + (-1)^2(2)\frac{1}{27} + (-1)^2(3)\frac{3}{27} + (3)^2(3)\frac{4}{27} = \frac{120}{27}.$$

   (c) $\mathbb{P}\{Z = 1\} = f_{X,Y}(-1, 2) + f_{X,Y}(0, 1) = \frac{1}{27}$.

3. (a) $f_Y(4) = \int_{s=0}^{1} \lambda e^{-\lambda(4-s)} ds = e^{-3\lambda} - e^{-4\lambda}$.

   (b) $f_Y(\frac{1}{3}) = \int_{s=0}^{1/3} \lambda e^{-\lambda((1/3)-s)} ds = 1 - e^{-(1/3)\lambda}$.

   (c)

$$f_Y(t) = \begin{cases} e^{-\lambda(t-1)} - e^{-\lambda t} & \text{if } t \geq 1 \\ 1 - e^{-\lambda t} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

   (d)

$$f_{X|Y}(s|4) = \frac{f_{X,Y}(s,4)}{f_Y(4)} = \frac{\lambda e^{-\lambda(4-s)}}{e^{-3\lambda} - e^{-4\lambda}}.$$

4. (a) $\mathbb{P}\{X < 10\} = 1 - e^{-30}$.

   (b) $\mathbb{P}\{X = 1\} = 1 - e^{-3}$.

   (c) $\mathbb{P}\{X \in (1, 10)\} = F_X(10-) - F_X(1) = e^{-3} - e^{-30}$.

   (d) $\mathbb{P}\{X \leq 5 \text{ and } X \in (1, 10)\} = \mathbb{P}\{X \in (1, 5]\} = F_X(5) - F_X(1) = e^{-3} - e^{-15}$.

   (e) $\tilde{F}(5) = (e^{-3} - e^{-15}) \,/\, (e^{-3} - e^{-30})$.

   (f) $\mathbb{P}\{X \leq 12 \text{ and } X \in (1, 10)\} = \mathbb{P}\{X \in (1, 10)\} = e^{-3} - e^{-30}$.

   (g) $\tilde{F}(12) = 1$.

   (h) $\mathbb{P}\{X \leq -2 \text{ and } X \in (1, 10)\} = \mathbb{P}(\emptyset) = 0$.

   (i) $\tilde{F}(-2) = 0$.

   (j) $\mathbb{P}\{X \leq 1 \text{ and } X \in (1, 10)\} = \mathbb{P}(\emptyset) = 0$.

   (k) $\tilde{F}(1) = 0$.

(l)

$$\tilde{F}(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{e^{-3}-e^{-3t}}{e^{-3}-e^{-30}} & \text{if } 1 \le t < 10 \\ 1 & \text{if } t \ge 10 \end{cases}$$

5. (a) $\mathbb{E}[e^{\theta Z_1}] = \exp\left[\frac{49}{2}\theta^2 + 3\theta\right]$.
   (b) $\mathbb{E}[e^{\theta Z_2}] = \exp\left[\frac{41}{2}\theta^2\right]$.