Math 361, Section E1, Fall 2002
Quiz 1, September 13 **Name:** ____________________

1. [10 points] Suppose that we have 15 women and 10 men in a room. We want to make a committee of 6. What is the probability that there is at least 2 women in the committee?

Answers

1.

$$1-\frac{\binom{10}{6}}{\binom{25}{6}}-\frac{\binom{10}{5}\binom{15}{1}}{\binom{25}{6}}$$

Math 361, Section E1, Fall 2002
Quiz 2, September 18 **Name:** ______________________

1. [10 points] A survey found that 40% of Chicago residents are male. The survey also found that 10% of men smoke and 15% of women smoke. Suppose a smoker is randomly picked. What is the probability that the smoker is a man?

1.

$$\mathbb{P}\{M|S\} = \frac{\mathbb{P}\{S|M\}\mathbb{P}\{M\}}{\mathbb{P}\{S|M\}\mathbb{P}\{M\} + \mathbb{P}\{S|W\}\mathbb{P}\{W\}} = \frac{0.1 \times 0.4}{0.1 \times 0.4 + 0.15 \times 0.6}$$
$$= \frac{4}{4+9} = \frac{4}{13}.$$

Math 361, Section E1, Fall 2002
Quiz 3, September 25 **Name:** ______________________

1. [10 points] Suppose that we have 11 boxes, labelled 1 through 11. Distribute 20 balls into the boxes. What is the probability that there are exactly 3 balls in the first 5 boxes?

Answers

1.

$$\frac{\binom{20}{3}5^3 6^{17}}{11^{20}}$$

Math 361, Section E1, Fall 2002
Quiz 4, September 30 **Name:** ____________________

1. [10 points] Suppose that we toss a fair six-sided die 3 times. Define

$$A \stackrel{\text{def}}{=} \{\text{face 2 appeared at least once}\}$$
$$B \stackrel{\text{def}}{=} \{\text{face 2 appeared exactly twice}\}$$

   (a) [5 points] What is $\mathbb{P}(B|A)$?

   (b) [5 points] Are the sets A and B independent?

Answers

1. Since $B \subset A$, $\mathbb{P}(B \cap A) = \mathbb{P}(B)$, so

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}.$$

We can compute that

$$\mathbb{P}(B) = 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$$

$$\mathbb{P}(A) = 1 - \mathbb{P}\{\text{face 2 doesn't appear at all}\} = 1 - \left(\frac{5}{6}\right)^3$$

(a)

$$\mathbb{P}(B|A) = \frac{(3)(5)}{6^3 - 5^3}$$

(b) Since $\mathbb{P}(A) < 1$, $\mathbb{P}(B|A) > \mathbb{P}(B)$, so $\mathbb{P}(B|A) \neq \mathbb{P}(A)$. Thus they are dependent.

Math 361, Section E1, Fall 2002
Quiz 5, October 9 **Name:** ____________________

1. [10 points] Let X be uniform on $\{1, 2, \dots 19, 20\}$; i.e., X has density

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{20} & \text{if } j \in \{1, 2 \dots 19, 20\} \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \max\{X, 14\}$. Compute the density of Y.

Answers

1.

$$f_Y(j) = \begin{cases} \frac{1}{20} & \text{if } j \in \{15, 16 \dots 20\} \\ \frac{14}{10} & \text{if } j = 14 \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Fall 2002
Quiz 6, October 11 **Name:** ____________________

1. 10 points Let X be uniform on $\{1, 2, \dots 19, 20\}$; i.e., X has density

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{20} & \text{if } j \in \{1, 2 \dots 19, 20\} \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \min\{X, 14\}$. Compute the density of Y.

1.

$$f_Y(j) = \begin{cases} \frac{1}{20} & \text{if } j \in \{1, 2 \ldots 13\} \\ \frac{7}{10} & \text{if } j = 14 \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Fall 2002
Quiz 7, October 28 **Name:** ____________________

1. [10 points] Let X and Y be independent random variables with distributions

$$f_X(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{11, 12 \dots\} \\ 1-(1-p)^{11} & \text{if } j = 10 \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) = \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$. Compute $f_Z(15)$.

Answers

1.

$$f_Z(j) = 5p^2(1-p)^{15} + p(1-(1-p)^{11})(1-p)^5.$$

Math 361, Section E1, Fall 2002
Quiz 8, October 30 **Name:** ____________________

1. [10 points] Suppose that X is a random variable with density

$$f_X(j) = \begin{cases} \frac{1}{10} & \text{if } j = -3 \\ \frac{1}{10} & \text{if } j = -1 \\ \frac{2}{10} & \text{if } j = 0 \\ \frac{3}{10} & \text{if } j = 2 \\ \frac{3}{10} & \text{if } j = 4 \\ 0 & \text{else.} \end{cases}$$

   (a) [4 points] Compute $\mathbb{E}[X]$

   (b) [4 points] Compute $\mathbb{E}[X^2]$

   (c) [2 points] Compute $\mathbb{E}[2X]$

Answers

1. (a) $\frac{14}{10}$
   (b) $\frac{70}{10}$
   (c) $\frac{28}{10}$

Math 361, Section E1, Fall 2002
Quiz 9, November 8 **Name:** ____________________

1. 10 points Suppose that X is a random variable with moment generating function

$$\mathbb{E}[e^{\theta X}] = \begin{cases} \frac{5}{5-\theta} & \text{if } \theta < 5 \\ \infty & \text{else.} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} X + 9$. What is the moment generating function of Y? In other words, compute $\mathbb{E}[e^{\theta Y}]$ for all $\theta \in \mathbb{R}$.

Answers

1.

$$\mathbb{E}[e^{\theta Y}] = \begin{cases} \frac{5e^{9\theta}}{5-\theta} & \text{if } \theta < 5 \\ \infty & \text{else.} \end{cases}$$

Math 361, Section E1, Fall 2002
Quiz 10, November 15 **Name:** ____________________

1. 10 points Suppose that X is a random variable with moment generating function

$$\mathbb{E}[e^{\theta X}] = \exp\left[\frac{16}{2}\theta^2 + 7\theta\right]$$

(a) 5 points Define $Y \stackrel{\text{def}}{=} X/10$. What is the moment generating function of Y?

(b) 5 points Let $Y_n \stackrel{\text{def}}{=} X/n$, and let φ_{Y_n} be the moment generating function of Y_n. Compute $\lim_{n\nearrow\infty} \varphi_{Y_n}(\theta)$ for each $\theta \in \mathbb{R}$.

1. (a)

$$\mathbb{E}[e^{\theta Y}] = \exp\left[\frac{16}{200}\theta^2 + \frac{7}{10}\theta\right]$$

(b)

$$\lim_{n \nearrow \infty} \varphi_{Y_n}(\theta) = \lim_{n \nearrow \infty} \exp\left[\frac{16}{2n^2}\theta^2 + \frac{7}{n}\theta\right] = e^0 = 1$$

for all $\theta \in \mathbb{R}$.

Math 361, Section E1, Fall 2002
Quiz 11, November 20 **Name:** ____________________

1. [10 points] Suppose that X is a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < -5 \\ \frac{1}{3} & \text{if } -5 \le t < 1 \\ 1 - e^{-40t} & \text{if } t \ge 1 \end{cases}$$

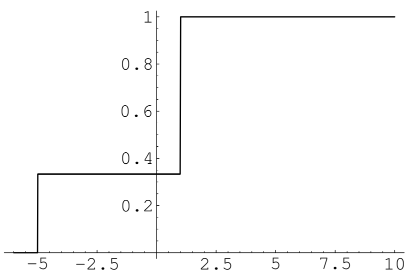
   (a) [2 points] Graph F_X.

   (b) [4 points] Compute $\mathbb{P}\{X = 1\}$

   (c) [4 points] Compute $\mathbb{P}\{-5 < X \le 10\}$

Answers

1. (a) See figure.



(b) $\mathbb{P}\{X=1\} = F_X(1) - F_X(1-) = 1 - e^{-40} - \frac{1}{3} = \frac{2}{3} - e^{-40}$.

(c) $\mathbb{P}\{-5 < X \le 10\} = F_X(10) - F_X(-5) = 1 - e^{-400} - \frac{1}{3} = \frac{2}{3} - e^{-400}$.

Math 361, Section E1, Fall 2002
Quiz 12, December 4 Name: ______________________

1. [10 points] Fix $\lambda > 0$. Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} \lambda^2 e^{-\lambda(s+t)} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Compute $\mathbb{P}\{X + Y \leq 7\}$.

Answers

1.
$$\mathbb{P}\{X+Y\le 7\} = \int_{s=0}^{7}\int_{t=0}^{7-s}\lambda^2 e^{-\lambda(s+t)}ds\,dt = 1-e^{-7\lambda}-7\lambda e^{-7\lambda}.$$

Math 361, Section E1, Fall 2002
Exam 1, October 2

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [25 points] Consider a box containing 7 white balls and 10 black balls. Pick the balls from the box one by one.
   (a) [10 points] What is the probability that the fourth ball is black and the sixth ball is white?
   (b) [10 points] What is the probability that the fourth ball is black?
   (c) [5 points] What is the probability that the sixth ball is white given that the fourth ball is black?

   Hint: you may want to "reserve" balls.

2. [15 points] An instructor gives her class 20 problems, telling them that the exam will consist of a random selection of 8 of them. By the time of the exam, a student knows how to do 15 out of the 20.
   (a) [8 points] What is the probability that the student knows how to do all of the questions on the exam?
   (b) [7 points] What is the probability that the student knows how to do at least 7 of the questions on the exam?

   Hint: compare this problem to a lottery problem.

3. [15 points] There are 10 restaurants in a town. Suppose seven students are hungry, and each student randomly chooses a restaurant.
   (a) [7 points] What is the probability that no two students go to the same restaurant?
   (b) [8 points] What is the probability that exactly three students go to the same restaurant?

4. [30 points] Suppose that each child born to Jim and Jane is equally likely to be a boy or girl, independently of the gender of all other children. Jim and Jane have 5 children. Compute the following probabilities:
   (a) [10 points] All children are of the same sex.
   (b) [5 points] Exactly 3 are girls
   (c) [10 points] youngest and oldest are boys
   (d) [5 points] At least one girl.

5. [15 points] Suppose that

$$\mathbb{P}(A \cap C|B) = 0.4 \quad \mathbb{P}(B) = 0.2 \quad \mathbb{P}(A \cap B \setminus C) = 0.03 \quad \text{and} \quad \mathbb{P}(A \setminus B) = 0.2$$

   Compute

(a) $\boxed{\text{5 points}}$ $\mathbb{P}(A \cap B)$

(b) $\boxed{\text{5 points}}$ $\mathbb{P}(A)$

(c) $\boxed{\text{5 points}}$ $\mathbb{P}(A \cup B)$

Answers

1. (a) Reserve a black ball and a white ball.

$$\frac{10 \times 7 \times (15)_4}{(17)_6}$$

   (b) Reserve a black ball.

$$\frac{10 \times (16)_3}{(17)_4}$$

   (c) Note that

$$\frac{10 \times (16)_3}{(17)_4} = \frac{10 \times (16)_5}{(17)_6}.$$

   Answer is

$$\frac{10 \times 7 \times (15)_4}{10 \times (16)_5} = \frac{7}{16}.$$

2. Define $q \stackrel{\text{def}}{=} 1/\binom{20}{8}$.
   (a) $\binom{15}{8}q$.
   (b) $\left(\binom{15}{8} + \binom{15}{7}\binom{5}{1}\right) q$.

3. Define $q \stackrel{\text{def}}{=} 1/10^7$.
   (a) $\binom{10}{7}7!q$.
   (b) $\binom{7}{3}\binom{10}{5}5!q$.

4. Set $q \stackrel{\text{def}}{=} (1/2)^5$.
   (a) $2q$
   (b) $\binom{5}{3}q$
   (c) 2^3q
   (d) $1 - q$.

5. (a)

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}(A \cap B \cap C) + \mathbb{P}(A \cap B \setminus C) \\ &= \mathbb{P}(A \cap C|B)\mathbb{P}(B) + \mathbb{P}(A \cap B \setminus C) = 0.4 \times 0.2 + 0.03 = 0.11.\end{aligned}$$

   (b) $\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \setminus B) = 0.11 + 0.2 = 0.31$.
   (c) $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \setminus B) = 0.4$.

Math 361, Section E1, Fall 2002
Exam 2, November 1

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [30 points] Suppose that X and Y are independent random variables with densities

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} (j+1)p^2(1-p)^j & \text{if } j \in \{0,1,2\dots\} \\ 0 & \text{else} \end{cases}$$

$$f_Y(j) \stackrel{\text{def}}{=} \begin{cases} p(1-p)^j & \text{if } j \in \{0,1,2\dots\} \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X+Y$.

   (a) [15 points] Compute the density of Z.

   (b) [15 points] Compute $f_{X|Z}(j|k)$ when $0 \le j \le k$.

   Hint: Remember that $\sum_{j=0}^N j = N(N+1)/2$.

2. [40 points] Let X be a random variable with density

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} p(1-p)^j & \text{if } j \in \{0,1,2\dots\} \\ 0 & \text{else} \end{cases}$$

Define the transformation

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 10-u & \text{if } u \le 10 \\ u-5 & \text{if } u > 10 \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

   (a) [10 points] Compute $\mathbb{P}\{Y=7\}$

   (b) [10 points] Compute $\mathbb{P}\{Y=3\}$

   (c) [10 points] Compute $\mathbb{P}\{Y=21\}$

   (d) [10 points] Compute the density of Y.

   Hint: Note that φ is discontinuous and that $\varphi(10)=0$. You may want to *carefully* graph it.

3. [20 points] Let X and Y be random variables with joint density

$$f_{X,Y}(j,k) \stackrel{\text{def}}{=} \begin{cases} \frac{p(1-p)^j}{j+1} & \text{if } 0 \le k \le j \\ 0 & \text{else} \end{cases}$$

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X = 6\}$.

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y = 3 | X = 6\}$.

4. $\boxed{\text{10 points}}$ Suppose that X is a random variable with

$$\mathbb{P}\{X = -3\} = \frac{1}{4}, \qquad \mathbb{P}\{X = 0\} = \frac{1}{4}, \qquad \text{and} \qquad \mathbb{P}\{X = 1\} = \frac{2}{4}.$$

Compute $\mathbb{E}[X^3]$.

Answers

1. (a)

$$f_Z(j) \stackrel{\text{def}}{=} \begin{cases} \binom{j+2}{2}p^2(1-p)^j & \text{if } j \in \{1,2\dots\} \\ 0 & \text{else} \end{cases}$$

(note that $\binom{j+2}{2} = (j+1)(j+2)/2$; the given representation is a more standard representation for negative binomials).

(b)

$$f_{X|Z}(j|k) = \frac{j+1}{(k+1)(k+2)/2} = \frac{\binom{j+1}{1}}{\binom{k+2}{2}}$$

if $0 \le j \le k$.

2. (a) $f_Y(7) = f_X(3) + f_X(12) = p\{(1-p)^3 + (1-p)^{12}\}$.

(b) $f_Y(3) = f_X(7) = p(1-p)^7$.

(c) $f_Y(21) = f_X(26) = p(1-p)^{26}$.

(d)

$$f_Y(j) = \begin{cases} p(1-p)^{10-j} & \text{if } j \in \{0,1,2,3,4,5\} \\ p\{(1-p)^{10-j} + (1-p)^{j+5}\} & \text{if } j \in \{6,7,8,9,10\} \\ p(1-p)^{j+5} & \text{if } j \in \{11,12\dots\} \\ 0 & \text{else} \end{cases}$$

3. (a) $p(1-p)^6$

(b) $\frac{1}{7}$

4.

$$(-3)^3\frac{1}{4} + 0^3\frac{1}{4} + 1^3\frac{2}{4} = \frac{-27+2}{4} = -\frac{25}{4}.$$

Math 361, Section E1, Fall 2002
Exam 3, December 6

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [35 points] Let X be a continuous random variable with density

$$f_X(t) = \begin{cases} 1 & \text{if } t \in (0,1] \\ 0 & \text{else} \end{cases}$$

(i.e., X is $U(0,1)$). Define the transformation

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 4 & \text{if } u \in (0, \frac{1}{10}] \\ -1 & \text{if } u \in (\frac{1}{10}, \frac{4}{10}] \\ 3 & \text{if } u \in (\frac{4}{10}, 1] \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

(a) [10 points] Compute F_Y, the cumulative distribution function of Y.

(b) [5 points] Is Y continuous, or is it discrete (Hint: in this case, it is one or the other).

(c) [10 points] Compute the density (i.e., the discrete density if Y is discrete, and the continuous density if Y is continuous).

(d) [10 points] Compute $\mathbb{E}[Y]$.

2. [31 points] Suppose that X and Y are independent continuous random variables with densities

$$f_X(s) \stackrel{\text{def}}{=} \begin{cases} \lambda e^{-\lambda s} & \text{if } s \ge 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \nu e^{-\nu t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

where λ and ν are positive parameters. Define $Z \stackrel{\text{def}}{=} \min\{X, Y\}$.

(a) [10 points] Compute $\mathbb{P}\{Y \ge X\}$.

(b) [10 points] Compute $\mathbb{P}\{Z > 5\}$.

(c) [3 points] Compute $\mathbb{P}\{Z \le 5\}$.

(d) [8 points] Compute F_Z, the cumulative distribution function of Z.

3. [34 points] Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} 6t & \text{if } s \ge 0,\ t \ge 0, \text{ and } s + t \le 1 \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$.

(a) 10 points Compute f_Y, the second marginal of $f_{X,Y}$.

(b) 10 points Compute $\mathbb{P}\left\{Z \le \frac{1}{3}\right\}$.

(c) 7 points Compute F_Z, the cumulative distribution function of Z. Hint: F_Z should be continuous.

(d) 7 points Compute f_Z, the density of Z.

Answers

1. (a)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < -1 \\ \frac{3}{10} & \text{if } -1 \le t < 3 \\ \frac{9}{10} & \text{if } 3 \le t < 4 \\ 1 & \text{if } t \ge 4 \end{cases}$$

   (b) Y is discrete.

   (c)

$$f_Y(j) = \begin{cases} \frac{3}{10} & \text{if } j = -1 \\ \frac{6}{10} & \text{if } j = 3 \\ \frac{1}{10} & \text{if } j = 4 \end{cases}$$

   (d)

$$\mathbb{E}[Y] = (-3)\frac{1}{10} + 3\frac{6}{10} + 4\frac{1}{10} = \frac{19}{10}.$$

2. (a)

$$\mathbb{P}\{Y \ge X\} = \int_{s=0}^{\infty}\int_{t=s}^{\infty} \lambda e^{-\lambda s}\nu e^{-\nu t} dt\, ds = \int_{s=0}^{\infty} \lambda e^{-\lambda s - \nu s} ds = \frac{\lambda}{\lambda+\nu}.$$

   (b)

$$\mathbb{P}\{Z > 5\} = \mathbb{P}\{X > 5\}\mathbb{P}\{Y > 5\} = \left(\int_{s=5}^{\infty} \lambda e^{-\lambda s} ds\right)\left(\int_{t=5}^{\infty} \nu e^{-\nu t} dt\right) = e^{-5(\lambda+\nu)}.$$

   (c)

$$\mathbb{P}\{Z \le 5\} = 1 - e^{-5(\lambda+\nu)}.$$

   (d)

$$F_Z(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-(\lambda+\nu)t} & \text{if } t \ge 0 \end{cases}$$

3. (a)

$$f_X(t) = \begin{cases} 6t(1-t) & \text{if } t \in [0,1] \\ 0 & \text{else} \end{cases}$$

   (b)

$$\mathbb{P}\left\{Z \le \frac{1}{3}\right\} = \int_{s=0}^{1/3}\int_{t=0}^{1/3-s} (6t) dt\, ds = 3\int_{s=0}^{1/3}\left(\frac{1}{3} - s\right)^2 ds = \left(\frac{1}{3}\right)^3.$$

   (c)

$$F_Z(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t^3 & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(d)

$$f_Z(t) = \begin{cases} 3t^2 & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Fall 2002
Final, December 21

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. 55 points Let X and Y be independent geometric random variables with common parameter p; i.e.,

$$f_X(j) = f_Y(j) = \begin{cases} p(1-p)^j & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

Define $Z_1 \overset{\text{def}}{=} \min\{X, Y+3\}$ and define $Z_2 \overset{\text{def}}{=} X + Y$.

(a) 10 points Compute $\mathbb{P}\{Z_1 \geq 14\}$

(b) 10 points Compute $\mathbb{P}\{Z_1 \geq j\}$ for all $j \in \{3, 4 \dots\}$.

(c) 10 points Compute $f_{Z_1}(j)$, the discrete density of Z_1, for $j \in \{3, 4 \dots\}$.

(d) 10 points Compute $\mathbb{P}\{X = 3 \text{ and } Z_2 = 5\}$.

(e) 15 points Compute the joint density f_{X,Z_2}.

2. 70 points Suppose that there are 7 restaurants in a town. These restaurants are labelled A through G. Suppose that there are 10 students who are hungry and that each one independently and randomly chooses a restaurant.

(a) 10 points What is the probability that all students go to restaurant A?

(b) 10 points What is the probability that all students go to the same restaurant?

(c) 10 points What is the probability that exactly 4 students go to restaurant A and exactly 3 students go to restaurant B?

(d) 10 points What is the probability that restaurant A has 4 students and all of the others have exactly 1 student?

Assume that three of the students are Debbie, Steve, and John. Assume that Debbie has been going out with both Steve and John (but that neither Steve nor John knows about the other).

(e) 10 points What is the probability that Debbie, Steve, and John will all end up in restaurant A (no doubt leading to a brawl since both Steve and John have anger-management issues)?

(f) 10 points What is the probability that Steve and John will end up in the same restaurant, but that Debbie will end up in a different restaurant (thus leading both Steve and John to break up with Debbie)?

(g) 10 points What is the probability that Debbie and Steve end up alone in a restaurant (thus allowing for a romantic dinner, leading to things getting out of hand later on that night)?

3. 25 points Let X be an $\mathbf{N}(0,1)$-random variable; i.e., it is a continuous random variable with density

$$f_X(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \qquad t \in \mathbb{R}$$

Let F_X be the cumulative distribution function of X. Define now $Y \stackrel{\text{def}}{=} e^X$.

(a) 10 points Compute F_Y, the cumulative distribution function for Y; you will probably have to use F_X to do this.

(b) 5 points Is Y continuous, or is it discrete (in this case, it is definitely one or the other)?

(c) 10 points Compute the density of Y (the discrete density if Y is discrete and the continuous density if Y is continuous).

Answers

1. (a)
$$\mathbb{P}\{Z_1 \geq 14\} = \mathbb{P}\{X \geq 14\}\mathbb{P}\{Y \geq 11\} = (1-p)^{14}(1-p)^{11} = (1-p)^{25}.$$
   (b) $\mathbb{P}\{Z_1 \geq j\} = (1-p)^{2j-3}$.
   (c) $f_{Z_1}(j) = \mathbb{P}\{Z_1 \geq j\} - \mathbb{P}\{Z_1 \geq j+1\} = (1-p)^{2j-3} - (1-p)^{2j-1}$.
   (d) $\mathbb{P}\{X = 3 \text{ and } Z_2 = 5\} = \mathbb{P}\{X = 3\}\mathbb{P}\{Y = 2\} = p^2(1-p)^5$
   (e)
$$f_{X,Z_2}(j,k) = \begin{cases} p^2(1-p)^k & \text{if } k \geq j \geq 0 \\ 0 & \text{else} \end{cases}$$

Let $q \stackrel{\text{def}}{=} \frac{1}{7^{10}}$.

   (a) q.
   (b) $7q$.
   (c) $\binom{10}{4}\binom{6}{3}5^3 q$.
   (d) $\binom{10}{4}(6!)q$.
   (e) $7^7 q = 1/7^3$.
   (f) $(7)_2 7^7 q = (7)_2/7^3$.
   (g) $(7)(6^8)q$

2. (a)
$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ F_X(\ln t) & \text{if } t \geq 0 \end{cases}$$
   (b) Continuous
   (c)
$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{t\sqrt{2\pi}} \exp\left[-\frac{t}{2}\right] & \text{if } t \geq 0 \end{cases}$$