

**Math 361, Section F1, Fall 2000**  
**Quiz 1, August 30**

**Name:** \_\_\_\_\_

1. 10 points Suppose that  $\mathbb{P}(A) = 0.5$ ,  $\mathbb{P}(B) = 0.3$ , and  $\mathbb{P}(A \cap B) = 0.2$ .
  - (a) 4 points Compute  $\mathbb{P}(A|B)$ .
  - (b) 4 points Compute  $\mathbb{P}(B|A)$ .
  - (c) 2 points Compute  $\mathbb{P}(A|A \cap B)$ .

## ANSWERS

1. (a)  $\mathbb{P}(A|B) = \frac{0.2}{0.3} = \frac{2}{3}$ .
- (b) Compute  $\mathbb{P}(B|A) = \frac{0.2}{0.5} = \frac{2}{5}$ .
- (c) Compute  $\mathbb{P}(A|A \cap B) = \frac{\mathbb{P}(A \cap A \cap B)}{\mathbb{P}(A \cap B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cap B)} = 1$ .

1. 10 points Consider a certain test for cancer. Suppose that 90% of those with cancer react positively to the test, while 5% of those without cancer also react positively to the test. Suppose also that 1% of people actually have cancer. Suppose you take the test and react positively. What is the probability that you actually have cancer?

## ANSWERS

1. Define the sets

$$P \stackrel{\text{def}}{=} \{\text{test positive}\} \quad \text{and} \quad C = \{\text{have cancer}\}.$$

Then

$$\mathbb{P}(P|C) = 0.9, \quad \mathbb{P}(P|C^c) = 0.05, \quad \mathbb{P}(C) = 0.01, \quad \text{and} \quad \mathbb{P}(C^c) = 0.99.$$

Then

$$\mathbb{P}(C|P) = \frac{\mathbb{P}(P|C)\mathbb{P}(C)}{\mathbb{P}(P|C)\mathbb{P}(C) + \mathbb{P}(P|C^c)\mathbb{P}(C^c)} = \frac{(0.9)(0.01)}{(0.9)(0.01) + (0.05)(0.99)}.$$

**Math 361, Section F1, Fall 2000**  
**Quiz 3, September 15**

**Name:** \_\_\_\_\_

1. 10 points A mathematics department consists of

- 15 full professors
- 10 associate professors
- 5 assistant professors.

A committee of 6 is selected at random from the faculty of the department. What is the probability that the committee will consist of

- 2 full professors
- 2 associate professors
- 2 assistant professors?

## ANSWERS

1.

$$\frac{\binom{15}{2} \binom{10}{2} \binom{5}{2}}{\binom{30}{6}}.$$

1. 10 points Suppose that we have a cumulative distribution function given by the formula

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{8} & \text{if } 0 \leq t < \frac{1}{3} \\ \frac{1}{9} + (t - \frac{1}{3})^2 & \text{if } \frac{1}{3} \leq t < \frac{2}{3} \\ \frac{1}{2} & \text{if } \frac{2}{3} \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

Compute

$$\mathbb{P} \left\{ \text{either } X = \frac{1}{3} \text{ or } X \geq \frac{3}{4} \right\}.$$

ANSWERS

1.

$$\begin{aligned}\mathbb{P}\left\{\text{either } X = \frac{1}{3} \text{ or } X \geq \frac{3}{4}\right\} &= \mathbb{P}\left\{X = \frac{1}{3}\right\} + \mathbb{P}\left\{X \geq \frac{3}{4}\right\} \\ &= \mathbb{P}\left\{X = \frac{1}{3}\right\} + \left(1 - \mathbb{P}\left\{X < \frac{3}{4}\right\}\right) = F_X(1/3) - F_X(1/3-) + 1 - F_X(3/4-) \\ &= \frac{1}{9} - \frac{1}{24} + 1 - \frac{1}{2}.\end{aligned}$$



1. 10 points Suppose that the random variable  $X$  has a geometric density with parameter  $1/8$ ; i.e.,

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{8} \left(1 - \frac{1}{8}\right)^j & \text{if } j \in \{0, 1, \dots\} \\ 0 & \text{else} \end{cases}$$

Define now

$$Y \stackrel{\text{def}}{=} X + 7.$$

Find the density of  $Y$ .

ANSWERS

1.

$$f_Y(j) = \begin{cases} \frac{1}{8} \left(\frac{7}{8}\right)^{j-7} & \text{if } j \in \{7, 8, 9 \dots\} \\ 0 & \text{else} \end{cases}$$

1. 10 points Suppose that we have a random variable whose density is given by

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{4} & \text{if } j = 0 \\ \frac{1}{8} & \text{if } j = 1 \\ \frac{1}{8} & \text{if } j = 3 \\ \frac{1}{2} & \text{if } j = 4 \end{cases}$$

- (a) 3 points Compute  $\mathbb{E}[X]$ .
- (b) 4 points Compute  $\mathbb{E}[X^2]$ .
- (c) 3 points Compute  $\mathbb{E}[X - 3]$ .

## ANSWERS

1. (a)

$$\mathbb{E}[X] = \frac{(0)(2) + (1)(1) + (3)(1) + (4)(4)}{8} = \frac{20}{8}.$$

(b)

$$\mathbb{E}[X^2] = \frac{(0)^2(2) + (1)(1) + (3)^2(1) + (4)^2(4)}{8} = \frac{74}{8}.$$

(c)  $\mathbb{E}[X - 3] = \mathbb{E}[X] - 3 = \frac{20}{8} - 3 = -\frac{4}{8}.$

1. 10 points Suppose that the continuous random variable  $X$  has a density

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 2 - t & \text{if } 1 < t \leq 2. \end{cases}$$

Compute

$$\mathbb{P} \left\{ \frac{1}{2} \leq X \leq \frac{3}{2} \right\}.$$

ANSWERS

1.

$$\begin{aligned}\mathbb{P}\left\{\frac{1}{2} \leq X \leq \frac{3}{2}\right\} &= \int_{1/2}^{3/2} f_X(s) ds = \int_{1/2}^1 s ds + \int_1^{3/2} (2-s) ds \\ &= 2 \int_{1/2}^1 s ds = s^2 \Big|_{s=1/2}^1 = 1 - \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

1. 10 points Suppose that the continuous random variable  $X$  has a density

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} 2t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

- (a) 5 points Compute the cumulative density  $F_X$  of  $X$ .
- (b) 5 points Define  $Y \stackrel{\text{def}}{=} X^3$ . Compute the density of  $Y$  (if one exists).

ANSWERS

1. (a)

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t^2 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 0 \leq t \leq 1 \end{cases}$$

(b)

$$f_Y(t) = \begin{cases} \frac{2}{3}t^{-1/3} & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases}$$



1. 10 points Suppose that the pair  $(X, Y)$  of continuous random variables has joint density

$$f_{X,Y}(s, t) \stackrel{\text{def}}{=} \begin{cases} \lambda^2 e^{-\lambda t} & \text{if } 0 \leq s \leq t \\ 0 & \text{else.} \end{cases}$$

Compute  $\mathbb{P}\{X \leq 7\}$ .

## ANSWERS

1.

$$\begin{aligned}\mathbb{P}\{X \leq 7\} &= \int_{s=-\infty}^7 \int_{t=-\infty}^{\infty} f_{X,Y}(s,t) dt ds = \int_{s=0}^7 \int_{t=s}^{\infty} \lambda^2 e^{-\lambda t} dt ds \\ &= \int_{s=0}^7 \lambda e^{-\lambda s} ds = 1 - e^{-7\lambda}.\end{aligned}$$

1. 10 points Suppose that the pair  $(X, Y)$  of continuous random variables has joint density

$$f_{X,Y}(s, t) \stackrel{\text{def}}{=} \begin{cases} \lambda^2 e^{-\lambda t} & \text{if } 0 \leq s \leq t \\ 0 & \text{else.} \end{cases}$$

Compute  $f_Y(7)$ .

ANSWERS

1.

$$f_Y(7) = \int_{s=-\infty}^{\infty} f_{X,Y}(s, 7) ds = \int_{s=0}^7 \lambda^2 e^{-7\lambda} ds = 7\lambda^2 e^{-7\lambda}.$$

1. 10 points Suppose that  $X$  and  $Y$  are independent continuous random variables. Assume that  $X$  is uniform on  $[0, 1]$ ; i.e.,

$$f_X(s) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 0 & \text{else} \end{cases}$$

and assume that  $Y$  is geometric with parameter  $\lambda > 0$ ; i.e.,

$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{else.} \end{cases}$$

Define  $Z \stackrel{\text{def}}{=} X + Y$ .

- (a) 5 points Compute  $f_Z(5)$ .
- (b) 5 points Compute  $f_Z(0.5)$ .

ANSWERS

1. (a)

$$\begin{aligned}
 f_Z(5) &= \int_{t=-\infty}^{\infty} f_X(t)f_Y(5-t)dt = \int_{t=-\infty}^{\infty} \chi_{[0,1]}(t)\chi_{[0,\infty)}(5-t)\lambda e^{-\lambda(5-t)}dt \\
 &= \int_{t=-\infty}^{\infty} \chi_{[0,1]\cap(-\infty,5]}(t)\lambda e^{-\lambda(5-t)}dt \\
 &= \int_{t=0}^1 \lambda e^{-\lambda(5-t)}dt = \int_{t=4}^5 \lambda e^{-\lambda t}dt = e^{-4\lambda} - e^{-5\lambda}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 f_Z(0.5) &= \int_{t=-\infty}^{\infty} f_X(t)f_Y(0.5-t)dt = \int_{t=-\infty}^{\infty} \chi_{[0,1]}(t)\chi_{[0,\infty)}(0.5-t)\lambda e^{-\lambda(0.5-t)}dt \\
 &= \int_{t=-\infty}^{\infty} \chi_{[0,1]\cap(-\infty,0.5]}(t)\lambda e^{-\lambda(0.5-t)}dt \\
 &= \int_{t=0}^{0.5} \lambda e^{-\lambda(0.5-t)}dt = \int_{t=0}^{0.5} \lambda e^{-\lambda t}dt = 1 - e^{-0.5\lambda}.
 \end{aligned}$$

1. 10 points Suppose that  $X$  is a random variable with moment generating function

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}[e^{\theta X}] = \exp[4\theta^2 + 9\theta].$$

What is the density of  $X$ ?

## ANSWERS

1.  $X$  is Gaussian with mean 9 and variance 16;

$$f_X(t) = \frac{1}{\sqrt{32\pi}} \exp\left[-\frac{(t-9)^2}{32}\right] \quad t \in \mathbb{R}$$



**Math 361, Section F1, Fall 2000**  
**Exam 1, September 20**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 35 points A mathematics department consists of

- 15 full professors
- 10 associate professors
- 5 assistant professors.

A committee of 6 is selected at random from the faculty of the department.

- (a) 5 points What is the probability that the committee will consist of 1 full professor, 2 associate professors, and 3 assistant professors?
- (b) 10 points What is the probability that the committee will consist of at least 3 full professors?
- (c) 10 points What is the probability that the committee will consist of exactly one assistant professor and at most one full professors?
- (d) 10 points What is the probability that the committee will consist of exactly one assistant professor given that it has at most one full professors?

2. 25 points There are 6 balls labelled  $A$  through  $F$  are in a box. We pick them out of the box one by one.

- (a) 5 points What is the probability that the third ball is ball  $B$ ?
- (b) 10 points What is the probability that the third ball is ball  $B$  and the fourth ball is ball  $A$ ?
- (c) 10 points What is the probability that the fourth ball is  $A$  given that the third ball is  $B$ ?

3. 40 points Suppose that we select a hand of five cards out of a standard deck of 52 cards. We assume that aces are high cards.

- (a) 10 points What is the probability that all of the cards are greater than 5?
- (b) 15 points What is the probability that at least two of the cards are greater than 10?
- (c) 15 points What is the probability that all of the cards are greater than 5 given that at least two of the cards are greater than 10?

ANSWERS

1. Set  $q \stackrel{\text{def}}{=} 1/\binom{30}{6}$ .

(a)  $15\binom{10}{2}\binom{5}{3}q$ .

(b)

$$1 - \mathbb{P}\{0,1, \text{ or } 2 \text{ full professors}\} = 1 - \left\{ \binom{15}{6} + 15\binom{15}{5} + \binom{15}{2}\binom{15}{4} \right\} q.$$

(c)

$$\left\{ 5\binom{10}{5} + 5 \cdot 15\binom{10}{4} \right\} q.$$

(d)

$$\frac{5\binom{10}{5} + 5 \cdot 15\binom{10}{4}}{\binom{15}{6} + 15\binom{15}{5}}.$$

2. (a)  $\frac{\binom{5}{2}}{\binom{6}{3}} = \frac{1}{6}$ .

(b)  $\frac{\binom{4}{2}}{\binom{6}{4}} = \frac{1}{30}$ .

(c)  $\frac{\binom{1/30}}{\binom{1/6}} = \frac{1}{5}$ .

3. Sequence: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A Set  $q \stackrel{\text{def}}{=} 1/\binom{52}{5}$ .

(a)  $\binom{36}{5}q$

(b)

$$\begin{aligned} & 1 - \mathbb{P}\{\text{no card is greater than } 10\} - \mathbb{P}\{1 \text{ card is greater than } 10\} \\ &= 1 - \mathbb{P}\{\text{all cards are } 10 \text{ or less}\} \\ &\quad - \mathbb{P}\{1 \text{ card is greater than } 10, 4 \text{ cards are } 10 \text{ or less}\} \\ &= 1 - \binom{36}{5}q - 16\binom{36}{4}q. \end{aligned}$$

(c)

$$\begin{aligned} & \mathbb{P}\{\text{all cards greater than } 5 \text{ and at least two cards greater than } 10\} \\ &= \mathbb{P}\{\text{all cards greater than } 5\} \\ &\quad - \mathbb{P}\{\text{all cards greater than } 5 \text{ and no cards greater than } 10\} \\ &\quad - \mathbb{P}\{\text{all cards greater than } 5, \text{ one card greater than } 10, \text{ and } 4 \text{ cards } 10 \text{ or less}\} \\ &= \mathbb{P}\{\text{all cards greater than } 5\} \\ &\quad - \mathbb{P}\{\text{all cards in } 6,7,8,9,10\} \\ &\quad - \mathbb{P}\{4 \text{ cards in } 6,7,8,9,10, \text{ and } 1 \text{ card in } J,Q,K,A\} \\ &= \binom{36}{5}q - \binom{20}{5}q - 16\binom{20}{4}q. \end{aligned}$$

Final answer is

$$\frac{\binom{36}{5} - \binom{20}{5} - 16\binom{20}{4}}{1 - \binom{36}{5} - 16\binom{36}{4}}.$$

Math 361, Section F1, Fall 2000  
Exam 2, October 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 35 points Let  $X$  be a geometric random variable with parameter  $p$ ; i.e.,

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} p(1-p)^j & j \in \{0, 1, \dots\} \\ 0 & \text{else.} \end{cases}$$

Define a new random variable

$$Y \stackrel{\text{def}}{=} 13 - |X - 20|.$$

Find the density  $f_Y$  of  $Y$ .

2. 35 points Suppose that  $X$  and  $Y$  have joint density

		Y			
		0	1	2	3
	-1	3/27	1/27	1/27	3/27
X	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words,  $f_{X,Y}(3,0) = \frac{1}{9}$ .

- (a) 5 points Compute  $f_X$ .
- (b) 5 points Compute  $f_Y$ .
- (c) 10 points Compute  $\mathbb{E}[XY]$ .
- (d) 5 points Compute  $\mathbb{E}[X]$ .
- (e) 10 points Compute  $\mathbb{E}[X^2]$ .
3. 30 points Suppose that a random variable  $X$  has moment generating function

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\exp[\theta X]] = \exp\left[\frac{9\theta^2}{2} + 4\theta\right]. \quad \theta \in \mathbb{R}$$

- (a) 10 points Compute  $\mathbb{E}[X]$ .
- (b) 10 points Compute  $\mathbb{E}[X^2]$ .
- (c) 10 points Compute the variance of  $X$ .

ANSWERS

1. Easiest to look at graph of  $t \mapsto 13 - |t - 20|$ . Analytically, we have as follows. If  $Y = j$ , then  $13 - |X - 20| = j$ , so  $|X - 20| = 13 - j$ . Thus, either  $X - 20 = 13 - j$  or  $X - 20 = j - 13$ ; i.e.,  $X = 33 - j$  or  $X = j + 7$ . Note that  $33 - j = j + 7$  if and only if  $j = 13$ , in which case  $33 - j = j + 7 = 20$ .

$$f_Y(j) = \begin{cases} f_X(20) & \text{if } j = 13 \\ f_X(33 - j) + f_X(j + 7) & \text{if } j \in \{0, 1, 2, \dots, 12\} \\ f_X(33 - j) & \text{if } j \in \{-1, -2, \dots\} \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} p(1 - p)^{20} & \text{if } j = 13 \\ p(1 - p)^{33-j} + p(1 - p)^{j+7} & \text{if } j \in \{0, 1, 2, \dots, 12\} \\ p(1 - p)^{33-j} & \text{if } j \in \{-1, -2, \dots\} \\ 0 & \text{else} \end{cases}$$

2. (a)

$$f_X(j) = \begin{cases} \frac{6}{27} & \text{if } j = -1 \\ \frac{12}{27} & \text{if } j = 0 \\ \frac{7}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$

- (b)

$$f_Y(j) = \begin{cases} \frac{12}{27} & \text{if } j = 0 \\ \frac{1}{27} & \text{if } j = 1 \\ \frac{4}{27} & \text{if } j = 2 \\ \frac{10}{27} & \text{if } j = 3 \\ 0 & \text{else} \end{cases}$$

- (c)

$$\mathbb{E}[XY] = \frac{(-1) \cdot 1 + (-2) \cdot 1 + (-3) \cdot 3 + 9 \cdot 4}{27} = \frac{24}{27}.$$

- (d)

$$\mathbb{E}[X] = \frac{(-1) \cdot 6 + 3 \cdot 7}{27} = \frac{25}{27}.$$

- (e)

$$\mathbb{E}[X^2] = \frac{(-1)^2 \cdot 6 + 3^2 \cdot 7}{27} = \frac{69}{27}.$$

3. We have that  $\dot{\varphi}_X(\theta) = (9\theta + 4)\varphi_X(\theta)$  and  $\ddot{\varphi}_X(\theta) = \{(9\theta + 4)^2 + 9\}\varphi_X(\theta)$  for all  $\theta \in \mathbb{R}$ .

- (a)  $\mathbb{E}[X] = \dot{\varphi}_X(0) = 4$ .

(b)  $\mathbb{E}[X^2] = \ddot{\varphi}_X(0) = 4^2 + 9 = 25.$

(c)  $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 25 - 4^2 = 9.$

**Math 361, Section F1, Fall 2000**  
**Exam 3, November 29**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.  
Maximum possible score: 100 Points

1. 47 points Let  $X$  be a uniform random variable on  $[0, 1]$ ; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

Define

$$Y_1 \stackrel{\text{def}}{=} X^3 \quad Y_2 \stackrel{\text{def}}{=} X^{-3} \quad \text{and} \quad Y_3 \stackrel{\text{def}}{=} X^0.$$

- (a) 5 points Compute  $F_X$ , the cumulative distribution function for  $X$ .
- (b) 9 points Find the cumulative distribution function for  $Y_1$ .
- (c) 5 points Find the density of  $Y_1$  if it exists. If it doesn't exist, state so.
- (d) 9 points Find the cumulative distribution function for  $Y_2$ .
- (e) 5 points Find the density of  $Y_2$  if it exists. If it doesn't exist, state so.
- (f) 9 points Find the cumulative distribution function for  $Y_3$ .
- (g) 5 points Find the density of  $Y_3$  if it exists. If it doesn't exist, state so.
2. 30 points Suppose that  $X$  and  $Y$  are two continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} e^{-t} & \text{if } 0 \leq s \leq t \\ 0 & \text{else.} \end{cases}$$

- (a) 20 points Compute the cumulative distribution function of  $X$ .
- (b) 10 points Compute the density of  $Y$ .
3. 23 points Suppose that  $X$  and  $Y$  are two independent continuous random variables with densities

$$f_X(s) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 0 & \text{else} \end{cases}$$
$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{if } -1 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

Define  $Z \stackrel{\text{def}}{=} X + Y$ .

- (a) 10 points Compute  $f_Z(0)$ .
- (b) 8 points Compute  $\mathbb{E}[Y]$ .
- (c) 5 points Compute  $\mathbb{E}[Y^2]$ .

ANSWERS

1. (a)

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

(b)

$$\begin{aligned} F_{Y_1}(t) &= \mathbb{P}\{X^3 \leq t\} = \mathbb{P}\{X \leq t^{1/3}\} = F_X(t^{1/3}) \\ &= \begin{cases} 0 & \text{if } t^{1/3} < 0 \\ t^{1/3} & \text{if } 0 \leq t^{1/3} < 1 \\ 1 & \text{if } t^{1/3} \geq 1 \end{cases} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases} \end{aligned}$$

(c)

$$f_{Y_1}(t) = \begin{cases} \frac{1}{3t^{2/3}} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

(d) Look at graph of  $t \mapsto t^{-3}$ .

$$F_{Y_2}(t) = \mathbb{P}\{X^{-3} \leq t\} = \begin{cases} \mathbb{P}\{X > t^{-1/3}\} & \text{if } t \geq 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 0 & \text{if } t < 1 \\ 1 - t^{-1/3} & \text{if } t \geq 1 \end{cases}$$

(e)

$$\begin{cases} \frac{1}{3t^{4/3}} & \text{if } t > 1 \\ 0 & \text{else} \end{cases}$$

(f)  $Y_3 \equiv 1$ .

$$F_{Y_3}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(g) No density.

2. Note that  $f_{X,Y}$  is nonzero only in an infinite triangle bounded by the vertical axis and the line through the origin of slope 1.

(a)

$$F_X(t) = \int_{u=-\infty}^t \int_{v=-\infty}^{\infty} f_{X,Y}(u,v) du dv = \int_{u=-\infty}^t \int_{v=-\infty}^{\infty} e^{-v} \chi_{\{0 \leq u \leq v\}} du dv.$$

If  $t < 0$ ,  $F_X(t) = 0$ . If  $t \geq 0$ ,

$$F_X(t) = \int_{u=0}^t \int_{v=u}^{\infty} e^{-v} dv du = \int_{u=0}^t e^{-u} du = 1 - e^{-t}.$$



Thus

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-t} & \text{if } t \geq 0 \end{cases}$$

(b)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t) ds = \int_{s=-\infty}^{\infty} e^{-t} \chi_{\{0 \leq s \leq t\}} ds.$$

If  $t < 0$ ,  $f_Y(t) = 0$ . If  $t \geq 0$ ,

$$f_Y(t) = \int_{s=0}^t e^{-t} ds = te^{-t}.$$

Thus

$$f_Y(t) = \begin{cases} 0 & \text{if } t > 0 \\ te^{-t} & \text{if } t \geq 0 \end{cases}$$

3. (a)

$$\begin{aligned} f_Z(t) &= \int_{s=-\infty}^{\infty} f_X(s) f_Y(t-s) ds = \frac{1}{2} \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s) \chi_{[-1,1]}(t-s) ds \\ &= \frac{1}{2} \int_{s=-\infty}^{\infty} \chi_{[0,1]}(s) \chi_{[t-1,t+1]}(s) ds = \frac{1}{2} \int_{s=-\infty}^{\infty} \chi_{[0,1] \cap [t-1,t+1]}(s) ds \\ &= \begin{cases} \frac{1}{2}(t+1) & \text{if } 0 \leq t+1 < 1 \\ \frac{1}{2} & \text{if } t-1 < 0 \text{ and } t+1 < 1 \\ \frac{1}{2}\{1 - (t-1)\} & \text{if } 0 < t-1 < 1 \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{1}{2}(t+1) & \text{if } -1 < t < 0 \\ \frac{1}{2} & \text{if } 0 < t < 1 \\ \frac{1}{2}(2-t) & \text{if } 1 < t < 2 \\ 0 & \text{else} \end{cases} \end{aligned}$$

(b)

$$\mathbb{E}[Y] = \int_{t=-1}^1 \frac{t}{2} dt = 0.$$

(c)

$$\mathbb{E}[Y^2] = \int_{t=-1}^1 \frac{t^2}{2} dt = \frac{t^3}{6} \Big|_{t=-1}^1 = \frac{2}{6} = \frac{1}{3}.$$

**Math 361, Section F1, Fall 2000**  
**Final, December 11**

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 150 Points

1. 55 points Let  $X$  and  $Y$  be two independent exponential random variables with parameters  $\alpha$  and  $\beta$ , respectively, where  $\beta > \alpha$ ; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \alpha e^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$
$$f_Y(t) \stackrel{\text{def}}{=} \begin{cases} \beta e^{-\beta t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

- (a) 10 points Find the joint density  $f_{X,Y}$ .
- (b) 15 points Compute  $\mathbb{P}\{Y \leq 3X\}$ .
- (c) 15 points Compute  $\mathbb{P}\{\min\{X, Y\} \geq 3\}$ .
- (d) 15 points Set  $Z \stackrel{\text{def}}{=} X + Y$ . Compute  $f_Z(t)$ .
2. 15 points Let  $X$  be a random variable with cumulative distribution function

$$F_X(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } t < 0 \\ (t + 1)/4 & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

- (a) 5 points Compute  $\mathbb{P}\{X \leq 1\}$ .
- (b) 5 points Compute  $\mathbb{P}\{X = 1\}$ .
- (c) 5 points Compute  $\mathbb{P}\{X \geq 0\}$ .
3. 15 points Suppose that we have a noisy data channel. Suppose that

$$\mathbb{P}\{0 \text{ sent}\} = \mathbb{P}\{1 \text{ sent}\} = 0.5,$$
$$\mathbb{P}\{1 \text{ received} | 1 \text{ sent}\} = 0.9, \quad \text{and} \quad \mathbb{P}\{0 \text{ received} | 0 \text{ sent}\} = 0.8.$$

- (a) 5 points Compute  $\mathbb{P}\{1 \text{ received} | 0 \text{ sent}\}$ .
- (b) 5 points Compute  $\mathbb{P}\{1 \text{ received}\}$ .
- (c) 5 points Compute  $\mathbb{P}\{1 \text{ sent} | 1 \text{ received}\}$ .
4. 20 points Suppose that

$$\mathbb{P}(A \cap C | B) = 0.04, \quad \mathbb{P}(A | B) = 0.2, \quad \mathbb{P}(C | B) = 0.2, \quad \mathbb{P}(B) = 0.5$$

- (a) 10 points Compute  $\mathbb{P}(A \cap B \cap C)$ .
- (b) 5 points Compute  $\mathbb{P}(C \cap B)$ .
- (c) 5 points Compute  $\mathbb{P}(A|B \cap C)$ .
5. 45 points Let  $X$  and  $Y$  be continuous random variables such that

$$f_{X,Y}(s,t) = \begin{cases} (t+1)e^{-(t+1)s} & \text{if } s \geq 0 \text{ and } 0 \leq t \leq 1 \\ 0 & \text{else.} \end{cases}$$

- (a) 10 points Compute the density of  $Y$ .
- (b) 10 points Compute  $f_{X|Y}(s|t)$ .
- (c) 15 points Compute  $\mathbb{E}[X|Y = t]$ .
- (d) 10 points Compute  $\mathbb{E}[X]$ .

ANSWERS

1. (a)

$$f_{X,Y}(s,t) = \begin{cases} \alpha\beta e^{-\alpha s - \beta t} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

(b)  $\frac{3\beta}{\alpha+3\beta}$

(c)  $e^{-3(\alpha+\beta)}$

(d)

$$f_Z(t) = \begin{cases} \frac{\alpha\beta}{\beta-\alpha}(e^{-\alpha t} - e^{-\beta t}) & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

2. (a) 1

(b)  $\frac{1}{2}$

(c) 1

3. (a) 0.2

(b) 0.55

(c)  $\frac{9}{11}$

4. (a) 0.02

(b) 0.1

(c) 0.2

5. (a)

$$f_Y(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$f_{X|Y}(s|t) = \begin{cases} (t+1)e^{-(t+1)s} & \text{if } s \geq 0 \text{ and } 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

(c)  $\mathbb{E}[X|Y = t] = \frac{1}{t+1}$

(d)  $\mathbb{E}[X] = \ln 2$