Show all work; partial credit will be given
Calculators should only be used on Questions 1c, 8, 9
Remember that I have to grade this; don’t over-reduce your answers
Remember that I have to grade this; be neat
255 points total.

1. **30 points** Suppose that in a typical lottery the probability of winning a prize is 1/100. Suppose that we play many lotteries.

   Suppose that we play 500 different lotteries. Let’s use the Poisson approximation.
   
   (a) **10 points** What is the probability of winning at least once?
   
   (b) **10 points** What is the probability of winning twice?
   
   (c) **10 points** Suppose now that we really like the rush of winning a lottery. How many lotteries should we play so that the probability of winning at least once is .3 or greater?

2. **40 points** Suppose that we have a coin such that $P\{\text{heads}\} = p$ for some fixed $p \in (0, 1)$. Flip the coin a number of times. Let $X_4$ be the position of the 4th heads and let $X_{10}$ be the position of the 10th heads.

   (a) **10 points** Describe what it means to have $X_4 = 13$ and $X_{10} = 25$.
   
   (b) **10 points** Compute $P\{X_4 = 13, X_{10} = 25\}$.
   
   (c) **10 points** Compute $P\{X_4 = 13|X_{10} = 25\}$.
   
   (d) **10 points** Compute $P\{X_i = i|X_{10} = 25\}$ for $i \in \{4, 5, \ldots, 14\}$

3. **30 points** (modelled on question 2.44). Six people are in a room. Three of them are Alex, Beth, and Charlie. They are randomly seated side by side on a bench.

   (a) **10 points** What is the probability that Alex and Beth are seated side by side?
   
   (b) **10 points** What is the probability that there are exactly two people between Alex and Beth?
   
   (c) **10 points** What is the probability that Alex, Beth, and Charlie are seated side by side?

4. **30 points** Suppose that $P(A) = .6$ \quad $P(B) = .7$ \quad and \quad $P(A|B) = .8$.

   (a) **5 points** Are $A$ and $B$ independent?
   
   (b) **10 points** Compute $P(A \cap B)$
   
   (c) **10 points** Compute $P(A \cup B)$
(d) 5 points Compute $\mathbb{P}((A \cup B)^c \cup (A \cap B))$

5. 50 points Let $X$ be an exponential random variable with parameter 1; i.e., it is continuous with density

$$f_X(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

Define

$$\varphi(z) \overset{\text{def}}{=} \begin{cases} 2 & \text{if } z \leq 2 \\ z - 2 & \text{if } 2 < z \leq 7 \\ 5 & \text{if } z > 7 \end{cases}$$

Define $Y \overset{\text{def}}{=} \varphi(X)$.

(a) 10 points Graph $\varphi$

(b) 10 points Compute $\mathbb{P}\{Y \leq \frac{1}{2}\}$

(c) 10 points Compute $\mathbb{P}\{Y \leq 3\}$.

(d) 10 points Compute the cumulative distribution function of $Y$.

(e) 10 points What is $\mathbb{P}\{Y = 2\}$?

6. 25 points Suppose that $X$ and $Y$ are two independent random variables with densities

$$f_X(s) = \begin{cases} e^{-s} & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) = \begin{cases} te^{-t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$.

(a) 10 points What is the joint density $f_{X,Y}$ of $X$ and $Y$?

(b) 10 points Compute $f_Z(7)$, where $f_Z$ is the density of $Z$.

(c) 5 points Compute $f_Z$.

7. 30 points (after question 4.38) Suppose that $\mathbb{E}[X] = 2$ and $\text{Var}(X) = 6$. Compute

(a) 10 points $\mathbb{E}[X^2]$.

(b) 10 points $\mathbb{E}[(3 + X)^2]$.

(c) 10 points $\text{Var}(3 + 2X)$.
8. 10 points Suppose that 55% are in favor of a proposed new law. If we sample 1000 people, what is the probability that we will in fact find that less than 500 are in favor of the law?

9. 10 points Let $X$ be a Gaussian random variable with mean 5 and variance 36. Find the value of $c$ such that $\mathbb{P}\{X \leq c\} = .4$. 

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1. If we play \( N \) lotteries, the number \( X \) of times we win is approximately Poisson with parameter \( N/100 \).

If \( N = 100 \), \( X \) is Poisson with parameter 5
(a) \( P\{X \geq 1\} = 1 - P\{X = 0\} = 1 - e^{-5} \).
(b) \( P\{X = 2\} = \frac{5^2}{2!} e^{-5} \).
(c) We want that \( 1 - e^{-N/100} \geq .3 \). Thus \( N \geq 100 \ln \frac{1}{.7} = 35 \).

2. (a) 13th and 25th coins are heads, 3 heads in the first 12, and 5 heads in tosses \( \{14, 15, \ldots, 24\} \).
(b) \( P\{X_{14} = 13, X_{10} = 25\} = \binom{12}{3} \binom{11}{5} p^{10}(1-p)^5 \).
(c) \( P\{X_{10} = 25\} = \binom{24}{9} (1-p)^{10}(1-p)^5 \), so
\[
P\{X_4 = 13|X_{10} = 20\} = \frac{\binom{12}{3} \binom{11}{5}}{\binom{24}{9}}.
\]
(d) \( P\{X_4 = i|X_{10} = 25\} = \frac{\binom{i-1}{3} \binom{24-i}{5}}{\binom{24}{9}}. \)

3. (a) \( 5! \times 2/6! = \frac{2}{6} = \frac{1}{3} \).
(b) \( 3 \times 2 \times 4!/6! = \frac{6}{50} = \frac{1}{8} \).
(c) \( 4! \times 3!/6! = \frac{3 \times 2}{6 \times 3} = \frac{1}{2} \).

4. (a) No, since \( P(A|B) = .6 \neq .2 = P(A) \).
(b) \( P(A \cap B) = P(A|B)P(B) = .8 \times .7 = .56 \).
(c) \( P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .7 - .56 = .74 \).
(d) \( P((A \cup B)^c \cup (A \cap B)) = P((A \cup B)^c) + P(A \cap B) = 1 - P(A \cup B) + P(A \cap B) = 1 - .74 + .56 = .82 \).

5. (a) \( \varphi \) is flat at 2 to the left of 2, jumps down to 0 at 2, increases at 45° from 2 to 7, and then stays flat.
(b) \( P\{Y \leq \frac{1}{2}\} = P\{2 \leq X \leq 2.5\} = \int_{s=2}^{2.5} f_X(s)ds = e^{-2} - 2^{-2.5} \).
(c) \( P\{Y \leq 3\} = P\{X \leq 5\} = \int_{s=-\infty}^{5} f_X(s)ds = 1 - e^{-5} \).

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(d) 
\[
F_Y(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\mathbb{P}\{2 \leq X \leq 2 + t\} & \text{if } 0 \leq t < 2 \\
\mathbb{P}\{X \leq 2 + t\} & \text{if } 2 \leq t < 7 \\
1 & \text{if } t \geq 7
\end{cases} = \begin{cases} 
0 & \text{if } t < 0 \\
e^{-2} - e^{-2-t} & \text{if } 0 \leq t < 2 \\
1 - e^{-2-t} & \text{if } 2 \leq t < 5 \\
1 & \text{if } t \geq 5
\end{cases}
\]

(e) 
\[
\mathbb{P}\{Y = 2\} = \mathbb{P}\{X \leq 2\} + \mathbb{P}\{X = 4\} = 1 - e^{-2}.
\]

6. (a) 
\[
f_{X,Y}(s,t) = \begin{cases} 
t e^{-s-t} & \text{if } s \geq 0 \text{ and } t \geq 0 \\
0 & \text{else}
\end{cases}
\]

(b) 
\[
f_Z(7) = \int_{t'=-\infty}^{\infty} f_X(7 - t') f_Y(t') dt' = \int_{t'=0}^{7} e^{-(7-t')}t' e^{-t'} dt' = e^{-7} \int_{t'=0}^{7} t' dt' = \frac{7^2}{2} e^{-7}.
\]

(c) 
\[
f_Z(t) = 0 \text{ if } t < 0. \text{ If } t > 0,
\]
\[
f_Z(t) = \int_{t'=-\infty}^{\infty} f_X(t - t') f_Y(t') dt' = \int_{t'=0}^{t} e^{-(t-t')}t' e^{-t'} dt' = e^{-t} \int_{t'=0}^{t} t' dt' = \frac{t^2}{2} e^{-t}.
\]
Thus 
\[
f_Z(t) = \begin{cases} 
\frac{t^2}{2} e^{-t} & \text{if } t \geq 0 \\
0 & \text{else}
\end{cases}
\]

7. (a) \(\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 6 + 4 = 10.\)

(b) \(\mathbb{E}[(3 + X)^2] = 9 + 6 \mathbb{E}[X] + \mathbb{E}[X^2] = 9 + 6 \times 2 + 10 = 31.\)

(c) \(\text{Var}(3 + X) = \mathbb{E}[(3 + X)^2] - \mathbb{E}[3 + X]^2 = 31 - (3 + 2)^2 = 31 - 25 = 6.\)

8. Let \(\xi_n = 1\) if the \(n\)-th student is in favor of the law and zero otherwise. Then \(\mathbb{P}\{\xi_n = 1\} = .55.\) By the central limit theorem, 
\[
Z \stackrel{\text{def}}{=} \frac{1}{\sqrt{1000 \times .55 \times .45}} \sum_{n=1}^{1000} (\xi_n - .5)
\]
is approximately Gaussian with mean 0 and variance 1. If \(N\) is a standard Gaussian, we thus have that 
\[
\mathbb{P}\left\{ \sum_{n=1}^{1000} \xi_n \leq 500 \right\} = \mathbb{P}\left\{ \frac{1}{\sqrt{1000 \times .55 \times .45}} \sum_{n=1}^{1000} (\xi_n - .55) \leq \frac{500 - 1000 \times .55}{\sqrt{1000 \times .55 \times .45}} \right\}
\approx \mathbb{P}\{N \leq -3.17\}
\approx \mathbb{P}\{N \geq 3.17\} = 1 - \Phi(3.17) = 1 - .9992 = .0002.
\]
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9. We can write $X = 6Z + 5$, where $Z$ is a standard normal. Thus we want

$$
.4 = P\{X \leq c\} = P\{6Z + 5 \leq c\} = P\left\{ Z \leq \frac{c - 5}{6} \right\} = \Phi\left( \frac{c - 5}{6} \right).
$$

Since $\Phi(x) \geq .5$ when $x \geq 0$, we know that $\frac{1}{6}(c - 5) < 0$. We thus calculate that

$$
.4 = P\left\{ Z \leq \frac{c - 5}{6} \right\} = P\left\{ Z \geq -\frac{c - 5}{6} \right\} = 1 - \Phi\left( \frac{5 - c}{6} \right)
$$

Hence we want that

$$
\Phi\left( \frac{5 - c}{6} \right) = 1 - .4 = .6.
$$

Hence

$$
\frac{5 - c}{6} = .26
$$

so $c = 4.844$. 