1. **20 points** This is question 7 on page 16 of the book.
   
   (a) **5 points** How many ways can we seat 3 boys and 3 girls in a row?
   
   (b) **5 points** How many ways can we seat 3 boys and 3 girls in a row if the boys and girls are each to sit together?
   
   (c) **5 points** How many ways can we seat 3 boys and 3 girls in a row if only the boys must sit together?
   
   (d) **5 points** How many ways can we seat 3 boys and 3 girls in a row if no two people of the same gender are allowed to sit together?

2. **15 points** At a certain college, 54% of the students are female. We also know that 10% of the students are majoring in computer science, and we know that 7% of the students are women majoring in computer science.
   
   (a) **10 points** Randomly select a computer science student. What is the probability that it is a woman?
   
   (b) **5 points** In this data, is gender independent of the decision to study computer science?

3. **12 points** Let $X$ be a random variable with probability mass function

   $$p(j) = \begin{cases} 
   \frac{2}{9} & \text{if } j = -1 \\
   \frac{1}{9} & \text{if } j = 0 \\
   \frac{2}{9} & \text{if } j = 3 \\
   \frac{4}{9} & \text{if } j = 7 \\
   0 & \text{else} 
   \end{cases}$$

   (a) **6 points** Compute $E[X]$.
   
   (b) **6 points** Compute $E[\min\{X, 1\}]$.

4. **40 points** We will here develop some ideas relating to the negative hypergeometric distribution. This is similar in some ways to a negative binomial, and you might keep those calculations in mind.

   Suppose that we have a box containing 20 black balls $\{B_1, B_2 \ldots B_{20}\}$ and 30 red balls $\{R_1, R_2 \ldots R_{30}\}$. Let’s pick balls out of the box, one by one.

   Let $X_7$ be the first time that we have 7 black balls.
(a) 10 points Write a specific configuration where \( X_7 = 13 \)

(b) 10 points Compute the probability of this configuration.

(c) 10 points Compute the probability that \( \mathbb{P}\{X_7 = 13\} \).

(d) 10 points What is the probability mass function of \( X_7 \)?

5. 66 points Let \( U \) be a continuous random variable which is uniformly distributed on \((0, 1)\); i.e., it has density

\[
f_U(t) = \begin{cases} 1 & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases}
\]

Define \( \varphi_1(u) \overset{\text{def}}{=} u^3 \) for \( u \in (0, 1) \), and define \( X_1 \overset{\text{def}}{=} \varphi_1(U) \).

(a) 5 points Graph \( \varphi_1 \)

(b) 7 points Compute \( \mathbb{P}\{X_1 \leq \frac{2}{3}\} \).

(c) 5 points Compute the cumulative of \( X_1 \).

(d) 4 points Does \( X_1 \) have a density? If so, compute it. If not, explain why.

Define \( \varphi_2(u) \overset{\text{def}}{=} u^{-3} \) for \( u \in (0, 1) \), and define \( X_2 \overset{\text{def}}{=} \varphi_2(U) \).

(e) 5 points Graph \( \varphi_2 \)

(f) 7 points Compute \( \mathbb{P}\{X_2 \leq 5\} \).

(g) 5 points Compute the cumulative of \( X_2 \).

(h) 4 points Does \( X_2 \) have a density? If so, compute it. If not, explain why.

Define \( \varphi_3(u) \overset{\text{def}}{=} -\frac{1}{3} \ln u \) for \( u \in (0, 1) \), and define \( X_3 \overset{\text{def}}{=} \varphi_3(U) \).

(i) 5 points Graph \( \varphi_3 \)

(j) 7 points Compute \( \mathbb{P}\{X_3 \leq 7\} \).

(k) 5 points Compute the cumulative of \( X_3 \).

(l) 4 points Does \( X_3 \) have a density? If so, compute it. If not, explain why.

(m) 3 points \( X_3 \) is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

6. 47 points Fix two parameters \( p \in (0, 1) \) and \( \lambda > 0 \). Consider two discrete random variables \( X \) and \( Y \) with joint probability mass function

\[
p_{X,Y}(i,j) = \begin{cases} \binom{j}{i}p^i(1-p)^{j-i}e^{-\lambda \frac{j}{j!}} & \text{if } j \in \{0, 1 \ldots \} \text{ and } i \in \{0, 1 \ldots j\} \\ 0 & \text{else} \end{cases}
\]
(as usual, we define \( \binom{n}{0} \overset{\rm def}{=} 1 \)). In other words,
\[
\Pr\{X = 2, Y = 7\} = \binom{7}{2} p^2 (1 - p)^5 e^{-\lambda} \frac{\lambda^7}{7!}.
\]

This is some combination of binomial and Poisson distributions, of course.

(a) 10 points Compute \( p_X(5) \overset{\rm def}{=} \Pr\{X = 5\} \). Hint: recall that \( \binom{j}{i} = \frac{j!}{i!(j-i)!} \). Recall also the Taylor formula for the exponential.

(b) 5 points Compute the probability mass function of \( X \).

(c) 3 points \( X \) is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

(d) 10 points Compute \( p_Y(7) \overset{\rm def}{=} \Pr\{Y = 7\} \) Hint: recall the binomial theorem.

(e) 5 points Compute the probability mass function of \( Y \).

(f) 3 points \( Y \) is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

(g) 4 points Are \( X \) and \( Y \) independent?

(h) 7 points Compute \( \Pr\{X = 5|Y = 7\} \).
Answers

1. (a) 6!
   (b) $2 \times 3! \times 3!$ (the 2 comes from which gender is left-most)
   (c) $4! \times 3!$ (group the boys as one unit, and then order within the unit)
   (d) $2 \times 3! \times 3!$ (the two comes from which gender is left-most).

2. $W = \{\text{woman}\}$ and $C = \{\text{computer science}\}$.
   (a) 
   $P(W|C) = \frac{P(W \cap C)}{P(C)} = \frac{7}{10} = .7$.
   (b) No; $P(W|C) = .7 \neq .54 = P(W)$.

3. (a) 
   $E[X] = \frac{(-1) \times 2 + 0 \times 1 + 3 \times 2 + 7 \times 4}{9} = \frac{32}{9}$.
   (b) 
   $E[\min\{X, 1\}] = \sum_{j} \min\{j, 1\}p(j)$
   $= \min\{-1, 1\} \frac{2}{9} + \min\{0, 1\} \frac{1}{9} + \min\{3, 1\} \frac{2}{9} + \min\{7, 1\} \frac{4}{9}$
   $= (-1) \frac{2}{9} + (0) \frac{1}{9} + (1) \frac{2}{9} + (1) \frac{4}{9} = \frac{-2 + 0 + 2 + 4}{9} = \frac{4}{9}$.

4. (a) $(B_4, B_2, R_9, R_{11}, R_3, B_3, R_7, B_{20}, B_{15}, R_{30}, B_8, R_1, B_1)$
   (b) 
   $\frac{1}{(50)_{13}}$.
   (c) 
   $\binom{12}{6} \frac{(20)_{7}(30)_{6}}{(50)_{13}}$.
   (d) 
   $p_{X_7}(j) = \begin{cases} \frac{(20)_{7}(30)_{j-7}}{(50)_{j-1} \cdot 6} & \text{if } j \in \{7, 8, \ldots, 37\} \\ 0 & \text{else} \end{cases}$

5. (a) increasing
   (b) 
   $\mathbb{P}\left\{X_1 \leq \frac{2}{3}\right\} = \mathbb{P}\left\{U \leq \left(\frac{2}{3}\right)^{1/3}\right\} = \left(\frac{2}{3}\right)^{1/3}$.

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(c) \[
F_{X_1}(t) = \mathbb{P}\{U^3 \leq t\} = \mathbb{P}\{U \leq t^{1/3}\} = \begin{cases} 
0 & \text{if } t < 0 \\
t^{1/3} & \text{if } 0 \leq t < 1 \\
1 & \text{if } t \geq 1 
\end{cases}
\]

(d) $X_1$ is continuous and has density

\[
f_{X_1}(t) = \begin{cases} 
\frac{1}{3}t^{2/3} & \text{if } t \in (0, 1) \\
0 & \text{else}
\end{cases}
\]

(e) decreasing

(f) \[
\mathbb{P}\{X_2 \leq 5\} = \mathbb{P}\left\{\frac{1}{U^3} \leq 5\right\} = \mathbb{P}\left\{U \geq \frac{1}{5^{1/3}}\right\} = 1 - \frac{1}{5^{1/3}}.
\]

(g) Since $U$ takes values in $(0, 1)$, $X_2$ takes values in $(1, \infty)$. Thus $F_{X_2}(t) = 0$ for $t \leq 1$. For $t > 1$, we have that $F_{X_2}(t) = \mathbb{P}\{\frac{1}{U^3} \leq t\} = \mathbb{P}\{U \geq t^{-1/3}\} = 1 - t^{-1/3}$. Thus

\[
F_{X_2}(t) = \begin{cases} 
0 & \text{if } t < 1 \\
1 - t^{-1/3} & \text{if } t \geq 1 
\end{cases}
\]

(h) $X_2$ is continuous and has density

\[
f_{X_2}(t) = \begin{cases} 
\frac{1}{3}t^{-4/3} & \text{if } t > 1 \\
0 & \text{else}
\end{cases}
\]

(i) decreasing

(j) \[
\mathbb{P}\{X_3 \leq 7\} = \mathbb{P}\{\ln U \geq -35\} = \mathbb{P}\{U \geq e^{-35}\} = 1 - e^{-35}.
\]

(k) Since $U$ takes values in $(0, 1)$, $X_3$ takes values in $(0, \infty)$. Thus $F_{X_3}(t) = 0$ for $t \leq 0$. For $t > 0$, we have that $F_{X_3}(t) = \mathbb{P}\{\ln U \geq -5t\} = \mathbb{P}\{U \geq 1 - e^{-5t}\} = 1 - e^{-5t}$. Thus

\[
F_{X_3}(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{e^{-5t}}{1 - e^{-5t}} & \text{if } t \geq 0 
\end{cases}
\]

(l) $X_3$ is continuous and has density

\[
f_{X_3}(t) = \begin{cases} 
5e^{-5t} & \text{if } t > 0 \\
0 & \text{else}
\end{cases}
\]
6. (a)

\[ 
\mathbb{P}\{X = 5\} = \sum_j \mathbb{P}\{X = 5, Y = j\} = \sum_{j=5}^{\infty} \binom{j}{5} p^j (1-p)^{5-j} e^{-\lambda} \frac{\lambda^j}{j!} 
\]

\[ 
= \frac{e^{-\lambda} p^5}{5!} \sum_{j=5}^{\infty} \frac{(1-p)^{5-j} \lambda^j}{(5-j)!} = \frac{e^{-\lambda} \lambda^5 (1-p)^5}{5!} \sum_{j=5}^{\infty} \frac{(\lambda-1)^{5-j}}{(5-j)!} 
\]

\[ 
= \frac{e^{-\lambda} \lambda^5}{5!} \sum_{j=0}^{\infty} \frac{(\lambda-1)^{j}}{j!} = \exp \left[ -\lambda + \lambda (1-p) \right] \frac{\lambda^5}{5!} 
\]

\[ 
= \exp \left[ -\lambda p \right] \frac{\lambda^5}{5!}. 
\]

(b)

\[ 
p_X(i) = \begin{cases} 
  e^{-\lambda} \frac{(\lambda p)^i}{i!} & \text{if } i \in \{0, \ldots \} \\
  0 & \text{else} 
\end{cases} 
\]

(c) \(X\) is Poisson.

(d)

\[ 
\mathbb{P}\{Y = 7\} = \sum_i \mathbb{P}\{X = i, Y = 7\} = \sum_{i=0}^{7} \binom{7}{i} p^i (1-p)^{7-i} e^{-\lambda} \frac{\lambda^7}{7!} = e^{-\lambda} \frac{\lambda^7}{7!}. 
\]

(e)

\[ 
p_Y(j) = \begin{cases} 
  e^{-\lambda} \frac{\lambda^j}{j!} & \text{if } j \in \{0, \ldots \} \\
  0 & \text{else} 
\end{cases} 
\]

(f) \(Y\) is Poisson

(g) They are not independent.

(h)

\[ 
\mathbb{P}\{X = 5|Y = 7\} = \frac{\mathbb{P}\{X = 5, Y = 7\}}{\mathbb{P}\{Y = 7\}} = \frac{p_{X,Y}(5,7)}{p_Y(7)} = \frac{\binom{5}{2} p^5 (1-p)^2 e^{-\lambda} \frac{\lambda^7}{7!}}{e^{-\lambda} \frac{\lambda^7}{7!}} = \binom{7}{5} p^5 (1-p)^2. 
\]