1. **34 points** Suppose that we have 20 people in a math department, which includes Professor Integrate and Professor Differentiate. We need a four-person committee to govern the department. One of the committee members will be the chair.

   (a) **7 points** How many ways are there to form the committee?

   (b) **7 points** How many ways are there to form the committee and select the chair?

   (c) **10 points** What is the probability that Professor Integrate will be on the committee?

   (d) **10 points** What is the probability that Professor Differentiate will be the chair of the committee and Professor Integrate will not be on the committee?

2. **40 points** This is inspired by Question 74 on page 121. Suppose that Fred and Jane roll a pair dice, with Fred starting first. They stop when either Fred rolls a 9 or Jane rolls a 4.

   (a) **5 points** What is the probability of the sum of two dice being 9?

   (b) **5 points** What is the probability of the sum of two dice being 4?

   (c) **10 points** What is the probability that Jane wins on her third toss?

   (d) **10 points** What is the probability that Jane wins on her n-th toss?

   (e) **10 points** What is the probability that Jane wins? Be as explicit as possible without getting into messy calculations.

3. **18 points** (Poisson approximation) This is inspired by question 51 on page 194. The expected number of typographical errors on a page is 0.01.

   (a) **9 points** What is the probability that the first page has 2 or less typographical errors?

   (b) **9 points** Assume that the first chapter has 10 pages. What is the probability that there are no typographical errors in the first chapter?

4. **58 points** Suppose that $X$ and $Y$ are continuous random variables with joint density function

$$f_{X,Y}(s, t) = \begin{cases} e^{-t} & \text{if } 0 \leq s \leq t \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$.

   (a) **2 points** Graph the region where $f_{X,Y}$ is positive

   (b) **10 points** Compute $f_X$.

   (c) **10 points** Compute $f_Y$.

   (d) **3 points** Are $X$ and $Y$ independent?
(e) 10 points Compute $f_{X|Y}(s|t)$ for $t > 0$.

(f) 3 points Verbally describe the distribution of $X$ if we know that $Y = 3$.

(g) 10 points Compute $f_{Y|X}(t|s)$ for $s > 0$.

(h) 10 points Compute $f_Z$ (be aware of your answer to part d).
1. (a) \( \binom{20}{4} \).
   (b) \( \binom{20}{4} \times 4 = 20 \times \binom{19}{3} \).
   (c) 
   \[ P\{\text{Integrate is on committee}\} = \frac{1 \times \binom{19}{3}}{\binom{20}{3}} = \frac{4}{20} = \frac{1}{5}. \]
   
   \[ P\{\text{Differentiate is chair and Integrate is not on committee}\} = \frac{1 \times \binom{13}{3}}{20 \times \binom{19}{3}} = \frac{1}{20 \binom{19}{3}} . \]

2. (a) 
   \[ P\{9\} = P\{(3, 6), (4, 5), (5, 4), (6, 3)\} = \frac{4}{36}. \]
   (b) 
   \[ P\{4\} = P\{(1, 3), (2, 2), (3, 1)\} = \frac{3}{36}. \]
   (c) 
   \[ P\{\text{Jane wins on her third toss}\} = \left(\frac{32}{36}\right)^3 \left(\frac{33}{36}\right)^2 \frac{3}{36}. \]
   (d) 
   \[ P\{\text{Jane wins on her n-th toss}\} = \left(\frac{32}{36}\right)^{n-1} \left(\frac{33}{36}\right)^n \frac{3}{36}. \]
   (e) 
   \[ P\{\text{Jane wins}\} = \sum_{n=1}^{\infty} \left(\frac{32}{36}\right)^{n-1} \left(\frac{33}{36}\right)^n \frac{3}{36} \sum_{n=1}^{\infty} \left(\frac{32}{36} \times \frac{33}{36}\right)^{n-1} \]
   \[ = \frac{33}{36} \times \frac{3}{36} \sum_{n=0}^{\infty} \left(\frac{32}{36} \times \frac{33}{36}\right)^n \]
   \[ = \frac{33 \times 3}{36^2 - 32 \times 33}. \]

3. If \( X \) is the number of errors (on one page), the Poisson approximation is that
   \[ P\{X = k\} = \begin{cases} e^{-0.01} \frac{(0.01)^k}{k!} & \text{for } k \in \{0, 1, \ldots\} \\ 0 & \text{else} \end{cases} \]
\[
P\{X \leq 2\} = 1 - P\{X = 0\} + P\{X = 1\} + P\{X = 2\} = e^{-0.01} \left\{ 1 + \frac{(0.01)}{1} + \frac{(0.01)^2}{2!} \right\} = 1.01005 e^{-0.01}.
\]

(b) For one page, \( P\{X = 0\} = e^{-0.01} \). Thus
\[
P\{\text{no errors in first chapter}\} = (e^{-0.01})^{10} = e^{-0.1}.
\]

(a) First quadrant, above the line of 45 degrees.

(b)
\[
f_X(s) = \int_{t=\infty}^\infty f_{X,Y}(s,t)dt = \begin{cases} \int_{t=s}^\infty e^{-t}dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}
\]

c)
\[
f_Y(t) = \int_{s=\infty}^\infty f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^t e^{-t}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} te^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}
\]

(d) No, since \( f_{X,Y}(s,t) \neq f_X(s)f_Y(t) \).

(e) For \( t > 0 \),
\[
f_{X|Y}(s|t) = \frac{f_{X,Y}(s,t)}{f_Y(t)} = \begin{cases} \frac{e^{-t}}{te^{-t}} & \text{if } 0 \leq s \leq t \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{t} & \text{if } 0 \leq s \leq t \\ 0 & \text{else} \end{cases}
\]

(f) Uniform on \((0,3)\)

(g) For \( s > 0 \),
\[
f_{Y|X}(t|s) = \frac{f_{X,Y}(s,t)}{f_X(s)} = \begin{cases} \frac{e^{-t}}{e^{-s}} & \text{if } 0 \leq s \leq t \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-(t-s)} & \text{if } 0 \leq s \leq t \\ 0 & \text{else} \end{cases}
\]

(h)
\[
f_Z(t) = \int_{s=\infty}^\infty f_{X,Y}(s,t-s)ds = \int_{s=\infty}^\infty e^{-(t-s)}\chi_{[0,t-s]}(s)ds
\]
\[
= \begin{cases} \int_{s=0}^{t/2} e^{-(t-s)}ds & \text{if } t > 0 \\ \int_{u=t/2}^t e^{-u}du & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-t/2} - e^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}
\]

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