Math 361, Section D13 and D14, Fall 2007
Final, December 12

1. **34 points** Suppose that we have 20 people in a math department, which includes Professor Integrate and Professor Differentiate. We need a four-person committee to govern the department. One of the committee members will be the chair.

   (a) **7 points** How many ways are there to form the committee?

   (b) **7 points** How many ways are there to form the committee and select the chair?

   (c) **10 points** What is the probability that Professor Integrate will be on the committee?

   (d) **10 points** What is the probability that Professor Differentiate will be the chair of the committee and Professor Integrate will not be on the committee?

2. **40 points** This is inspired by Question 74 on page 121. Suppose that Fred and Jane roll a pair dice, with Fred starting first. They stop when either Fred rolls a 9 or Jane rolls a 4.

   (a) **5 points** What is the probability of the sum of two dice being 9?

   (b) **5 points** What is the probability of the sum of two dice being 4?

   (c) **10 points** What is the probability that Jane wins on *her* third toss?

   (d) **10 points** What is the probability that Jane wins on *her* n-th toss?

   (e) **10 points** What is the probability that Jane wins? Be as explicit as possible without getting into messy calculations.

3. **18 points** (Poisson approximation) This is inspired by question 51 on page 194. The expected number of typographical errors on a page is 0.01.

   (a) **9 points** What is the probability that the first page has 2 or less typographical errors?

   (b) **9 points** Assume that the first chapter has 10 pages. What is the probability that there are no typographical errors in the first chapter?

4. **58 points** Suppose that X and Y are continuous random variables with joint density function

$$f_{X,Y}(s,t) = \begin{cases} e^{-t} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

   Define $Z \overset{\text{def}}{=} X + Y$.

   (a) **2 points** Graph the region where $f_{X,Y}$ is positive

   (b) **10 points** Compute f_X.

   (c) **10 points** Compute f_Y.

   (d) **3 points** Are X and Y independent?

(e) 10 points Compute $f_{X|Y}(s|t)$ for $t > 0$.

(f) 3 points Verbally describe the distribution of X if we know that $Y = 3$.

(g) 10 points Compute $f_{Y|X}(t|s)$ for $s > 0$.

(h) 10 points Compute f_Z (be aware of your answer to part d).

Answers

1. (a) $\binom{20}{4}$.

   (b) $\binom{20}{4} \times 4 = 20 \times \binom{19}{3}$.

   (c)

$$\mathbb{P}\{\text{Integrate is on committee}\} = \frac{1 \times \binom{19}{3}}{\binom{20}{3}} = \frac{4}{20} = \frac{1}{5}.$$

$$\begin{aligned}\mathbb{P}\{\text{Differentiate is chair and Integrate is not on committee}\} &= \frac{1 \times \binom{13}{3}}{20 \times \binom{19}{3}} \\ &= \frac{1}{20}\frac{(18)_3}{(19)_3}.\end{aligned}$$

2. (a)

$$\mathbb{P}\{9\} = \mathbb{P}\{(3,6),(4,5),(5,4),(6,3)\} = \frac{4}{36}.$$

   (b)

$$\mathbb{P}\{4\} = \mathbb{P}\{(1,3),(2,2),(3,1)\} = \frac{3}{36}.$$

   (c)

$$\mathbb{P}\{\text{Jane wins on her third toss}\} = \left(\frac{32}{36}\right)^3 \left(\frac{33}{36}\right)^2 \frac{3}{36}.$$

   (d)

$$\mathbb{P}\{\text{Jane wins on her n-th toss}\} = \left(\frac{32}{36}\right)^{n-1} \left(\frac{33}{36}\right)^n \frac{3}{36}.$$

   (e)

$$\begin{aligned}\mathbb{P}\{\text{Jane wins}\} &= \sum_{n=1}^{\infty} \left(\frac{32}{36}\right)^{n-1} \left(\frac{33}{36}\right)^n \frac{3}{36}\frac{33}{36} \times \frac{3}{36}\sum_{n=1}^{\infty}\left(\frac{32}{36} \times \frac{33}{36}\right)^{n-1} \\ &= \frac{33}{36} \times \frac{3}{36}\sum_{n=0}^{\infty}\left(\frac{32}{36} \times \frac{33}{36}\right)^n = \frac{\frac{33}{36} \times \frac{3}{36}}{1 - \frac{32}{36} \times \frac{33}{36}} \\ &= \frac{33 \times 3}{36^2 - 32 \times 33}\end{aligned}$$

3. If X is the number of errors (on one page), the Poisson approximation is that

$$\mathbb{P}\{X = k\} = \begin{cases} e^{-0.01}\frac{(0.01)^k}{k!} & \text{for } k \in \{0,1\dots\} \\ 0 & \text{else} \end{cases}$$

(a)

$$\begin{aligned}\mathbb{P}\{X \le 2\} &= 1 - \mathbb{P}\{X = 0\} + \mathbb{P}\{X = 1\} + \mathbb{P}\{X = 2\} \\ &= e^{-0.01}\left\{1 + \frac{(0.01)}{1} + \frac{(0.01)^2}{2!}\right\} = 1.01005e^{-0.01}.\end{aligned}$$

(b) For one page, $\mathbb{P}\{X = 0\} = e^{-0.01}$. Thus

$$\mathbb{P}\{\text{no errors in first chapter}\} = (e^{-0.01})^{10} = e^{-0.1}.$$

(a) First quadrant, above the line of 45 degrees.

(b)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=s}^{\infty} e^{-t}dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

(c)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^{t} e^{-t}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} te^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

(d) No, since $f_{X,Y}(s,t) \ne f_X(s)f_Y(t)$.

(e) For $t > 0$,

$$f_{X|Y}(s|t) = \frac{f_{X,Y}(s,t)}{f_Y(t)} = \begin{cases} \frac{e^{-t}}{te^{-t}} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{t} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

(f) Uniform on $(0,3)$

(g) For $s > 0$,

$$f_{Y|X}(t|s) = \frac{f_{X,Y}(s,t)}{f_X(s)} = \begin{cases} \frac{e^{-t}}{e^{-s}} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-(t-s)} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

(h)

$$\begin{aligned}f_Z(t) &= \int_{s=-\infty}^{\infty} f_{X,Y}(s,t-s)ds = \int_{s=-\infty}^{\infty} e^{-(t-s)}\chi_{[0,t-s]}(s)ds \\ &= \begin{cases} \int_{s=0}^{t/2} e^{-(t-s)}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} \int_{u=t/2}^{t} e^{-u}du & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-t/2} - e^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}\end{aligned}$$