Math 361, Section D13 and D14, Fall 2007
Exam 3, November 12

1. [50 points] Suppose that X is a continuous uniform$(0,1)$ random variable; i.e., it has density

$$f_X(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Define $Y \overset{\text{def}}{=} X^3$.

   (a) [10 points] Compute $\mathbb{P}\left\{Y \le \frac{1}{5}\right\}$.

   (b) [15 points] Compute the cumulative distribution F_Y of Y

   (c) [15 points] Compute the density f_Y of Y, if it exists. If the density does not exist, explain why.

   (d) [10 points] Compute $\mathbb{E}[Y]$.

2. [20 points] This is essentially question 21 on page 249 of the book. We will model the weight (in pounds) of 25-year old men as normal with mean $\mu = 170$ and variance $\sigma^2 = 25$. You should use the attached table of values of Gaussian integrals. Go as far as possible without doing messy math by hand.

   (a) [10 points] What percentage of 25-year old men weigh more than 165 pounds?

   (b) [10 points] What percentage of 25-year old men over 165 pounds are less than 180 pounds?

3. [30 points] This is essentially question 7 on page 248 of the book. The density of a continuous random variable X is given by

$$f_X(t) = \begin{cases} a + bx^3 & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

If $\mathbb{E}[X] = \frac{3}{5}$, compute a and b.

Answers

1. (a)

$$\mathbb{P}\left\{Y \le \tfrac{1}{5}\right\} = \mathbb{P}\left\{X^3 \le \tfrac{1}{5}\right\} = \mathbb{P}\left\{X \le \left(\tfrac{1}{5}\right)^{1/3}\right\} = \int_{t=-\infty}^{(1/5)^{1/3}} f_Y(t)dt = (1/5)^{1/3}.$$

(b)

$$F_Y(t) = \mathbb{P}\{Y \le t\} = \mathbb{P}\left\{X^3 \le t\right\} = \mathbb{P}\left\{X \le t^{1/3}\right\} = \int_{s=-\infty}^{t^{1/3}} f_Y(s)ds$$
$$= \begin{cases} 0 & \text{if } t^{1/3} < 0 \\ t^{1/3} & \text{if } 0 \le t^{1/3} < 1 \\ 1 & \text{if } t^{1/3} \ge 1 \end{cases} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(c)

$$f_Y(t) = F_Y'(t) = \begin{cases} \frac{1}{3}t^{-2/3} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

(d)

$$\mathbb{E}[Y] = \mathbb{E}[X^3] = \int_{t=-\infty}^{\infty} t^3 f_X(t)dt = \int_{t=0}^{1} t^3 dt = \frac{1}{4}$$

or equivalently

$$\mathbb{E}[Y] = \int_{t=-\infty}^{\infty} t f_Y(t)dt = \int_{t=0}^{1} t\left(\frac{1}{3}t^{-2/3}\right)dt = \frac{1}{3}\int_{t=0}^{1} t^{1/3}dt = \frac{1}{4}$$

2. Let ξ be normal with mean 0 and variance 1.

(a)

$$\mathbb{P}\{X \ge 165\} = \mathbb{P}\{5\xi + 170 \ge 165\} = \mathbb{P}\{\xi \ge -1\} = \mathbb{P}\{-\xi \ge -1\}$$
$$= \mathbb{P}\{\xi \le 1\} = 0.8413$$

(b)

$$\mathbb{P}\{X \le 180 | X \ge 165\} = \frac{\mathbb{P}\{165 \le X \le 180\}}{\mathbb{P}\{X \ge 165\}}$$

We calculate that

$$\mathbb{P}\{165 \le X \le 180\} = \mathbb{P}\{X \le 180\} - \mathbb{P}\{X < 165\}$$

We then compute that

$$\mathbb{P}\{X \le 180\} = \mathbb{P}\{5\xi + 170 \le 180\} = \mathbb{P}\{\xi \le 2\} = 0.9772.$$

We also have that

$$\mathbb{P}\{X < 165\} = 1 - \mathbb{P}\{X \geq 165\} = 1 - 0.8413.$$

Thus the final answer is

$$\frac{0.9772 - 1 + 0.8413}{0.8413} = \frac{0.8185}{0.8413}.$$

3. We need that

$$\int_{t=-\infty}^{\infty} f_X(t)dt = 1 \qquad \text{and} \qquad \int_{t=-\infty}^{\infty} t f_X(t)dt = \frac{3}{5}.$$

In other words,

$$1 = \int_{x=0}^{1} (a + bx^3)dx = a + \frac{b}{4}$$
$$\frac{3}{5} = \int_{x=0}^{1} x(a + bx^3)dx = \frac{a}{2} + \frac{b}{5}.$$

This gives us the two equations

$$4 = 4a + b \qquad \text{and} \qquad 6 = 5a + 2b.$$

The first equation is equivalent to $8 = 8a + 2b$, so we can subtract the second equation from this to get that $2 = 3a$; i.e., $a = \frac{2}{3}$. This then implies that $b = 4 - 4a = \frac{4}{3}$.