Math 361, Section D13 and D14, Fall 2007
Exam 2, October 19

1. [30 points] Suppose that X is a random variable with moment generating function

$$M(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\exp[\theta X]] = \exp\left[5(e^{2\theta}-1)+3\theta\right]. \qquad \theta \in \mathbb{R}$$

   (a) [10 points] Compute $\mathbb{E}[X]$.

   (b) [10 points] Compute $\mathbb{E}[X^2]$.

   (c) [10 points] Compute the variance of X.

2. [20 points] Let X be geometric with parameter p; i.e.,

$$\mathbb{P}\{X=j\} = \begin{cases} (1-p)^{j-1}p & \text{for } j \in \{1,2,\dots\} \\ 0 & \text{else} \end{cases}$$

   Let's also define a new random variable $Y \stackrel{\text{def}}{=} \max\{X, 10\}$.

   (a) [10 points] Compute $\mathbb{E}[\max\{X, 10\}]$. You don't need to compute the final summation.

   (b) [10 points] Compute $\mathbb{P}\{Y = 10\}$. You do need to simplify as much as possible.

3. [20 points] This is close to question 13 on page 188. A salesman has two appointments. Each appointment will independently lead to a sale with probability p. Any sale made is equally likely to be either for \$500 or \$1,000. Let X be the total dollar amount of sales.

   (a) [10 points] Find the probability mass function of X

   (b) [10 points] Compute $\mathbb{E}[X]$.

4. [10 points] Suppose that X is a random variable with mean 20 and variance 5. Find a bounded interval (a, b) such that

$$\mathbb{P}\{X \notin (a,b)\} \le \frac{5}{100}.$$

5. [20 points] This is essentially question 65 on page 195. Each of the 50,000 students in a university has a certain disease with probability 1/100. To prevent an epidemic, blood from all of the students is taken and combined and then the combined of blood is tested.

   (a) [10 points] What is the probability that the combined blood will test positive (i.e., at least one person has the disease)?

   (b) [10 points] The university should shut down if at least 50 people have the disease. Suppose that the combined blood tests positive. What is the (approximate) probability that the university needs to be shut down? You don't need to carry out the final numerical computation.

Answers

1. We first compute that

$$M'(\theta) = (10e^{2\theta} + 3)M(\theta) \qquad \text{and} \qquad M''(\theta) = (20e^{2\theta})M(\theta) + (10e^{2\theta} + 3)^2 M(\theta)$$

for all $\theta \in \mathbb{R}$.

(a) $\mathbb{E}[X] = M'(0) = 13$.

(b) $\mathbb{E}[X^2] = M''(0) = 20 + 13^2 = 189$.

(c) Variance of X is $20 + 13^2 - 13^2 = 20$.

2. (a)

$$\mathbb{E}[\max\{X, 10\}] = \sum_{j=1}^{\infty} \max\{j, 10\}(1-p)^{j-1}p = 10\sum_{j=1}^{10}(1-p)^{j-1}p + \sum_{j=1}^{10} j(1-p)^{j-1}p.$$

(b)

$$\mathbb{P}\{Y = 10\} = \mathbb{P}\{X \le 10\} = \sum_{j=1}^{10}(1-p)^{j-1}p = p\sum_{j=0}^{9}(1-p)^j = p\frac{1-(1-p)^{10}}{1-(1-p)}$$
$$= 1 - (1-p)^{10}.$$

3. Consider a single sale. The probability of making no sale is $1-p$. The probability of making a sale of \$500 is $p/2$. The probability of making a sale of \$1,000 is also $p/2$.

(a)

$$\mathbb{P}\{X = j\} = \begin{cases} (1-p)^2 & \text{for } j = 0 \\ (1-p)p & \text{for } j = 500 \\ p(1-p) + \frac{p^2}{4} & \text{for } j = 1000 \\ \frac{p^2}{2} & \text{for } j = 1500 \\ \frac{p^2}{4} & \text{for } j = 2000 \end{cases}$$

(b)

$$\mathbb{E}[X] = 500p(1-p) + 1000p(1-p) + 250p^2 + 750p^2 + 1000p^2.$$

4. By Chebychev's inequality,

$$\mathbb{P}\{|X - 20| \ge L\} \le \frac{5}{L^2}.$$

We want that $\frac{5}{L} \le \frac{5}{100}$; take $L = 10$. Taking $a = 20 - 10 = 10$ and $b = 20 + 10 = 30$, we have that

$$\mathbb{P}\{X \notin (10, 30)\} = \mathbb{P}\{|X - 20| \ge 10\} \le \frac{5}{100} = 0.05.$$

5. This is a Poisson approximation problem. We let X be the number of people with the disease. The distribution of X will approximately be Poisson with parameter

$$50,000 \times \frac{1}{100} = 500.$$

(a) $\mathbb{P}\{X \geq 1\} = 1 - \mathbb{P}\{X = 0\} = 1 - e^{-500}$.

(b)

$$\mathbb{P}\{X \geq 50 | X \geq 1\} = \frac{\mathbb{P}\{X \geq 50\}}{\mathbb{P}\{X \geq 1\}} = (1 - e^{-500})^{-1} \sum_{j=50}^{\infty} e^{-500} \frac{(500)^j}{j!}.$$

Alternately, since there are only 50,000 students at the university, the sum in the numerator can have upper limit 50,000.