SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. 20 points A pair of dice is rolled until either a 3 or a 6 appears.
   (a) 10 points What is the probability that the 3 appears before the 6 and the first
       three is on the 17-th toss?
   (b) 10 points What is the probability that the 3 appears before the 6?

2. 20 points Four cards are taken from a standard deck of cards. What is the probability
   that they are
   (a) 10 points of different face values.
   (b) 10 points of different suits.

3. 30 points (roughly taken from question 31 on p. 57) Suppose that a 3-person basketball
   team consists of a guard, a center, and a forward. Suppose that there are 5 such teams
   (i.e., team 1, team 2, team 3, team 4, and team 5). Suppose that we randomly pick 3
   players.
   (a) 10 points What is the probability of picking a pre-existing team (i.e., team 1,
       team 2, team 3, team 4, or team 5)?
   (b) 10 points What is the probability of picking a “playable” team (i.e., a guard, a
       center, and a forward)?
   (c) 10 points What is the probability that all three chosen players are guards?

4. 50 points Suppose that $X$ and $Y$ are independent Poisson random variables with pa-
   rameters $\lambda$ and $\nu$; i.e., they are discrete random variables with probability mass functions
   $$p_X(j) = \begin{cases} e^{-\lambda} \frac{\lambda^j}{j!} & \text{if } j \in \{0, 1 \ldots\} \\ 0 & \text{else} \end{cases}$$

   $$p_Y(j) = \begin{cases} e^{-\nu} \frac{\nu^j}{j!} & \text{if } j \in \{0, 1 \ldots\} \\ 0 & \text{else} \end{cases}$$

   and are independent. Define $Z \overset{\text{def}}{=} X + Y$.
   (a) 10 points Compute $P\{Z = 5\}$ (hint: remember the binomial theorem)
   (b) 10 points Compute the probability mass function of $Z$.
   (c) 10 points Compute $E[e^{5X}]$
   (d) 10 points Compute $E[e^{\theta X}]$ for all $\theta \in \mathbb{R}$.
   (e) 10 points Compute $E[X]$. 
5. **30 points** Suppose that $X$ and $Y$ are continuous random variables with joint density

$$f_{X,Y}(s, t) = \begin{cases} 2e^{-2s-t} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

(a) **15 points** Compute $P\{X < Y\}$

(b) **15 points** Compute $P\{X \leq 5\}$. 
Answers

1. Let $p_3 \overset{\text{def}}{=} \frac{2}{20}$ be the probability of throwing a 3 and let $p_6 \overset{\text{def}}{=} \frac{5}{20}$ be the probability of throwing a 6.
   (a) $(1 - p_3 - p_6)^{16} p_3$.
   (b) $\sum_{j=1}^{\infty} (1 - p_3 - p_6)^{j-1} p_3 = \frac{p_3}{p_3 + p_6}$.

2. (a) $\frac{(13)^4}{(5)^2}$.
   (b) $\frac{(13)^4}{(3)^2}$.

3. Define $q \overset{\text{def}}{=} \frac{1}{(\frac{15}{3})}$.
   (a) $5q$.
   (b) $5^3 q$.
   (c) $\left(\frac{5}{3}\right) q$.

4. (a) 
   \[
   \mathbb{P}\{Z = 5\} = \sum_{j=0}^{5} p_X(j)p_Y(5 - j) = e^{-\lambda - \nu} \sum_{j=0}^{5} \frac{\lambda^j}{j!} \frac{\nu^{5-j}}{(5-j)!} 
   = \frac{1}{5!} e^{-\lambda - \nu} \sum_{j=0}^{5} \binom{5}{j} \lambda^j \nu^{5-j} = \frac{(\lambda + \nu)^5}{5!} e^{-\lambda - \nu}.
   \]
   (b) 
   
   \[
   p_X(j) = \begin{cases} 
   e^{-\lambda-\nu} \frac{(\lambda+\nu)^j}{j!} & \text{if } j \in \{0, 1 \ldots\} \\
   0 & \text{else}
   \end{cases}
   \]
   (c) 
   
   \[
   \mathbb{E}[e^{5X}] = \sum_{j=0}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{(\lambda e^5)^j}{j!} = e^{-\lambda} \exp[\lambda e^5] = \exp[\lambda(e^5 - 1)].
   \]
   (d) $\mathbb{E}[e^{\theta X}] = \exp[\lambda(e^{\theta} - 1)]$.
   (e) If $\varphi_X(\theta) \overset{\text{def}}{=} \mathbb{E}[e^{\theta X}] = \exp[\lambda(e^\theta - 1)]$, then $\varphi_X'(\theta) = \lambda e^\theta \exp[\lambda(e^\theta - 1)]$ for all $\theta \in \mathbb{R}$, so $\mathbb{E}[X] = \varphi_X'(0) = \lambda$. 

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5. (a)

\[
P\{X < Y\} = \int_{s=-\infty}^{\infty} \int_{t=s}^{\infty} f_{X,Y}(s,t) \, dt \, ds = \int_{s=0}^{\infty} \int_{t=s}^{\infty} 2e^{-2s-t} \, dt \, ds
\]

\[
= \int_{s=0}^{\infty} 2e^{-2s} \int_{t=s}^{\infty} e^{-t} \, dt \, ds = \int_{s=0}^{\infty} 2e^{-2s-s} \, ds = 2 \int_{s=0}^{\infty} e^{-3s} \, ds = \frac{2}{3}.
\]

(b)

\[
P\{X \leq 5\} = \int_{s=-\infty}^{5} \int_{t=-\infty}^{\infty} f_{X,Y}(s,t) \, dt \, ds = \int_{s=0}^{5} \int_{t=0}^{\infty} 2e^{-2s-t} \, dt \, ds
\]

\[
= 2 \int_{s=0}^{5} e^{-2s} \int_{t=0}^{\infty} e^{-t} \, dt \, ds = 2 \int_{s=0}^{5} e^{-2s} \, ds = 1 - e^{-10}.
\]