SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. **40 points** Suppose that we have a red coin and a blue coin. Suppose that both coins are biased; \( \mathbb{P}\{\text{heads}\} = p \) and \( \mathbb{P}\{\text{tails}\} = q \), where \( p \) and \( q \) are some parameters. Suppose now that we repeatedly flip both coins (at the same time, one in the left hand and one in the right hand) until the coins disagree. Let \( N \) denote the first time that both coins disagree (i.e., if \( N = 2 \), then both coins agree on the first flip and then disagree on the second flip). Let \( X = H \) if the red coin shows heads when the coins first disagree (i.e., on the \( N \)-th flip), and let \( X = T \) if the red coin shows tails when the coins first disagree (i.e., on the \( N \)-th flip).

   (a) **10 points** For a single flip, compute the probability that the coins disagree
   (b) **10 points** Compute \( \mathbb{P}\{N = 3\} \).
   (c) **10 points** Compute \( \mathbb{P}\{X = H \text{ and } N = 3\} \).
   (d) **10 points** Compute \( \mathbb{P}\{X = H\} \). Compute the answer explicitly.

2. **40 points** (roughly taken from problem 11 on page 185) Teams A and B play a series of independent games against each other. The first team to win 4 games is declared the winner of the series. Suppose that in each game, the probability that A wins is \( p \) (and consequently the probability that B wins is \( 1 - p \)).

   (a) **10 points** What is the probability that A wins the series in exactly 7 games?
   (b) **10 points** What is the probability that A wins the series?
   (c) **10 points** Compute the probability that A wins the first game and wins the series.
   (d) **10 points** Compute the probability that A wins the series given that it wins the first game.

3. **10 points** Suppose that a continuous random variable \( X \) has density

\[
   f_X(t) = \begin{cases} 
   2(1 - t) & \text{if } t \in (0, 1) \\
   0 & \text{else}
   \end{cases}
\]

Compute \( \mathbb{E}[X] \).

4. **60 points** Suppose that \( X \) and \( Y \) are independent exponential random variables with parameters 2 and 7; i.e., they are both continuous random variables with densities

\[
   f_X(t) = \begin{cases} 
   2e^{-2t} & \text{if } t \geq 0 \\
   0 & \text{else}
   \end{cases}
\]

\[
   f_Y(t) = \begin{cases} 
   7e^{-7t} & \text{if } t \geq 0 \\
   0 & \text{else}
   \end{cases}
\]
and are independent. Define

\[ Z \overset{\text{def}}{=} \min\{X, 3Y\}. \]

(a) \textbf{10 points} Compute \( P\{X \geq 13\} \).

(b) \textbf{10 points} Compute \( P\{Y \geq t\} \) for all \( t \geq 0 \).

(c) \textbf{10 points} Compute \( P\{Z \geq 4\} \) (hint: you first should clearly understand what it means in terms of \( X \) and \( Y \) when \( Z \geq 4 \)).

(d) \textbf{10 points} Compute \( P\{Z \geq t\} \) for all \( t \geq 0 \).

(e) \textbf{10 points} Compute the cumulative distribution of \( Z \).

(f) \textbf{10 points} Compute \( P\{Z \geq 4 \text{ and } X \leq 20\} \).
1. (a) $\mathbb{P}\{\text{red heads and blue tails}\} + \mathbb{P}\{\text{blue heads and red tails}\} = pq + (1 - p)(1 - q)$.
(b) $\mathbb{P}\{N = 3\} = (1 - pq - (1 - p)(1 - q))^2(pq + (1 - p)(1 - q))$.
(c) $\mathbb{P}\{X = H \text{ and } N = 3\} = (1 - pq - (1 - p)(1 - q))^2pq$.
(d) 
$$
\mathbb{P}\{X = H\} = \sum_{j=1}^{\infty} \mathbb{P}\{X = H \text{ and } N = j\}
= \sum_{j=1}^{\infty} (1 - pq - (1 - p)(1 - q))^{j-1}pq = \frac{pq}{pq + (1 - p)(1 - q)}.
$$

2. (a) The 7th game must be won by A. In the preceding 6 games, A must win 3 of them (and B must win 3). The answer is $\binom{6}{3}p^4(1 - p)^3$.
(b) If A wins, it must do so in at least 4 games and in at most 7 games. The answer is 
$$
\sum_{j=4}^{7} \binom{j}{2}p^4(1 - p)^{j-4}.
$$
(c) $\sum_{j=4}^{7} \binom{j}{2}p^4(1 - p)^{j-4}$.
(d) $\sum_{j=4}^{7} \binom{j}{2}p^4(1 - p)^{j-5}$.

3. 
$$
\mathbb{E}[X] = \int_{t=-\infty}^{\infty} tf_X(t)dt = 2 \int_{0}^{1} t(1 - t)dt
= 2 \int_{0}^{1} \{t - t^2\}dt = 2 \left(\frac{t^2}{2} - \frac{t^3}{3}\right) \bigg|_{0}^{1} = 2 \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.
$$

4. (a) 
$$
\mathbb{P}\{X \geq 13\} = \int_{13}^{\infty} f_X(t)dt = \int_{13}^{\infty} 2e^{-2t}dt = e^{-26}.
$$
(b) For $t \geq 0$,
$$
\mathbb{P}\{Y \geq t\} = e^{-7t}.
$$
(c) 
$$
\mathbb{P}\{Z \geq 4\} = \mathbb{P}\{X \geq 4 \text{ and } Y \geq 4/3\} = \mathbb{P}\{X \geq 4\} \mathbb{P}\{Y \geq 4/3\} = e^{-8}e^{-28/3}.
$$
(d) For $t \geq 0$,
$$
\mathbb{P}\{Z \geq t\} = \mathbb{P}\{X \geq t\} \mathbb{P}\{Y \geq t/3\} = e^{-2t-7t/3}.
$$
(e) 

\[ F_Z(t) = \begin{cases} 
1 - \exp[-(2 + 7/3)t] & \text{if } t \geq 0 \\
0 & \text{else}
\end{cases} \]

(f) 

\[ \mathbb{P}\{Z \geq 4 \text{ and } X < 20\} = \mathbb{P}\{4 \leq X < 20\} \mathbb{P}\{Y \geq 4/3\} \]
\[ = (\mathbb{P}\{X \geq 4\} - \mathbb{P}\{X \geq 20\}) \mathbb{P}\{Y \geq 4/3\} = (e^{-8} - e^{10}) e^{-28/3}. \]