SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. **25 points** Suppose that a random variable $X$ has moment generating function

$$\varphi_X(\theta) = \mathbb{E} [e^{\theta X}] = \begin{cases} \frac{7}{\theta - 7} & \text{if } \theta < 7 \\ \infty & \text{if } \theta \geq 7 \end{cases}$$

(a) **10 points** Compute $\mathbb{E}[X]$.

(b) **10 points** Compute $\mathbb{E}[X^2]$.

(c) **5 points** Compute the variance of $X$.

2. **30 points** Suppose that we toss a sequence of biased coins ($\mathbb{P}\{H\} = p$). Let $X$ be the position of the first heads. Compute

(a) **10 points** $\mathbb{P}\{\text{the second heads appears on the 10th toss}\}$.

(b) **10 points** $\mathbb{P}\{X = 7 \text{ and the second heads appears on the 10th toss}\}$.

(c) **10 points** $\mathbb{P}\{X = 7 | \text{the second heads appears on the 10th toss}\}$.

3. **10 points** (This is essentially Question 5 on p. 228) Suppose that the demand for gasoline at a certain gas station is a continuous random variable with density

$$f_X(t) = \begin{cases} 2(1 - t) & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases}$$

Suppose that the owner wants to buy a new gasoline tank for the station. Find the capacity $C$ of the tank so that the gas station will be sold out with probability 0.01.

4. **20 points** (This is essentially question 20 in Chapter 4) Consider a roulette strategy. A roulette wheel can come up either red (R) or black (B). We have that

$$\mathbb{P}\{R\} = \frac{18}{38} \quad \text{and} \quad \mathbb{P}\{B\} = \frac{20}{38}.$$ 

On each game, we can bet $1 on red. If it comes up red, we get our original dollar back and get one more dollar (winnings of $1). If it comes up black, we lose our original dollar (winnings of $-1). Consider the following strategy. Bet on red. If it comes up red, we quit. If it comes up black, we bet on red on the next two games (and then quit). Let $X$ be our total winnings.

(a) **10 points** Compute $\mathbb{P}\{X = 1\}$.

(b) **10 points** Compute $\mathbb{E}[X]$ (do not do the final computation).
5. **15 points** Let $X$ be a geometric random variable with parameter $p$; i.e., it has probability mass function

$$p_X(j) = \begin{cases} (1 - p)^j p & \text{if } j \in \{0, 1, \ldots\} \\ 0 & \text{else} \end{cases}$$

(hint: you may want to remember that $\sum_{j=0}^{\infty} \alpha = (1 - \alpha)^{-1}$ if $|\alpha| < 1$).

(a) **10 points** Compute $\mathbb{P}\{X \geq 7\}$.

(b) **5 points** Compute $\mathbb{P}\{X \geq 7 | X \geq 3\}$. 

R. Sowers
1. Note that if $\theta < 2$,

$$
\phi_X(\theta) = \frac{7}{(7-\theta)^2} \quad \text{and} \quad \phi_X(t) = 2 \frac{7}{(7-\theta)^3}
$$

(a) $E[X] = \phi_X(0) = \frac{7}{7} = \frac{1}{7}$.
(b) Compute $E[X^2] = 2 \frac{7}{7^2} = \frac{2}{49}$.
(c) $\text{Var}(X) = E[X^2] - E[X]^2 = \frac{2}{49} - \frac{1}{49} = \frac{1}{49}$.

2. (a) $9p^2(1-p)^8$.
(b) $p^2(1-p)^8$.
(c) 

$$
\frac{p^2(1-p)^8}{9p^2(1-p)^8} = \frac{1}{9}.
$$

3. We want $C \in (0,1)$ such that

$$
0.01 = \int_C f_X(t) dt = \int_C 2(1-t) dt = (1-C)^2
$$

so $C = 0.9$.

4. (a) $P\{X = 1\} = \frac{18}{38} + \frac{20}{38} \left(\frac{18}{38}\right)^2$.
(b) 

$$
E[X] = (1)\frac{18}{38} + (-1) \cdot 2 \cdot \frac{18}{38} \left(\frac{20}{38}\right)^2 + (-3) \left(\frac{20}{38}\right)^3 + (1)\frac{20}{38} \left(\frac{18}{38}\right)^2.
$$

5. (a) $P\{X \geq 7\} = \sum_{j=7}^{\infty} (1-p)^j p = (1-p)^7$.
(b) 

$$
P\{X \geq 7|X \geq 3\} = \frac{(1-p)^7}{(1-p)^3} = (1-p)^4.
$$