1. **5 points** Suppose that $X$ and $Y$ are random variables with joint characteristic function

$$E[\exp[i\theta X + i\psi Y]] = \exp \left[ 2i\theta + 5i\psi - 5\theta^2/2 - 9\psi^2/2 - 3\psi \theta \right].$$

(a) What is the mean of $X$?
(b) What is the variance of $X$?
(c) What is the covariance $\sigma_{X,Y}$ between $X$ and $Y$?
1. Set $\varphi_{X,Y}(\theta, \psi) \overset{\text{def}}{=} \mathbb{E}[\exp[i\theta X + i\psi Y]]$. Could just use fact that $(X, Y)$ must be jointly Gaussian.

(a) Note that

$$\frac{\partial \varphi_{X,Y}}{\partial \theta}(\theta, \psi) = (2i - 5\theta - 3\psi)\varphi_{X,Y}(\theta, \psi).$$

Thus $\mathbb{E}[iX] = \frac{\partial \varphi_{X,Y}}{\partial \theta}(0, 0) = 2i$, so $\mathbb{E}[X] = 2$.

(b) We have that

$$\frac{\partial^2 \varphi_{X,Y}}{\partial \theta^2}(\theta, \psi) = \{(2i - 5\theta - 3\psi)^2 - 5\}\varphi_{X,Y}(\theta, \psi).$$

Thus $-\mathbb{E}[X^2] = \frac{\partial^2 \varphi_{X,Y}}{\partial \theta^2}(0, 0) = -4 - 5 = -9$, so $\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 9 - 4 = 5$.

(c) We have that

$$\frac{\partial \varphi_{X,Y}}{\partial \psi}(\theta, \psi) = \{(5i - 9\psi - 3\theta)\varphi_{X,Y}(\theta, \psi)$$

$$\frac{\partial^2 \varphi_{X,Y}}{\partial \theta \partial \psi}(\theta, \psi) = \{(2i - 5\theta - 3\psi)(5i - 9\psi - 3\theta) - 3\}\varphi_{X,Y}(\theta, \psi).$$

From the first, we get that $\mathbb{E}[iY] = \frac{\partial \varphi_{X,Y}}{\partial \psi}(0, 0) = 5i$, so $\mathbb{E}[Y] = 5$. From the second we get that $-\mathbb{E}[XY] = \frac{\partial^2 \varphi_{X,Y}}{\partial \theta \partial \psi}(0, 0) = -10 - 3$, so $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 12 - 2 \cdot 5 = 3$. 

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