SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. **20 points** Suppose that $X$ and $Y$ are independent Gaussian random variables with mean zero and variance one. Define $Z \equiv 3X + Y$.
   
   (a) 
   (b) **10 points** Compute either the moment generating function or characteristic function of the pair $X$ and $Z$.
   (c) **10 points** Compute the variance $\sigma_{Z,Z}$ of $Z$ and the covariance $\sigma_{X,Z}$ of $X$ and $Z$.

2. **30 points** Suppose that $X$ is a Gaussian random variable with mean zero and variance 1. Define $Z = X^2$.
   
   (a) **15 points** Find the cumulative distribution function of $Z$.
   (b) **15 points** Does $Z$ have a density? Why or why not? If so, find it.

3. **30 points** Suppose that the random variable $X$ has uniform distribution on $(0,1)$. Given that $X = t$ (where $0 < t < 1$), the random variable $Y$ is exponential with parameter $1 + t$; i.e.,

   $$f_{Y|X}(s|t) = \begin{cases} (1 + t)e^{-(1+t)s} & \text{if } s \geq 0 \\ 0 & \text{else} \end{cases}$$

   (a) **10 points** Compute $\mathbb{E}[Y|X = t]$ for $0 < t < 1$.
   (b) **5 points** Compute $\mathbb{E}[Y]$.
   (c) **15 points** Compute the joint density $f_{X,Y}$

4. **20 points** Suppose that $X$ and $Y$ have joint density function

   $$f_{X,Y}(s,t) = \begin{cases} c/s^3 & \text{if } s > 1 \text{ and } 0 < t < s \\ 0 & \text{else.} \end{cases}$$

   (a) **5 points** Find $c$
   (b) **15 points** Find the marginal density of $X$; i.e., find $f_X$
1. (a)
\begin{align*}
\varphi_{X,Z}(\theta, \psi) &= \mathbb{E}[\exp \{ i\theta X + i\psi Z \}] = \mathbb{E}[\exp \{ i\theta X + i\psi(3X + Y) \}] \\
&= \mathbb{E}[\exp \{ i(\theta + 3\psi)X \exp \{ i\psi Y \} \}] \\
&= \exp \left[ -\frac{1}{2}(\theta + 3\psi)^2 - \frac{1}{2}\psi^2 \right] = \exp \left[ -\frac{1}{2}\theta^2 - \frac{10}{2}\psi^2 - 3\theta \psi \right]
\end{align*}

(b) \( \mathbb{E}[X] = \mathbb{E}[Z] = 0 \), so
\[ \mathbb{E}(X - 0)(Z - 0) = \mathbb{E}[XZ] = \mathbb{E}(X(3X + Y)) = 3\mathbb{E}[X^2] + \mathbb{E}[X]\mathbb{E}[Y] = 3. \]

2. (a)
\begin{align*}
F_Z(t) &= \mathbb{P}\{X^2 \leq t\} = \begin{cases} 
0 & \text{if } t < 0 \\
\mathbb{P}\{|X| \leq \sqrt{t}\} & \text{if } t \geq 0
\end{cases} \\
&= \begin{cases} 
0 & \text{if } t < 0 \\
\int_{s=-\sqrt{t}}^{\sqrt{t}} f_X(s)ds & \text{if } t \geq 0
\end{cases} = \begin{cases} 
0 & \text{if } t < 0 \\
\int_{s=-\sqrt{t}}^{\sqrt{t}} \frac{\exp\left[-\frac{s^2/6}{\sqrt{6t}}\right]}{\sqrt{6\pi t}} ds & \text{if } t \geq 0
\end{cases}
\]

(b) Yes; \( F_Z \) is differentiable;
\[ f_Z(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{\exp\left[-\frac{t/6}{\sqrt{6t}}\right]}{\sqrt{6\pi t}} & \text{if } t \geq 0
\end{cases} \]

3. (a)
\[ \mathbb{E}[Y \mid X = t] = \int_{s \in \mathbb{R}} sf_{Y \mid X}(s \mid t)ds = \int_{s=0}^{\infty} s(1 + t)e^{-(1+t)s}ds = \frac{1}{1 + t} \]

(b)
\[ \mathbb{E}[Y] = \int_{t=0}^{1} \mathbb{E}[Y \mid X = t]dt = \int_{t=0}^{1} \frac{1}{1 + t} dt = \ln \frac{1}{2}. \]

(c)
\[ f_{X,Y}(s, t) = f_{Y \mid X}(t \mid s)f_X(s) = \begin{cases} 
(1 + s)e^{-(1+s)t} & \text{if } t \geq 0 \text{ and } s \in [0, 1] \\
0 & \text{else}
\end{cases} \]

4. (a)
\[ 1 = \int_{\mathbb{R}^2} f_{X,Y}(s, t)ds dt = c \int_{s=1}^{\infty} \int_{t=0}^{s} \frac{1}{s^3} dt ds = c \int_{s=1}^{\infty} \frac{1}{s^2} ds = 2c; \]

\[ c = \frac{1}{2}. \]
(b) 

\[ f_X(s) = \int_{t \in \mathbb{R}} f_{X,Y}(s, t) dt = \begin{cases} \int_{t=0}^{s} \frac{1}{2s} ds & \text{if } s \geq 1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{2s^2} & \text{if } s \geq 1 \\ 0 & \text{else} \end{cases} \]