SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. \textbf{20 points} A lottery is held, and \( n \) tickets are sold. You buy 3 tickets, and there are 5 winning tickets. Let

\[
A \overset{\text{def}}{=} \{ \text{you win at least one prize} \} \\
B \overset{\text{def}}{=} \{ \text{you win exactly one prize} \}
\]

(a) \textbf{5 points} Compute \( \mathbb{P}(A) \)
(b) \textbf{5 points} Compute \( \mathbb{P}(B) \)
(c) \textbf{5 points} Compute \( \mathbb{P}(A \cap B) \)
(d) \textbf{3 points} Compute \( \mathbb{P}(A|B) \)
(e) \textbf{2 points} Are \( A \) and \( B \) independent?

2. \textbf{47 points} Suppose that 10 balls are distributed into 10 boxes. Let

\[
A \overset{\text{def}}{=} \{ \text{exactly one box is empty} \} \\
B \overset{\text{def}}{=} \{ \text{box 1 is empty} \}.
\]

(a) \textbf{15 points} What is \( \mathbb{P}(A) \)?
(b) \textbf{10 points} What is \( \mathbb{P}(B) \)?
(c) \textbf{12 points} What is \( \mathbb{P}(A \cap B) \)?
(d) \textbf{5 points} What is \( \mathbb{P}(A|B) \)?
(e) \textbf{5 points} What is \( \mathbb{P}(B|A) \)?

3. \textbf{20 points} Suppose you know that

\[
\mathbb{P}(A) = 0.6 \quad \text{and} \quad \mathbb{P}(B) = 0.3
\]

and that \( A \) and \( B \) are independent.

(a) \textbf{5 points} Compute \( \mathbb{P}(A \cap B) \).
(b) \textbf{5 points} Compute \( \mathbb{P}(A \setminus B) \).
(c) \textbf{5 points} Compute \( \mathbb{P}(A \cup B) \).
(d) \textbf{5 points} Compute \( \mathbb{P}((A \cup B) \setminus (A \cap B)) \).

4. \textbf{13 points} At a certain prestigious institution of higher learning, 60\% of the students are female and 40\% are male. It is known that 10\% of the men have tattoos and 15\% of the women have tattoos.
(a) [5 points] What is the probability that a randomly selected student is both male and has a tattoo?

(b) [5 points] If you randomly select a student who has a tattoo, what is the probability that the student will be male?

(c) [3 points] Are masculinity and tattoo-ness independent?
Answers

1. (a) 
   \[1 - \mathbb{P}\{\text{no prizes}\} = 1 - \frac{\binom{n-5}{3}}{\binom{n}{3}}.\]

(b) 
   \[\mathbb{P}(B) = \frac{(n-5)\binom{9}{2}}{\binom{n}{3}}.\]

(c) \(B \subset A\), so 
   \[\mathbb{P}(A \cap B) = \mathbb{P}(B) = \frac{(n-5)\binom{9}{2}}{\binom{n}{3}}.\]

(d) \(\mathbb{P}(A|B) = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.\)

(e) no, since \(\mathbb{P}(A|B) = 1 \neq \mathbb{P}(A)\).

2. (a) \(\mathbb{P}(A) = \binom{10}{2}(10)_9/10^{10}\).

(b) \(\mathbb{P}(B) = 9^{10}/10^{10}\).

(c) \(\mathbb{P}(A \cap B) = \binom{10}{2}9!/10^{10}\).

(d) \(\mathbb{P}(A|B) = \binom{10}{2}9!/9^{10}\).

(e) \(\mathbb{P}(B|A) = 9!/10^{9} = \frac{1}{10}\).

3. (a) \(\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = (0.6)(0.3) = 0.18.\)

(b) \(\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.6 - 0.18 = 0.42.\)

(c) \(\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.6 + 0.3 - 0.18 = 0.72.\)

(d) \(\mathbb{P}((A \cup B) \setminus (A \cap B)) = \mathbb{P}(A \cup B) - \mathbb{P}(A \cap B) = 0.72 - 0.18 = 0.54.\)

4. (a) \(\mathbb{P}(M \cap T) = \mathbb{P}(T|M)\mathbb{P}(M) = (0.1)(0.4) = 0.04.\)

(b) 
   \[\mathbb{P}(M|T) = \frac{\mathbb{P}(M \cap T)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T|M)\mathbb{P}(M)}{\mathbb{P}(T|M)\mathbb{P}(M) + \mathbb{P}(T|F)\mathbb{P}(F)} = \frac{(0.4)(0.1)}{(0.4)(0.1) + (0.15)(0.6)} = \frac{4}{13}.\]

(c) no; \(\mathbb{P}(M|T) = \frac{4}{13} \neq 0.4 = \mathbb{P}(M)\).