Assume that we have three (random) investments $X_1$, $X_2$, and $X_3$. We use the standard notation that

$$\bar{r}_i \overset{\text{def}}{=} \mathbb{E}[X_i] \quad \text{and} \quad \sigma_{i,j} \overset{\text{def}}{=} \mathbb{E}[(X_i - \bar{r}_i)(X_j - \bar{r}_j)]$$

for all $i$ and $j$ in $\{1, 2, 3\}$. We assume that

$$\bar{r}_1 = 1 \quad \bar{r}_2 = 5 \quad \bar{r}_3 = 7$$

$$\sigma_{1,1} = 2, \quad \sigma_{2,2} = 2, \quad \sigma_{3,3} = 3$$

$$\sigma_{1,2} = \sigma_{2,1} = 3/2, \quad \sigma_{1,3} = \sigma_{3,1} = 2, \quad \sigma_{2,3} = \sigma_{3,2} = 0.$$

1. **20 points** Find two funds, convex combinations of which give the efficient frontier.

**Answer:** Row reduce the matrix

$$
\begin{pmatrix}
2 & 3/2 & 2 & 1 & 1 \\
3/2 & 2 & 0 & 1 & 5 \\
2 & 0 & 3 & 1 & 7
\end{pmatrix}
$$

to get

$$
\begin{pmatrix}
1 & 0 & 0 & 10/11 & 178/11 \\
0 & 1 & 0 & -2/11 & -106/11 \\
0 & 0 & 1 & -3/11 & -93/11
\end{pmatrix}.
$$

Normalize the last two columns to get the two portfolios

$$
\begin{pmatrix}
2 \\
-2/5 \\
-3/5
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
-178/21 \\
106/21 \\
31/7
\end{pmatrix}.$$
2. (a) 20 points Find the portfolio which gives a minimum variance.
Answer:
\[
\begin{pmatrix}
 2 \\
-2/5 \\
-3/5 \\
\end{pmatrix}
\]

(b) 20 points Find the rate of return of this portfolio.
Answer: \(-21/5\).

(c) 10 points Find the variance of this portfolio.
Answer: \(11/5\).

3. Suppose that you want to achieve a rate of return of \(r = 4\).
   (a) 20 points Identify the minimum-variance portfolio which will give you this return.
   Answer:
   \[
   \begin{pmatrix}
   43/124 \\
   57/124 \\
   6/31 \\
   \end{pmatrix}
   \]

   (b) 10 points Find the variance of this portfolio.
   Answer: \(755/496\).