Convergence

Question: For what $\theta$ is $f^*$ (the Legendre transform of $f$) flat? 

Let $f(x) = x^4$. Then

$$f^*(\theta) = \max_{x>0} \{ x \sup_{x<0} \{ \theta x - x^2 \} \}$$

$$\sup_{x<0} \{ \theta x - x^2 \} = \sup_{x>0} (\theta - 1)x = \begin{cases} \infty & \text{if } \theta > 1 \\ 0 & \text{if } \theta \leq 1 \end{cases}$$

$$\sup_{x<0} \theta x = \begin{cases} \infty & \text{if } \theta < 0 \\ 0 & \text{if } \theta > 0 \end{cases}$$

Thus

$$f^*(\theta) = \begin{cases} \infty & \text{if } 0 \leq \theta < 1 \\ 0 & \text{else} \end{cases}$$

Thus, it looks like corners in $f$ $\Rightarrow$ flat in $f^*$. 

\[ \text{fig} \]
Let's next see that \( f(x) = x^+ \) can be used to represent more general discontinuities.

**Lemma:** Fix \( f: \mathbb{R} \to (-\infty, \infty] \) and let \( \hat{f} \) be its Legendre Transform. If we define \( f^* \) in the following ways, we have the following Legendre transforms.

a) \( f^*(x) \overset{\text{def}}{=} f(x) + c \quad \hat{f}^*(\theta) = \hat{f}(\theta) - c \)  
   (translate)

b) \( f^*(x) = f(ax) \quad \hat{f}^*(\theta) = \hat{f}(\theta a) \)  
   (scale)

c) \( f^*(x) = f(x) + x \quad \hat{f}^*(\sigma) = \hat{f}(\sigma - 1) \)  
   (add linear part)

d) \( f^*(x) = f(x + \alpha) \quad \hat{f}(\theta) = \hat{f}(\theta - \alpha) \)

**Proof:**

a) is obvious. For b),

\[
\hat{f}^*(\theta) = \sup_x \left\{ \theta x - f(ax) \right\} = \hat{f}(\theta a)
\]

For c),

\[
\hat{f}^*(\sigma) = \sup_x \left\{ \sigma x - x - f(x) \right\} = \sup_x \left\{ \sigma x - f(x) \right\}
\]

\[
= \hat{f}(\sigma - 1).
\]
For \( \alpha \),
\[
\hat{f}^\alpha(x) = \sup \left\{ \Theta x - f(x+\alpha) \right\} \\
= \sup \left\{ \Theta(x+\alpha) - f(x+2\alpha) \right\} - \Theta \alpha \\
= \hat{f}(\Theta) - \Theta \alpha \\
\]

Let's next see what the L-F transform means.
Assume that \( f \) is convex & smooth.
\[
\hat{f}(\Theta) = \sup \left\{ \Theta x - f(x) \right\}
\]
Assuming supremum is attained, \( \Theta = f'(x) \); \( \Theta \) is the slope of \( f \) at \( x \). Then \( \hat{f}(\Theta) = \Theta x - f(x) \); or
\[
f(x) = \Theta x - \hat{f}(\Theta);
\]
The line \( t \rightarrow \Theta t - \hat{f}(\Theta) \) is tangent to \( f \) at \( x \); \( -\hat{f}(\Theta) \) is its y-intercept.
Graphical Way to Compute $\hat{f}(\theta)$

Find a line of slope $\theta$ which "touches $f$ from below" (supporting hyperplane). The $y$-intercept at this line is $-f(\theta)$.

**Example**

$f(x) = x^+$

- For $\theta \in (0, 1)$, $y$-intercept is $\theta$: $\hat{f}(\theta) = 0$ if $0 < \theta < 1$.
- For $\theta \notin (0, 1)$, need to push "supporting hyperplane" down to $-\infty$: $\hat{f}(\theta) = \infty$ if $\theta \in [-1, 1]$.

$\hat{f}(\theta) = 0$

$\theta < 0$

$\theta > 1$
Example: \( f(x) = |x-3| \)

- If \( x > 1 \), \( f(x) = x \)
- If \( x < 1 \), \( f(x) = 0 \)
- If \( x = 1 \), \( f(x) = \infty \)

Look for line of slope \( \theta \) which passes through \((3,0)\)

**Line is:** \( y - 0 = \theta (x-3) \); \( y = \theta (x-3) \)

\( x = 0 \Rightarrow y = -3\theta \)