CONTRACTION PRINCIPLE

**Proposition 0.1.** Suppose that \( \{X^\varepsilon; \varepsilon > 0\} \) is a collection of random variables defined on some probability space \( (\Omega, \mathcal{F}, P) \) and taking values in some Polish space \( E \). Suppose furthermore that \( \{X^\varepsilon; \varepsilon > 0\} \) has a large deviations principle with rate function \( I \). Suppose that \( E' \) is a second Polish space and suppose that \( \psi \) is a continuous map from \( E \) to \( E' \). Define the \( E' \)-valued random variables \( Y^\varepsilon \overset{\text{def}}{=} \psi(X^\varepsilon) \) for all \( \varepsilon > 0 \). Then \( \{Y^\varepsilon; \varepsilon > 0\} \) has a large deviations principle with rate function

\[
I'(y) \overset{\text{def}}{=} \inf \{I(x) : \psi(x) = y\}.
\]

**Proof.** We break the calculation up into three steps.

**Upper Bound.** Fix a closed subset \( F \) of \( E' \). Then

\[
\lim_{\varepsilon \to 0} \varepsilon \ln P\{Y^\varepsilon \in F\} = \lim_{\varepsilon \to 0} \varepsilon \ln P\{X^\varepsilon \in \psi^{-1}(F)\} \leq - \inf \{I(x) : x \in \psi^{-1}(F)\} = - \inf \{I'(y) : y \in F\}.
\]

**Lower Bound.** Fix an open subset \( G \) of \( E' \). Then

\[
\lim_{\varepsilon \to 0} \varepsilon \ln P\{Y^\varepsilon \in G\} = \lim_{\varepsilon \to 0} \varepsilon \ln P\{X^\varepsilon \in \psi^{-1}(G)\} \geq - \inf \{I(x) : x \in \psi^{-1}(G)\} = - \inf \{I'(y) : y \in G\}.
\]

**Compactness.** Since \( \psi \) is continuous, it suffices to show that for any \( s \geq 0 \),

\[
\Phi'(s) \overset{\text{def}}{=} \{y \in E' : I'(y) \leq s\} = \psi(\Phi(s)).
\]

First, we show that \( \psi(\Phi(s)) \subset \Phi'(s) \). If \( x \in \Phi(s) \), then \( I(x) \leq s \), so if we define \( y \overset{\text{def}}{=} \psi(x) \), we see that \( x \in \{x' \in E : \psi(x') = y\} \), so \( I'(y) \leq I(x) \leq s \), so indeed \( \psi(\Phi(s)) \subset \Phi'(s) \).

Next, we show that \( \Phi'(s) \subset \psi(\Phi(s)) \). Fix \( y \in \Phi'(s) \). Thus, for each \( n \in \mathbb{N} \), there is an \( x_n \in E \), such that \( \psi(x_n) = y \) and

\[
I(x_n) \leq I'(y) + \frac{1}{n} \leq s + \frac{1}{n}.
\]

Since \( \Phi(s + 1) \) is compact, there is a convergent subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) with limit point \( x^* \). By continuity, \( \psi(x^*) = \lim_{k} \psi(x_{n_k}) = y \). Fixing any \( \delta > 0 \), we see that for \( k \) large enough that \( n_k \geq 1/\delta \), \( x_{n_k} \in \Phi(s + 1/n_k) \subset \Phi(s + \delta) \). Since \( \Phi(s + \delta) \) is closed, we thus have that \( x^* \in \Phi(s + \delta) \); i.e., \( I(x^*) \leq s + \delta \). Thus in fact, \( I(x^*) \leq s \). Hence \( \psi(x^*) = y \) and \( x^* \in \Phi(s) \), so \( y \in \psi(\Phi(s)) \). \( \Box \)