REJECTION SAMPLING

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Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a collection of i.i.d. discrete random variables with common probability mass function p_X. Consider a second probability mass function $\hat{p}$ such that, for some $M>0$, $\hat{p}(i)\le Mp(i)$ for all i. Let $\{U_n\}_{n\in\mathbb{N}}$ be an i.i.d. collection of $U(0,1)$ random variables which are independent of the X_n's. Define

$$\tau \overset{\text{def}}{=} \min\left\{n\ge 1: U_n<\frac{\hat{p}(X_n)}{Mp_X(X_n)}\right\}$$

and define $\hat{X}\overset{\text{def}}{=}X_\tau$. I claim that $\hat{X}$ has probability mass function $\hat{p}$. Indeed, for any i,

$$\begin{aligned}\mathbb{P}\{\hat{X}=i\} &= \sum_{n=1}^\infty \mathbb{P}\{\hat{X}=i,\tau=n\}\\ &= \sum_{n=1}^\infty \mathbb{P}\left\{X_n=i, U_{n'}\ge \frac{\hat{p}(X_{n'})}{Mp_X(X_{n'})}\text{ for } n'\le n-1,\ U_n<\frac{\hat{p}(X_n)}{Mp_X(X_n)}\right\}\\ &= \sum_{n=1}^\infty \mathbb{P}\left\{X_n=i, U_n<\frac{\hat{p}(X_n)}{Mp_X(X_n)}\right\}\prod_{1\le n'\le n-1}\mathbb{P}\left\{U_{n'}\ge\frac{\hat{p}(X_{n'})}{Mp_X(X_{n'})}\right\}\end{aligned}$$

Note that conditioning first on $X_{n'}$, we have that

$$\begin{aligned}\mathbb{P}\left\{U_{n'}\ge\frac{\hat{p}(X_{n'})}{Mp_X(X_{n'})}\right\} &= \sum_i\left\{\int_{u=\hat{p}(i)/(Mp_X(i))}^1 du\right\}p_X(i)\\ &= \sum_i\left\{1-\frac{\hat{p}(i)}{Mp_X(i)}\right\}p_X(i) = 1-\frac{1}{M}\sum_i\hat{p}(i) = 1-\frac{1}{M}\end{aligned}$$

$$\mathbb{P}\left\{X_n=i,U_n<\frac{\hat{p}(X_n)}{Mp_X(X_n)}\right\} = \left\{\int_{u=0}^{\hat{p}(i)/(Mp_X(i))}du\right\}p_X(i) = \frac{\hat{p}(i)}{Mp_X(i)}p_X(i) = \frac{\hat{p}(i)}{M}$$

Thus

$$\mathbb{P}\{\hat{X}=i\} = \sum_{n=1}^\infty\frac{1}{M}\hat{p}(i)\left(1-\frac{1}{M}\right)^{n-1} = \frac{\frac{1}{M}\hat{p}(i)}{\frac{1}{M}} = \hat{p}(i).$$

0.1. **Example.** Let's use rejection sampling to simulate a negative binomial random variable using a geometric random variable. Fix $K\in\{1,2\dots\}$ and $p\in(0,1)$ and define

$$\hat{p}(i) = \begin{cases}\binom{i-1}{K-1}(1-p)^{i-K}p^K & \text{if } i\in\{K,K+1\dots\}\\ 0 & \text{else}\end{cases}$$

Now let $p_*\in(0,p)$ so that

$$\frac{1-p}{1-p_*}=\frac{1}{\alpha}\in(0,1)$$

Let X be binomial with parameter p_*; i.e.,

$$p_X(i) = \begin{cases}(1-p_*)^{n-1}p_* & \text{if } n\in\{1,2\dots\}\\ 0 & \text{else.}\end{cases}$$

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For each $i \in \{K, K+1 \dots\}$, we then have that

$$\begin{aligned}\frac{\hat{p}(i)}{p_X(i)} &= \frac{\binom{i-1}{K-1}(1-p)^{i-K}p^K}{(1-p_*)^{i-1}p_*} \\ &\le \frac{1-p_*}{(1-p)^K}\frac{i^K}{(K-1)!}\left(\frac{1-p}{1-p_*}\right)^i \\ &= \alpha(1-p)^{1-K}\frac{i^K}{(K-1)!}\left(\frac{1}{\alpha}\right)^i\end{aligned}$$

Using the Taylor expansion for the exponential, we have that

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \ge x.$$

Thus $x^K \le e^{Kx}$. For any $\delta > 0$ and nonnegative integer i, we thus have that

$$(i\delta)^K \le \exp[K\delta i]$$

and thus

$$i^K \le \frac{e^{K\delta i}}{\delta^K}.$$

Taking $\delta \overset{\text{def}}{=} \frac{1}{K}\ln\alpha$ so that $e^{K\delta} = \alpha$, we have that

$$i^K \le \frac{\alpha^i}{\left(\frac{1}{K}\ln\alpha\right)^K} = \left(\frac{K}{\ln\alpha}\right)^K\alpha^i.$$

Hence

$$\frac{\hat{p}(i)}{p_X(i)} \le M \overset{\text{def}}{=} \frac{\alpha(1-p)^{1-K}}{(K-1)!}\left(\frac{K}{\ln\alpha}\right)^K = \frac{1}{(K-1)!}\frac{1-p_*}{(1-p)^K}\left(\frac{K}{\ln((1-p_*)/(1-p))}\right)^K.$$

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