We want to solve

\[ dX_t = b(X_t)dt + \sigma(X_t)dw_t \]

\[ X_0 = x_0 \]

In other words, we want to solve the integral equation

\[ X_t = x_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dw_s \]

Let's recall how this works with ODE's.

Consider the ODE

\[ \dot{Z}_t = b(Z_t) \]

\[ Z_0 = Z_0 \]

\[ Z_t = Z_0 + \int_0^t b(Z_s)ds \]

\( b \) is bounded & Lipschitz-continuous:

\[ |b(x) - b(y)| \leq \beta |x - y| \]

\( \forall x \neq y \)
Richard iterations

\[ Z^1_+ = Z_0 \]
\[ Z^2_+ = Z_0 + b(Z_0)_+ = Z_0 + \int_0^1 b(Z_0) \, ds \]
\[ \vdots \]

\[ Z^{n+1}_+ = Z_0 + \int_0^1 b(Z^n_0) \, ds \]

Note: If \( \{x_n\} \) CIR and

\[ \sum_{j=1}^{\infty} |X_{j+1} - X_j| < \infty, \]

then if \( m, u \equiv n \),

\[ |X_m - X_u| \leq \sum_{j=n}^{m-1} |X_{j+1} - X_j| \]

\[ = \sum_{j=n}^{\infty} |X_{j+1} - X_j| \]

Since

\[ \lim_{m, u \to \infty} |X_m - X_u| < \infty, \]
So \( \{x_n\} \) converges (\( K \) is a Banach space)

Goal:

\[
\sum_{n=1}^{\infty} \left\{ \sup_{t \in T} |z_u^n - z_t^n| \right\} < \infty.
\]

Since \( C([0, T]; \mathbb{R}) \) is Banach, there is

\( z^* \in C([0, T]; \mathbb{R}) \) such that

\[
\lim_{u \to \infty} \sup_{t \in T} |z_u - z^*_t| = 0.
\]

Take limits of \( z_u^n \) as \( u \to \infty \). Then we get that \( z^* \) satisfies desired integral equation.

\[
z_t^3 - z_t^1 = b(z) +
\]

\[
|z_t^2 - z_t^1| < B +
\]
\[ Z_{+}^n - Z_{+}^n = \int_{0}^{+} b(Z_{+}^n) - b(Z_{+}^n) \right) \, ds \]

\[ |Z_{+}^n - Z_{+}^n| \leq \int_{0}^{+} |b(Z_{+}^n) - b(Z_{+}^n)| \, ds \]

\[ = \beta \int_{0}^{+} |Z_{+}^n - Z_{+}^n| \, ds \]

\[ |Z_{+}^n - Z_{+}^n| \leq \beta \int_{0}^{+} |B_{+} \, ds = \frac{B \beta}{2} \]

\[ |Z_{+}^n - Z_{+}^n| \leq \frac{B \beta^3}{2n^2} \cdot z_2 \cdot \frac{3n^3}{2n^2} \]

\[ \sup_{0 \leq t \leq T} |Z_{+}^n - Z_{+}^n| \leq B \frac{\beta^{n+1}}{n!} \]

\[ \sup_{0 \leq t \leq T} |Z_{+}^n - Z_{+}^n| \leq B \frac{\beta^{n+1}}{n!} \]
Thus there is \( \hat{Z} \in C([0,1]) \) such that
\[
\lim_{u \to \infty} \sup_{t \in \mathbb{T}_+} |Z^u_t - \hat{Z}_t| = 0
\]
Take limits of \( \rho \) as \( u \to \infty \). Continuity of \( b \) implies that
\[
\hat{Z}_t = Z_0 + \int_0^t b(\hat{Z}_s) \, ds
\]
Uniqueness: Suppose that someone comes up with a different solution, i.e. \( \tilde{Z} \in C([0,1]) \) such that
\[
\tilde{Z}_t = Z_0 + \int_0^t b(\tilde{Z}_s) \, ds.
\]
Does \( Z^u = \tilde{Z} \)?

Then
\[
Z^u_t - \tilde{Z}_t = \int_0^t \{b(Z^u_s) - b(\tilde{Z}_s)\} \, ds
\]
\[ |\vec{Z}_T - \hat{\vec{Z}}_T| \leq \int_0^+ |b(\vec{Z}_s) - b(\hat{\vec{Z}}_s)| \, ds \]

\[ \leq \beta \int_0^+ |\vec{Z}_s - \hat{\vec{Z}}_s| \, ds \]

Lipschitz coefficient

Grüss-type inequality; Set

\[ R_T = \int_0^+ |\vec{Z}_s - \hat{\vec{Z}}_s| \, ds \]

Then

\[ \dot{R}_T = |\vec{Z}_T - \hat{\vec{Z}}_T| \leq \beta R_T \]

\[ R_0 = 0 \]

Then

\[ (R_T e^{-\beta t})' = (\dot{R}_T - \beta R_T) e^{-\beta t} \leq 0 \]

\[ R_T e^{-\beta t} \leq R_0 e^{-\beta \times 0} = 0 \]

\[ R_T = 0 \]

\[ R_T = 0 \Rightarrow |\vec{Z}_T - \hat{\vec{Z}}_T| = \dot{R}_T \leq \beta R_T = 0 \]
Back to SDE's.

Simplest case

\[ dX_t = \sigma(X_t) \, dw_t \]
\[ X_0 = x_0 \]

\[ X_t = x_0 + \int_0^t \sigma(X_s) \, dw_s \]

\( \sigma \) bounded & Lipschitz

\( |\sigma(x)| \leq M \)

\( |\sigma(x) - \sigma(y)| \leq L \cdot |x-y| \)

Iterations:

\[ X_{t}^1 = x_0 \]

\[ X_{t}^2 = x_0 + \int_0^t \sigma(X_s^1) \, dw_s = x_0 + \int_0^t \sigma(X_s^0) \, dw_s \]

\[ X_{t}^3 = x_0 + \int_0^t \sigma(X_s^2) \, dw_s \]

\[ \vdots \]

\[ X_{t}^{n+1} = x_0 + \int_0^t \sigma(X_s^n) \, dw_s \]

Clearly, \( X^1 \in \overline{\Theta} \). Then \( X^2 \in \overline{\Theta} \) \( \Rightarrow \) \( X^n \in \overline{\Theta} \)

\( X^t \) is continuous, adapted, so all stochastic integrals are well-defined.
\[ X_t^2 - X_t^1 = \sigma(x_t) \mathcal{W}_t \]

\[ \mathbb{E}(|X_t^2 - X_t^1|^2) \leq \|\sigma\| t \]

\[ X_t^3 - X_t^2 = \int_0^t \{ \sigma(X_s^3) - \sigma(X_s^4) \} \, d\mathcal{W}_s \]

\[ \mathbb{E}(|X_t^3 - X_t^2|^2) = \mathbb{E}\left( \int_0^t \{ \sigma(X_s^3) - \sigma(X_s^4) \} \, d\mathcal{W}_s \right)^2 \]

\[ = \mathbb{E}\left( \int_0^t (\sigma(X_s^3) - \sigma(X_s^4))^2 \, ds \right) \]

\[ \leq \mathbb{L}_0^2 \int_0^t \mathbb{E}(|X_s^3 - X_s^4|^2) \, ds \]

\[ = \mathbb{L}_0^2 \|\sigma\|^2 \int_0^t s \, ds = \frac{\mathbb{L}_0^2 \|\sigma\|^2 t^2}{2} \]

\[ \mathbb{E}(|X_t^4 - X_t^3|^2) = \mathbb{E}\left( \int_0^t \{ \sigma(X_s^3) - \sigma(X_s^4) \} \, d\mathcal{W}_s \right)^2 \]

\[ = \mathbb{E}\left( \int_0^t (\sigma(X_s^3) - \sigma(X_s^4))^2 \, ds \right) \]

\[ \leq \mathbb{L}_0^2 \int_0^t \mathbb{E}(|X_s^3 - X_s^4|^2) \, ds \]
\[ \mathbb{E} \left[ \left| X_{n+1} - X_n \right|^2 \right] \leq \frac{\|\sigma\|^2 \left( L_0^2 + t \right)^n}{n!} \]

We actually need to show that

\[ \sum_{n=1}^{\infty} \mathbb{E} \left[ \left| X_{n+1} - X_n \right|^2 \right] \frac{1}{n^2} < \infty \]

and

\[ \mathbb{E} \left[ \left| X_{n+1} - X_n \right|^2 \right] \leq \frac{\|\sigma\|^2 }{L_0} \left( \frac{L_0^2 + t}{\sqrt{n!}} \right)^{n/2} \]

By ratio test,
\[
\lim_{n \to \infty} \frac{(L_n + t)^{2n}}{(n+1)!} = \lim_{n \to \infty} \left( L_n t \right) \frac{n!}{(n+1)!}
\]

\[
= L_0 + \lim_{n \to \infty} \frac{1}{n+1} = 0.
\]

Thus

\[
\sum_{n=1}^{\infty} \frac{|L_n|^2 (L_n + t)^{2n}}{2^n n!} < \infty.
\]

So

\[
\sum_{n=1}^{\infty} \left( E \left[ \frac{1}{X_{n+1}^4} \right] \right)^{1/2} < \infty.
\]

By Doob's inequality, thus

\[
\sum_{n=1}^{\infty} E \left[ \sup_{t \geq T} \left| X_{n-1}^4 - X_n^4 \right| \right]^{1/2} < \infty.
\]
\[ \varepsilon^2 \sum_{n=1}^{\infty} \mathbb{E} \left[ \frac{1}{X_T^n - X_T^{n+1} - X_T^{n+2}} \right] \leq 0. \]

Since \( L^2(\mathbb{R}; C([0,1])) \) is a Banach space, \( C([0,1]) \)-valued random variables there is a continuous process \( \{X^*_t; 0 \leq t \leq T\} \) such that

\[ \lim_{n \to \infty} \mathbb{E} \left( \sup_{0 \leq t \leq T} \left| X^n_t - X^*_t \right|^2 \right)^{\frac{1}{2}} = 0. \]

Thus in particular, for any \( t \in [0,1] \),

\[ \lim_{n \to \infty} \mathbb{E} \left( \left| X^n_t - X^*_t \right|^2 \right)^{\frac{1}{2}} = 0. \]

So \( X^*_t \) is \( \mathcal{F}_t \)-measurable (it is the pointwise limit of \( \mathcal{F}_t \)-measurable functions).

Furthermore, since \( X^*_t \) is continuous,

\[ \int_0^T \sigma(X^*_s) \, dW_s \]

is well-defined for all \( t \in [0,1] \).
We also have

\[
\begin{align*}
E \left( \int_0^+ \sigma(X_s^*) \, dW_s - \int_0^+ \sigma(X_s) \, dW_s \right)^2 \\
= E \left( \int_0^+ \left( \sigma(X_s^*) - \sigma(X_s) \right)^2 \, dW_s \right)^2 \\
= E \left( \int_0^+ \left( \sigma(X_s^*) - \sigma(X_s) \right)^2 \, ds \right) \\
\xrightarrow{u \to \infty} \int_0^+ E \left( \left| X_s^* - X_s \right|^2 \right) \, ds \quad \xrightarrow{u \to \infty} 0
\end{align*}
\]

Thus

\[
X_T^* = X_0 + \int_0^+ \sigma(X_s^*) \, dW_s
\]

Uniqueness: If

\[
\hat{X}_T = X_0 + \int_0^+ \sigma(\hat{X}_s) \, dW_s
\]

then

\[
X_T^* - \hat{X}_T = \int_0^+ \left( \sigma(X_s^*) - \sigma(\hat{X}_s) \right) \, dW_s
\]
\[ E[|X_T - \hat{X}_T|^2] = E[\left( \int_0^T \{ o(X_s^*) - o(\hat{X}_s) \} dW_s \right)^2] \]

\[ \leq E\left[ \int_0^T \{ o(X_s^*) - o(\hat{X}_s) \}^2 ds \right] \]

\[ \leq L \int_0^T E[|X_s^* - \hat{X}_s|^2] ds \]

Same calculation as for ODE's shows that

\[ E[|X_T - \hat{X}_T|^2] = 0. \]

Extension to

\[ dX_t = b(X_t) dt + \sigma(X_t) dW_t \]

\[ X_0 = X_0 \]

in one or more dimensions is combination of above calculations.