# Ito's formula

Note Title 2/6/2007

Let's next differentiate in time.

<u>Ito's formula</u>: $f \in C_b^2(\mathbb{R})$

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)\,dW_s$$

$$+ \frac{1}{2}\int_0^t f''(W_s)\,ds$$

Pf: $t_j^{(N)} = \frac{jt}{N}$

$$f(W_t) - f(W_0) = \sum_{j=0}^{N-1} \left\{ f(W_{t_{j+1}^{(N)}}) - f(W_{t_j^{(N)}}) \right\}$$

$$= I_1^{(N)} + I_2^{(N)} + I_3^{(N)}$$

recall: $f(y) \sim f(x) + f'(x)(y-x) + \frac{1}{2} f''(x)(y-x)^2 + O(|y-x|^3)$

where

$$I_1^{(N)} = \sum_{j=0}^{N-1} f'(W_{t_j^{(N)}}) \{ W_{t_{j+1}^{(N)}} - W_{t_j^{(N)}} \}$$

$$I_2^{(N)} = \frac{1}{2} \sum_{j=0}^{N-1} f''(W_{t_j^{(N)}}) \left( W_{t_{j+1}^{(N)}} - W_{t_j^{(N)}} \right)^2$$

$$I_3^{(N)} = \sum_{j=0}^{N-1} R(W_{t_j^{(N)}}, W_{t_{j+1}^{(N)}})$$

where $R(x,y) \overset{\text{def}}{=} f(y) - f(x) - f'(x)(y-x)$

$$- \frac{1}{2} f''(x)(y-x)^2$$

$$I_1^{(N)} \longrightarrow \int_0^t f'(W_s)\,dW_s$$

$$|I_3^{(N)}| \le K \sum_{j=0}^{N-1} |W_{t_{j+1}^{(N)}} - W_{t_j^{(N)}}|^3$$

so $$\mathbb{E}\left[\,|I_3^{(N)}|\,\right] \le K \sum_{j=0}^{N-1} \mathbb{E}\left[\,|W_{t_{j+1}^{(N)}} - W_{t_j^{(N)}}|^3\right]$$

$$\le K N \left(1/\sqrt{N}\right)^3 = \frac{K}{\sqrt{N}} \longrightarrow 0$$

as $N \to \infty$

Finally,

$$I_2^{(N)} = \frac{1}{2} \sum_{j=0}^{N-1} \frac{1}{2} f''(W_{t_j^{(N)}}) \left\{ t_{j+1}^{(N)} - t_j^{(N)} \right\}$$

$$+ \frac{1}{2} \tilde{I}^{(N)}$$

where

$$\tilde{I}^{(N)} = \sum_{j=0}^{N-1} f''(W_{t_j^{(N)}}) \left\{ \underbrace{\left(W_{t_{j+1}^{(N)}} - W_{t_j^{(N)}}\right)^2 - t_{j+1}^{(N)} - t_j^{(N)}}_{\Delta_j^{(N)}} \right\}$$

$$\mathbb{E}\left((\tilde{I}^{(N)})^2\right) = A^{(N)} + B^{(N)}$$

$$\mathbb{E}\left[\Delta_j^{(N)}\right] = 0$$

where

$$A^{(N)} = \sum_{0 \le j < k \le N-1} \mathbb{E}\left[ f''(W_{t_j^{(N)}}) f''(W_{t_k^{(N)}}) \Delta_j^{(N)} \Delta_k^{(N)} \right]$$

$$B^{(N)} = \sum_{j=0}^{N-1} \mathbb{E}\left[ \left(f''(W_{t_j^{(N)}})\right)^2 \left(\Delta_j^{(N)}\right)^2 \right]$$

note:

$$(W_t - W_s)^2 - (t-s) = W_t^2 + W_s^2 - 2W_t W_s - t + s$$

$$= \{W_t^2 - t\} - \{W_s^2 - s\}$$

$$-2(W_t - W_s) W_s$$

$$\mathbb{E}\left( (W_t - W_s)^2 - (t-s) \mid \mathcal{F}_s \right)$$

$$= \mathbb{E}\left[ W_t^2 - t \mid \mathcal{F}_s \right] - \{W_s^2 - s\}$$

$$-2 W_s \{ \mathbb{E}[W_t \mid \mathcal{F}_s] - W_s \}$$

$$= 0$$

$$\mathbb{E}[A^{(N)}] = \sum_{0 \le j < k \le N-1} \mathbb{E}\Big[f''(W_{t_j}) f''(W_{t_k}) \Delta_j^{(N)} \underbrace{\mathbb{E}[\Delta_k^{(N)} \mid \mathcal{F}_{t_j}]}_{=0}\Big]$$

$$\mathbb{E}\left[\{(W_t - W_s)^2 - (t-s)\}^2\right]$$

$$= (t-s)^2\, \mathbb{E}\left[\Big\{\Big(\underbrace{\frac{W_t - W_s}{\sqrt{t-s}}}_{\eta^2}\Big)^2 - 1\Big\}^2\right]$$

Thus

$$\mathbb{E}[B^{(N)}] \le K\, N \left(\frac{1}{N}\right)^2 = K/N \longrightarrow 0$$

More generally,

$$f(M_t) = f(M_0) + \int_0^t f'(M_s)\, dM_s + \frac{1}{2}\int_0^t f''(M_s)\, d\langle M\rangle_s$$

$$d f(M_t) = f'(M_t)\, dM_t + \frac{1}{2} f''(M_t)\, d\langle M\rangle_t$$

$$f(t, M_t) = f(0, M_0) + \int_0^t \frac{\partial f}{\partial s}(s, M_s)\, ds$$

$$+ \int_0^t \frac{\partial f}{\partial m}(s, M_s)\, dM_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial m^2}(s, M_s)\, d\langle M\rangle_s$$

$$f(X_t) = f(X_0) + \sum_{j=1}^{n} \int_0^t \frac{\partial f}{\partial x_j}(X_s)\, dX_s^j$$

$$+ \frac{1}{2} \sum_{1 \le i, j \le n} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\, d\langle X^i, X^j\rangle_s$$

$$= d\langle M^i, M^j\rangle_s$$

$$= \frac{1}{2}\{d\langle M^i + M^j\rangle - d\langle M^i - M^j\rangle\}$$

$X_t = (X_t^1, X_t^2 \sim X_t^n)$

$X_t^1 = M_t^1 + A_t^1$

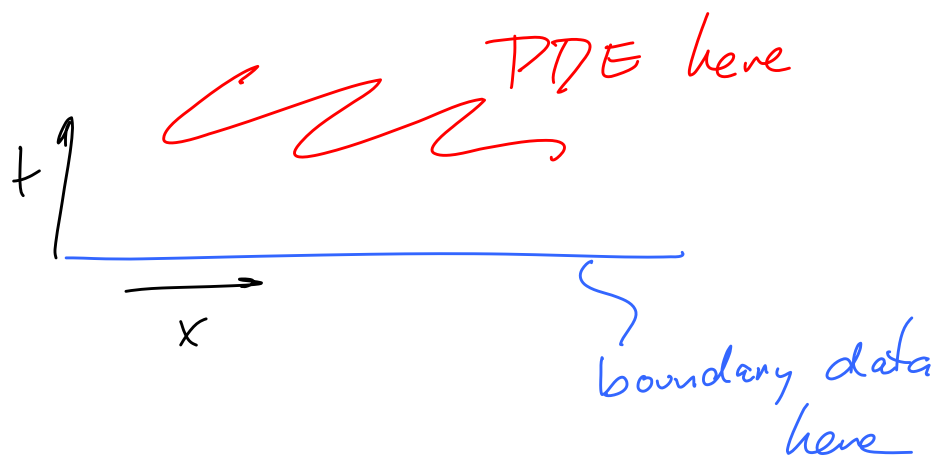
$X_t^2 = M_t^2 + A_t^2$

bounded variation, Predictable

## Application: Feynman Kac for parabolic PDE's

Assume that $u \in C^2([0,\infty)\times\mathbb{R})$ satisfies the PDE

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \qquad t>0,\ x\in\mathbb{R} \\ u(0,x) &= f(x) \qquad x\in\mathbb{R} \end{aligned}\right\} \text{parabolic PDE}$$




Fix $T>0$, $\hat{x}\in\mathbb{R}$. Consider

$$\xi_t = u(T-t, \underbrace{W_t+\hat{x}}_{W_t^{\hat{x}}})$$

$$d\xi_t = \cancel{-\frac{\partial u}{\partial s}(T-t, W_t^{\hat{x}})\,dt + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}}(T-t, W_t^{\hat{x}})\,dt$$
$$+ \frac{\partial u}{\partial x}(T-t, W_T^{\hat{x}})\,dW_t$$

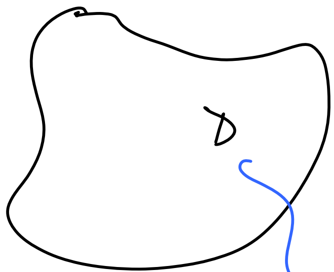
$$\xi_0 = u(T, \hat{x}) \qquad \xi_T = u(0, W_T^{\hat{x}}) = f(W_T^{\hat{x}})$$

$$\mathbb{E}[\xi_T] - \mathbb{E}[\xi_0] = \underbrace{\mathbb{E}\left[\int_0^T \frac{\partial u}{\partial x}(T-s, W_s^{\hat{x}})\, dW_s\right]}_{=0}$$

$$u(T, \hat{x}) = \mathbb{E}\left[f(W_T^{\hat{x}})\right]$$

$$= \int_{\mathbb{R}} \frac{f(z) \exp\left(-\frac{(z-\hat{x})^2}{2T}\right)}{\sqrt{2\pi T}} dz$$

Application: Feynman Kac for elliptic PDE's



Assume $u \in C^\infty(D)$ satisfies

$$\Delta u(x) = 0 \quad x \in D$$

$$u(x) = f(x) \quad x \in \partial D$$

D bounded, smooth boundary, closed

Fix $x \in \mathbb{R}$.

$$T(\omega) = \inf\{r \geq 0 : W_r(\omega) + x \notin D\}$$

$$\{\omega \in \Omega ; T(\omega) < t\} = \overbrace{\bigcup_{\substack{s < t \\ s \in \mathbb{Q}}} \underbrace{\{\omega \in \Omega : W_s(\omega) + x \notin D\}}_{\in \mathcal{F}_s \subset \mathcal{F}_t}}^{\text{countable union of } \mathcal{F}_t\text{-sets}}$$

Note: If $T(\omega) < 7$, then $r \mapsto W_r(\omega) + x$ is in $\mathbb{R}^2 \setminus D$ by some time before 7. Since $\mathbb{R}^2 \setminus D$ is open and the paths of W are continuous, then $W_{r^*}(\omega)+x \in \mathbb{R}^2 \setminus D$ for some $r^* \in \mathbb{Q}$ with $r^* < \tau$

Definition of stopping time: $\{\tau \le t\} \in \mathcal{F}_t$ all $t \ge 0$

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Idea: The decision to stop before time $\tau$ can be made on the basis of information up to time $\tau$.

Drive north from UIUC

take first exit after Rantoul ← a stopping time

take last left before Chicago ← not a stopping time; requires knowledge of future; must drive to Chicago & then backtrack.

$$W^x_\tau = W^x_{\tau(\omega)}(\omega)$$ ~ evaluate random trajectory at random time

$$u(W_T^x) - u(x) = M_T + \frac{1}{2}\int_0^T \overbrace{\Delta u(W_r^x)}^{0}\,dr$$

$f(W_T^x)$

$M_T = \int_0^t u'(W_s)\,dW_s$

We claim that $\mathbb{E}[M_T] = \mathbb{E}(M_0) = 0$; then

$$\mathbb{E}[f(W_T^x)] - \mathbb{E}(u(x)) = 0$$

$$u(x) = \mathbb{E}[f(W_T^x)]$$

Consequence: $u(x) \le \max_{x \in \partial D} f(x)$;

maximum principle

Note that by martingale property,

$$\mathbb{E}[M_T] = \mathbb{E}[\mathbb{E}[M_T | \mathcal{F}_0]] = \mathbb{E}(M_0)$$

so $M$ is constant mean.

evaluate the curve $r \mapsto M_r(\omega)$ at a fixed time

We claim that in our case, we also get that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0] = 0$$

This is a simple form of Optional Sampling, which we will not prove here.

## Pf of $\mathbb{E}[M_\tau] = 0$ in our case

$$\tau_N(\omega) \overset{\text{def}}{=} \frac{\lceil \tau(\omega) N \rceil}{N}$$ integer ceiling

$$\{\omega \in \Omega : \tau_N(\omega) \le j/N\} = \{\omega \in \Omega : \lceil \tau(\omega) N \rceil \le j\}$$

$$= \{\omega \in \Omega : \tau(\omega) N \le j\}$$

$$= \{\omega \in \Omega : \tau(\omega) \le j/N\} \in \mathcal{F}_{j/N}$$

$\tau_N(\omega) \downarrow \tau(\omega)$ for all $\omega \in \Omega$ such that $\tau(\omega) < \infty$.

Set $f_s(\omega) = u(W_s(\omega))$

$$M_{T_N(\omega)}(\omega) = \sum_{j=0}^{\infty} \chi_{\{T_N = j/N\}} \int_0^{j/N} f_s \, dW_s$$

$$= \sum_{j=0}^{\infty} \chi_{\{T_N = j/N\}} \sum_{k=0}^{j-1} \int_0^{\infty} \chi_{[k/N, \frac{k+1}{N})}(s) \, f_s \, dW_s$$

$$= \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \chi_{\{T_N = j/N\}} \int_0^{\infty} \chi_{[k/N, \frac{k+1}{N})}(s) \, f_s \, dW_s$$

$$= \sum_{k=0}^{\infty} \chi_{\underbrace{\{T_N \geq \frac{k+1}{N}\}}} \int_0^{\infty} \chi_{[k/N, \frac{k+1}{N})}(s) \, f_s \, dW_s$$

$$= \Omega \setminus \underbrace{\{T_N \leq k/N\}}_{\in \mathcal{F}_{k/N}} \in \mathcal{F}_{k/N}$$

$$= \int_0^{\infty} f_s \underbrace{\sum_{k=0}^{\infty} \chi_{\{T_N > k/N\}} \chi_{[k/N, \frac{k+1}{N})}(s)}_{\in \mathcal{P}} \, dW_s$$

Thus $\mathbb{E}(M_{\tau_n}) = 0$ and by Ito isometry,

$$\mathbb{E}\left( (M_{\tau_n} - M_{\tau})^2 \right) = \mathbb{E}\left[ \int_{\tau}^{\tau_n} (u'(W_s))^2 \, ds \right]$$

$$\xrightarrow{n \to \infty} 0$$ if $u'$ is bounded & $E[\tau] < \infty$.

Assumption that $u \in C^\infty(D) \Rightarrow u'$ is bounded.

Assumption that $D$ is bounded $\Rightarrow E[\tau] < \infty$.

If you think a bit, this works since

$$\mathbb{E}\left[ \langle M \rangle_{\tau} \right] < \infty.$$

**Application:** Levy's characterization of BM

B.M. $W$ is a continuous martingale with $\langle W \rangle_t = t$

**Levy:** B.M. is only such martingale. If $M$ is a martingale with $\langle M \rangle_t = t$, then $M$ is a B.M.

**Pf:** $f_\theta(x) = e^{i\theta x}$ $\qquad f_\theta'(x) = i\theta e^{i\theta x}$

$$f_\theta''(x) = -\theta^2 f_\theta(x)$$

$$f_\theta(M_t) = f(M_0) + \underbrace{\int_0^t f_\theta'(M_r)\,dM_r}_{\text{martingale}} - \frac{1}{2}\theta^2 \int_0^t f(M_r)\,dr$$

$$\mathbb{E}[f_\theta(M_t) \mid \mathcal{F}_s] = f(M_0) + \int_0^s f_\theta'(M_r)\,dM_r$$

$$- \frac{1}{2}\theta^2 \int_0^t \mathbb{E}[f(M_r) \mid \mathcal{F}_s]\,dr$$

$$= f_\theta(M_s) - \frac{1}{2}\theta^2 \int_s^t \mathbb{E}[f(M_r) \mid \mathcal{F}_s]\,dr$$

$$\mathbb{E}\left[f_\theta(M_t) \mid \mathcal{F}_s\right] = f_\theta(M_s)\, e^{-\frac{\theta^2}{2}(t-s)}$$

$$\mathbb{E}\left[e^{i\theta M_t} \mid \mathcal{F}_s\right] = e^{i\theta M_s}\, e^{-\frac{\theta^2}{2}(t-s)}$$

$$\mathbb{E}\left[e^{i\theta(M_t - M_s)} \mid \mathcal{F}_s\right] = e^{-\frac{\theta^2}{2}(t-s)}$$

$M_t - M_s$ is independent of $\mathcal{F}_s$ &

$M_t - M_s$ is $\mathcal{N}(0, t-s)$.

Implication: If $M$ is a continuous martingale, set

$$\tau_t(\omega) = \underbrace{\inf\{r \geq 0 : \langle M\rangle_s(\omega) > t\}}$$

$s \mapsto \langle M\rangle_s(\omega)$ is nondecreasing

find its right-continuous inverse

$$B_t(\omega) = M_{T_t(\omega)}(\omega)$$

This is a martingale and

$$\langle B \rangle_t(\omega) = \langle M \rangle_{T_t(\omega)}(\omega) = t$$

$\Rightarrow$ B is a B.M.

$$M_t(\omega) = B_{\langle M \rangle_t(\omega)}(\omega)$$

<u>Some</u> B.M.

# Application: Girsanov's formula

Suppose $\xi$ is $U(0,1)$. Suppose $f:(0,1) \to \mathbb{R}_+$ is such that $\int_0^1 f(z)\,dz = 1$. Define

$$\tilde{\mathbb{P}}(A) = \mathbb{E}[\chi_A f(\xi)]$$

$$\tilde{\mathbb{E}}(\varphi(\xi)) = \mathbb{E}(\varphi(\xi) f(\xi))$$

$$= \int_0^1 \varphi(z) f(z)\,dz$$

Under $\tilde{\mathbb{P}}$, $\xi$ has law $f$

Another example: $\eta$ is $\mathcal{N}(0,1)$.

$$\mathbb{E}(e^{(a+b_1)\eta}) = \exp\left(\frac{(a+b_1)^2}{2}\right)$$

$$= \exp\left(\frac{a^2}{2} + ab_1 - \frac{b^2}{2}\right)$$

$$\mathbb{E}\left[\exp\left[b_n(\eta - a)\right] \exp\left(a\eta - \frac{a^2}{2}\right)\right]$$

$$= \exp\left(-b^2/2\right)$$

Set

$$\tilde{\mathbb{P}}(A) = \mathbb{E}\left(\chi_A \exp\left(a\eta - a^2/2\right)\right)$$

Note:

$$\tilde{\mathbb{P}}(\Omega) = \mathbb{E}\left[\exp(0)\, e^{a\eta - a^2/2}\right] = 1$$

$$\tilde{\mathbb{E}}\left(\exp\left(\theta_n(\eta - a)\right)\right) = \exp\left(-\frac{a^2}{2}\right)$$

$\Longrightarrow$ $\eta - a$ is $\mathcal{N}(0,1)$ under $\tilde{\mathbb{P}}$.

Girsanov: Fix $h \in \mathcal{P}$ (bounded). Set

$$\tilde{W}_t = W_t - \int_0^t h_s\, ds \qquad 0 \le t \le 1$$

finite horizon!

$$M_t = \exp\left[ \int_0^t h_s \, dW_s - \frac{1}{2} \int_0^t h_s^2 \, ds \right] \qquad 0 \le t \le 1$$

1) $M$ is a martingale

2) $\tilde{\mathbb{P}}(A) = \mathbb{E}[\chi_A M_1]$ is a probability measure

3) $\tilde{W}$ is a B.M. under $\tilde{\mathbb{P}}$.

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$f(x) = e^x \qquad f'(x) = e^x \qquad f''(x) = e^x$

$$dM_t = M_t \left\{ h_t \, dW_t - \frac{1}{2} h_t^2 \, dt \right\} + \frac{1}{2} M_t \underbrace{h_t^2 \, dW_t}_{d\left\langle \int_0^t h_s dW_s - \frac{1}{2} \int_0^t h_s^2 ds \right\rangle}$$

$$= M_t h_t \, dW_t$$

$$M_t = 1 + \int_0^t M_s h_s \, dW_s \leftarrow M \text{ is a martingale}$$

$$M > 0, \qquad \mathbb{E}[M_1] = \mathbb{E}\left[ \mathbb{E}(M_1 | \mathcal{F}_0) \right] = \mathbb{E}[M_0] = 1$$

$\longrightarrow$ $\tilde{\mathbb{P}}$ is a probability measure

Fix $0 \le s \le t$, $A \in \mathcal{F}_s$

$$\widetilde{\mathbb{E}}(\widetilde{W}_t \chi_A) = \mathbb{E}(\widetilde{W}_t M_1 \chi_A) = \mathbb{E}[\chi_A \widetilde{W}_t \mathbb{E}[M_1 | \mathcal{F}_t]]$$

$$= \mathbb{E}[\chi_A \widetilde{W}_T M_t]$$

$$d(\widetilde{W}_T M_t) = M_t(dW_t - h_t dt)$$

$$+ \widetilde{W}_T M_t h_t dW_t + d\langle \widetilde{W}, M \rangle_t$$

$$= M_t dW_t - M_t h_t dt$$

$$+ \widetilde{W}_T M_t h_t dW_t - M_t h_t dt$$

$$= M_t (1 + \widetilde{W}_T h_t) dW_t \longleftarrow \widetilde{W}M \text{ is a martingale}$$

$$\mathbb{E}[\widetilde{W}_T M_t | \mathcal{F}_s) = \widetilde{W}_s M_s$$

$$\widetilde{\mathbb{E}}[\widetilde{W}_t \chi_A] = \mathbb{E}[\chi_A \mathbb{E}[\widetilde{W}_t M_t | \mathcal{F}_s)]$$

$$= \mathbb{E}[\chi_A \widetilde{W}_s M_s]$$

Since

$$\tilde{\mathbb{E}}[\tilde{W}_t \chi_A] = \mathbb{E}[\tilde{W}_s \chi_A M_t] = \mathbb{E}[\tilde{W}_s \chi_A \mathbb{E}[M_t / \mathcal{F}_s]]$$

$$= \mathbb{E}[\chi_A \tilde{W}_s M_s],$$

we have that

$$\tilde{\mathbb{E}}(\tilde{W}_t \chi_A) = \tilde{\mathbb{E}}(\tilde{W}_s \chi_A) \Rightarrow \tilde{\mathbb{E}}(\tilde{W}_t / \mathcal{F}_s) = \tilde{W}_s$$

$$\Rightarrow \tilde{W} \text{ is a } \tilde{\mathbb{P}}\text{-martingale}$$

Similar calculation:

$$d\tilde{W}_t^2 = 2\tilde{W}_t dW_t - 2\tilde{W}_t h_t dt + dt$$

$$\tilde{\mathbb{E}}[(\tilde{W}_t^2 - t)\chi_A] = \mathbb{E}[(\tilde{W}_t^2 - t) M_t]$$

$$d((\tilde{W}_t^2 - t) M_t) = M_t (2\tilde{W}_t dW_t \cancel{-2\tilde{W}_t h_t dt} \cancel{+dt} \cancel{-dt})$$

$$+ (\tilde{W}_t^2 - t) M_t h_t dW_t$$

$$+ 2\tilde{W}_t M_t h_t dt$$

$$= M_t (2\tilde{W}_t + \tilde{W}_t^2 h_t - t h_t) dW_t$$

$$\Rightarrow \quad \widetilde{\mathbb{E}}\left((\widetilde{W}_t^2 - t)\chi_A\right) = \widetilde{\mathbb{E}}\left((\widetilde{W}_s^2 - s)\chi_A\right)$$

$$\Rightarrow \quad \widetilde{\mathbb{E}}\left(\widetilde{W}_t^2 - t \mid \mathcal{F}_s\right) = \widetilde{W}_s^2 - s$$

$\Rightarrow$ $\widehat{W}_t^2 - t$ is a $\widehat{\mathbb{P}}$ - martingale

$\widetilde{W}$ is a $\widehat{\mathbb{P}}$ - B.M.