We here consider an exit problem.

Namely, fix \( a < b \) and consider the SDE

\[
\text{d}X_t = b(X_t) \text{d}t + \sigma(X_t) \text{d}W_t
\]

\[X_0 = x_0\]

where \( x_0 \in (a, b) \). Set

\[\tau = \inf \{ t \geq 0 : X_t \notin (a, b) \}\]

We are interested in

\[
\mathbb{P} \{ \tau < \infty \} = 0
\]

\[
\mathbb{P} \{ X_\tau = a, \tau < \infty \} = 0
\]

\[
\mathbb{P} \{ X_\tau = b, \tau < \infty \} = 0
\]

Let's start our calculations on \( [a, \beta) \subset (a, b) \) (i.e. \( a < \alpha < \beta < b \)). Then \( \sigma \) is bounded from below on \([a, \beta)\).

Set

\[\tau' = \inf \{ t > 0 : X_t \notin (a, b) \}\]
First step: exit time; $X$ must exit $[a, b]$

Suppose that we have a bounded solution of

$$\frac{1}{2} \sigma^2(x) u''(x) + b(x) u'(x) = 1$$

Then

$$d u(X_t) = u'(X_t) b(X_t) dt + u'(X_t) \sigma(X_t) dW_t + \frac{1}{2} u''(X_t) \sigma^2(X_t) dt$$

= $dt + d W_t$

i.e. $u(X_t) = u(X_0) + t + W_t$

Thus $T' \wedge L$ is a bounded stopping time; we also assume that

$u$ is bounded on $[a, b]$, which we will check below

$$|E [ T' \wedge L ] | = |E [ u(X_{T' \wedge L})] - u(x) - E [ M_{T' \wedge L} ] |$$

$$\leq 2 \sup_{x \in [a, b]} |u(x)|$$

$\xrightarrow{L} \sup \Rightarrow |E [ T' \wedge L ] | \leq 2 \sup_{x \in [a, b]} |u(x)|$
Back to PDE

\[ \sigma^2(x) u''(x) + b(x) u'(x) = 1 \]

\[ u''(x) + \frac{2 b(x)}{\sigma^2(x)} u'(x) = \frac{2}{\sigma^2(x)} \]

Integrating factor:

\[ \zeta(x) = \exp \left[ \int_{x_0}^{x} \frac{2 b(z)}{\sigma^2(z)} \, dz \right] \]

Then

\[ \zeta'(x) = \frac{2 b(x)}{\sigma^2(x)} \zeta(x) \]

\[ (u' \zeta)'(x) = u''(x) \zeta(x) + u'(x) \zeta'(x) \]

\[ = \zeta(x) \left\{ u''(x) + \frac{2 b(x)}{\sigma^2(x)} u'(x) \right\} \]

\[ = \frac{2 \zeta(x)}{\sigma^2(x)} \]

Thus

\[ u'(x) \zeta(x) = 2 \int_{x_0}^{x} \frac{\zeta(z)}{\sigma^2(z)} \, dz + \text{set } = 0; \text{ we want a solution} \]

\[ u'(x) = 2 \frac{1}{\zeta(x)} \int_{x_0}^{x} \frac{\zeta(z)}{\sigma^2(z)} \, dz \]
\[ u(x) = 2 \int_{x_0}^{x} \frac{1}{\beta(w)} \int_{x_0}^{w} \frac{\beta(z)}{\sigma^2(z)} \, dz \, dw + \int_{x_0}^{x} \frac{1}{\beta(w)} \int_{x_0}^{w} \frac{\beta(z)}{\sigma^2(z)} \, dz \, dw \]

\[ \text{set} = 0 \]

Explicitly check that

\[ \sup_{\alpha \leq x \leq \beta} |u(x)| < \infty \]

(\text{and} \quad \sup_{\alpha \leq x \leq \beta} |\dot{u}(x)| < \infty).

---

**Second Step**space

Solve

\[ \frac{1}{2} \sigma^2(x) u''(x) + b(x) u'(x) = 0 \]

\[ u(a) = 1 \quad u(b) = 0 \]

Then

\[ \Delta u(x_t) = \Delta M_t \]

\[ u(\chi_{T_L}') = u(x_0) + M_{T_L} \]
\[ u(x_0) = \mathbb{E}[u(x_{T_x})] \xrightarrow{L} \mathbb{E}[u(x_{T_x})] \]

\[ = \mathbb{P} \{ X_{T_x} = \beta \} \]

\[ \text{if } u \text{ is bounded} \]

\[ u''(x) + \frac{2b(x)}{\sigma^2(x)} u'(x) = 0 \]

\[ (u e^{\frac{b}{\sigma^2}})'(x) = 0 \]

\[ u'(x) e^{\frac{b}{\sigma^2}(x)} = C, \]

\[ u'(x) = C e^{-\frac{b}{\sigma^2}(x)} \]

\[ u(x) = C \int_x^\infty e^{-\frac{b}{\sigma^2}(z)} \, dz \]

\[ u(x) = \frac{\int_x^\infty e^{-\frac{b}{\sigma^2}(z)} \, dz}{\int_{x_0}^\infty e^{-\frac{b}{\sigma^2}(z)} \, dz} \text{ ~ check: } u \text{ is bounded} \]

Set

\[ P(x) = \int_{x_0}^x e^{\frac{b}{\sigma^2}(z)} \, dz = \int_{x_0}^x e^{\frac{b}{\sigma^2}(z)} \left[ - \int_{x_0}^z \frac{2b(u)}{\sigma^2(u)} \, du \right] \, dz, \]

then

\[ u(x) = \frac{P(x) - P(x_0)}{P(\beta) - P(\alpha)} \]
\[
\mathbb{P}_\beta X_{\tau_\alpha} = \beta \beta = \frac{\mathbb{P}(x_\alpha) - \mathbb{P}(\alpha)}{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}
\]

\[
\mathbb{P}_\beta X_{\tau_\alpha} = \alpha \beta = \frac{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}
\]

**Interpretation**

\[
\mathbb{P}_\beta X \text{ hits } \beta \text{ before } \alpha \beta = \frac{\mathbb{P}(\gamma) - \mathbb{P}(\alpha)}{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}
\]

\[
\mathbb{P}_\beta X \text{ hits } \alpha \text{ before } \beta \beta = \frac{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}
\]

**If** \( \mathbb{P}(\alpha) > -\infty \text{ and } \mathbb{P}(\beta) < \infty \)

**Let** \( \alpha < a, \beta > b \)

\[
\mathbb{P}_\beta X \text{ hits } b \text{ before } a \beta = \frac{\mathbb{P}(\beta) - \mathbb{P}(\alpha)}{\mathbb{P}(b) - \mathbb{P}(\alpha)}
\]

\[
\mathbb{P}_\beta X \text{ hits } a \text{ before } b \beta = \frac{\mathbb{P}(b) - \mathbb{P}(\alpha)}{\mathbb{P}(b) - \mathbb{P}(\alpha)}
\]
\[ P \mathcal{E} (T < \omega^3 \cap \mathcal{E} \text{ hits } b \text{ before } a) \]
\[ + P \mathcal{E} (\text{ hits } a \text{ before } b) = 1 \]

If \( P(a) = -\infty \) and \( P(b) = \infty \), then \( a \top a \)

\[ P \mathcal{E} x \text{ hits } b \text{ before } a \ 3 = 1 \]
\[ P \mathcal{E} x \text{ hits } a \text{ before } b \ 3 = 0 \]
\[ \beta \top \beta \]
\[ P \mathcal{E} x \text{ hits } b \text{ before } a \ 3 = 1 \]
\[ P \mathcal{E} x \text{ hits } a \text{ before } b \ 3 = 0 \]
\[ P \mathcal{E} T < \omega^3 = 1 \]

If \( P(a) = -\infty \), \( P(b) = \infty \), \( \top \top \top \top \)

\[ P \mathcal{E} x \text{ hits } a \text{ before } b \ 3 = 1 \]
\[ P \mathcal{E} x \text{ hits } b \text{ before } a \ 3 = 0 \]
\[ P \mathcal{E} T < \omega^3 = 1 \]
If \( P(a) = -\infty \), \( P(b) = \infty \),

\[
\frac{P(\infty) - P(a)}{P(b) - P(a)} = \mathbb{P}\{X \text{ hits } b \text{ before } a\}
\]

2. \( \mathbb{P}\{\sup_{0 \leq t \leq T} X_t = b\} = 1 \)

3. \( \mathbb{P}\{\sup_{0 \leq t \leq T} X_t > b\} = 0 \)

4. \( \mathbb{P}\{\inf_{0 \leq t \leq T} X_t = a\} = 1 \)

5. \( \mathbb{P}\{\inf_{0 \leq t \leq T} X_t > a\} = 0 \)

Thus,

6. \( \mathbb{P}\{\sup_{0 \leq t \leq T} X_t < b\} = \mathbb{P}\{\inf_{0 \leq t \leq T} X_t > a\} = 0 \)

\( \mathbb{P}\{T < \infty\} = 0 \)
Thus with probability one

\[ T = \infty, \]

\[ \inf_{t \to \infty} X_t = a \]

\[ \sup_{t \to \infty} X_t = b \]