Strong Markov Property

Note Title 9/30/2005

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Definition A random variable $T : \Omega \to \{0, 1, 2 \ldots\}$ is called a stopping time if, for each $n \geq 0$, $\{T \leq n\}$ is measurable $X_0, X_1, \ldots X_n$

i.e. $\{T \leq n\} = \{(X_0, X_1 \sim X_n) \in A\}$ for some $A \subset I^n$

Example H^A

$$\{H^A \leq 5\} = \bigcup_{j=0}^{5} \{X_j \in A\}$$

$I \times I \times A \times I \ldots$ jth position

$$= \bigcup_{j=0}^{5} \{(X_0 \sim X_5) \in \hat{A}_j\}$$

$$= \{(X_0 \sim X_5) \in \bigcup_{j=0}^{5} A_j\}$$

The basic idea of a stopping time is that you can decide to "stop" now based on what you know up to now.

Idea of stopping times vs. non-stopping times:

Drive

τ = time you hit first intersection

ρ = time you hit last intersection before hitting I-74

τ depends on information up to present

ρ depends on future; you need to see into future to know that it is the last

more later

Further examples of stopping times

- Hitting times after stopping times

  Assume τ_1 is a stopping time. Set

  $$T_2 = \inf\{u \geq T_1 : X_u \in A\}$$ show that this is a stopping time

  $$\{T_2 \leq 20\} = \{T_1 \leq 20 \text{ \& } X \text{ visits } A \text{ after } T_1 \text{ but before } 20\}$$

- Constants $T \equiv 5$.

  $$\{T \leq n\} = \{5 \leq n\} = \begin{cases} \phi & \text{if } n < 5 \\ \Omega & \text{if } n \geq 5 \end{cases}$$

  either way, $\{T \leq n\}$ is measurable $(X_0 \sim X_n)$

- Minimum of two stopping times.

  For example, $T_2 = \min\{T_1, 5\}$ stopping time

What is not a stopping time

- Last hitting times

$$L_A = \sup\{n \le 10 : X_n \in A\}$$ last time to hit A before 10

$$\{L_A \le 4\} = \bigcap_{n=5}^{10} \{X_n \notin A\}$$

The value of stopping times is that we can restart Markov processes after them.

The full theorem requires some notation which allows us to manipulate the information of $(X_0, X_1 \sim X_\tau)$, where τ is a stopping time. The problem is that

$(X_0, X_1 \sim X_\tau)$ involves a random number of random variables.

We let $\mathcal{Q}_T$ be the collection of subsets of Ω of the form

$$B = \{X_0 \in A_0, X_1 \in A_1 \cdots X_{T \wedge n} \in A_n\}$$

for some subsets $A_0, A_1 \cdots A_n$ of I. One can show that $\mathcal{Q}_T \subset \mathcal{F}$.

<u>Theorem</u> Assume X is Markov(λ, P) & T is a stopping time. Conditional on $T < \infty$ & $X_T = i$,

$X_{T(\omega)}(\omega)$

$\hat{X}^T$ is Markov(δ_i, P) & independent of any $B \in \mathcal{Q}_T$

$\hat{X}^T_n = X_{T+n}$

Pf

$$\mathbb{P}\{\hat{X}_0^T = i_0, \hat{X}_1^T = i_1, \sim \hat{X}_m^T = i_m, B \mid X_T = i, T<\infty\}$$

$$= \frac{\mathbb{P}\{\hat{X}_0^T = i_0, \hat{X}_1^T = i_1, \sim \hat{X}_m^T = i_m, B, X_T = i, T<\infty\}}{\mathbb{P}\{X_T = i, T<\infty\}} \quad S\mathcal{A}$$

$$\mathcal{A} = \sum_{k=0}^{\infty} \mathbb{P}\{\underbrace{\hat{X}_0^k = i_0, \hat{X}_1^k = i_1, \sim \hat{X}_m^k = i_m}_{A_1}, \underbrace{B \cap \{T=k\}}_{A_2}, \underbrace{X_k = i}_{A_3}\}$$

(1) $B \cap \{T = k\}$

$$= \{X_0 \in A_0, X_1 \in A_1 \sim \underbrace{X_{k\wedge n}}_{k \wedge n \le k} \in A_n\} \cap \Big(\underbrace{\{T \le k\}}_{X_0 \sim X_k\text{-meas}} \setminus \underbrace{\{T \le k-1\}}_{X_0 \sim X_{k-1}\text{-meas}}\Big)$$

$$\Rightarrow B \cap \{T = k\} = \{(X_0, X_1 \sim X_k) \in \hat{A}\}$$

(2) Regular Markov property tells us that

$$\mathbb{P}(A_1 \cap A_2 \mid A_3) = \mathbb{P}(A_1 \mid A_3)\,\mathbb{P}(A_2 \mid A_3)$$

i.e.

$$\frac{\mathbb{P}(A_1 \cap A_2 \cap A_3)}{\cancel{\mathbb{P}(A_3)}} = \frac{\mathbb{P}(A_1 \cap A_3)}{\cancel{\mathbb{P}(A_3)}} \; \frac{\mathbb{P}(A_2 \cap A_3)}{\mathbb{P}(A_3)}$$

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1 \mid A_3)\,\mathbb{P}(A_2 \cap A_3)$$

$$\star = \sum_{k=0}^{\infty} \delta_{i i_0} P_{i_0 i_1} \sim P_{i_{m-1} i_m} \mathbb{P}(B \cap \{T=k\}, X_k = i)$$

$$= \delta_{i i_0} P_{i_0 i_1} \sim P_{i_{m-1} i_m} \mathbb{P}(B, T<\infty, X_T = i)$$

$$\mathbb{P}\{\hat{X}^T_0 = i_0,\ \hat{X}^T_1 = i_1 \sim \hat{X}^T_m = i_m, B \mid X_T = i, T<\infty\}$$

$$= \delta_{i i_0} P_{i_0 i_1} \sim P_{i_{m-1} i_m} \mathbb{P}(B \mid T<\infty, X_T = i)$$

1 Let $B = \Omega$; then we get that conditionally

on $X_T = i$, $T<\infty$, $\hat{X}^T$ is Markov(δ_i, P);

$$\mathbb{P}\{\hat{X}^T_0 = i_0, \hat{X}^T_1 = i_1 \sim \hat{X}^T_m = i_m \mid X_T = i, T<\infty\}$$

$$= \delta_{i i_0} P_{i_0 i_1} \sim P_{i_{m-1} i_m}$$

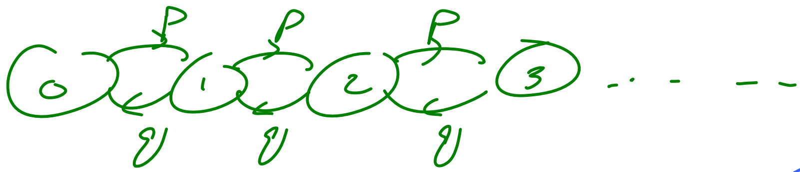
2 Thus $\mathbb{P}\{\hat{X}^T_0 = i_0, \hat{X}^T_1 = i_1 \sim \hat{X}^T_m = i_m, B \mid X_T = i, T<\infty)$

conditional independence $\left\{ \begin{aligned} &= \mathbb{P}\{\hat{X}^T_0 = i_0 \sim \hat{X}^T_m = i_m \mid X_T = i, T<\infty\} \\ &\times \mathbb{P}(B \mid X_T = i, T<\infty) \end{aligned} \right.$

$\boxed{\Sigma}$

We can sometimes compute distributions of hitting times.

Example Gambler's Ruin



$H = H^{\{0\}}$ Compute $\phi(\alpha) = \mathbb{E}_1[e^{-\alpha H}]$

by Widder's Uniqueness thm, this uniquely characterizes the law of H

$$\phi(\alpha) = \mathbb{E}_1[e^{-\alpha H} | X_1 = 0]\, q + \mathbb{E}_1[e^{-\alpha H} | X_1 = 2]\, p$$

$$\mathbb{E}_1[e^{-\alpha H} | X_1 = 0] = e^{-\alpha}$$

$H = 1$ on $\{X_1 = 0\}$

If $X_1 = 2$, then $H \geq 1$ (in fact it takes at least 3 steps to go to 0). Thus

$$\mathbb{E}_1[e^{-\alpha H} | X_1 = 2] = \mathbb{E}_1[e^{-\alpha H} \chi_{\{H \geq 1\}} | X_1 = 2]$$

$$= e^{-\alpha}\, \mathbb{E}_1[e^{-\alpha (H-1)} \chi_{\{H-1 \geq 0\}} | X_1 = 2]$$

For $k \geq 0$,

$$\mathbb{P}_1\{H-1=k \mid X_1=2\}$$

$$= \mathbb{P}_1\{H=k+1 \mid X_1=2\}$$

$$= \mathbb{P}_1\{X_1>0, X_2>0 \sim X_{k-1}>0, X_k=0 \mid X_1=2\}$$

$$= \mathbb{P}_1\{\hat{X}_0>0, \hat{X}_1>0 \sim \hat{X}_{k-1}>0, \hat{X}_k=0 \mid X_1=2\}$$

$$= \mathbb{P}_2\{H=k\}$$

Thus

$$\mathbb{E}_1[e^{-\alpha H} \mid X_1=2) = \mathbb{E}_1[e^{-\alpha H}\chi_{\{H\geq 1\}} \mid X_1=2]$$

$$= e^{-\alpha}\,\mathbb{E}_1[e^{-\alpha(H-1)}\chi_{\{H-1\geq 0\}} \mid X_1=2]$$

$$= e^{-\alpha}\,\mathbb{E}_2[e^{-\alpha H}\chi_{\{H\geq 0\}}] = e^{-\alpha}\,\mathbb{E}_2[e^{-\alpha H}]$$

Now define

$$T = \inf\{u \geq 0 : X_u = 1\}.$$

for $k \geq j$,

$$\mathbb{P}\{M=k,\ T=j\}$$

$$= \mathbb{P}\{X_{l_1} > 1 \text{ for } 0 \leq l_1 < j,\ X_j = 1,$$

$$X_{l_2} > 0 \text{ for } j \leq l_2 < k,\ X_k = 0\}$$

$$= \mathbb{P}_2\{\hat{X}^j_{l_2} > 0 \text{ for } 0 \leq l_2 < k-j,\ \hat{X}^j_{k-j} = 0,$$

$$X_{l_1} > 1 \text{ for } 0 \leq l_1 < j \mid X_j = 1\}\,\mathbb{P}_2\{X_j = 1\}$$

$$= \mathbb{P}_1\{X_{l_2} > 0 \text{ for } 0 \leq l < k-j,\ X_{k-j} = 0\}$$

$$\times\ \mathbb{P}_2\{X_l > 1 \text{ for } 0 \leq l < j,\ X_j = 1\}$$

$$= \mathbb{P}_1\{M = k-j\}$$

by spatial homogeneity,

$$\mathbb{P}_2\{X_\ell > 1 \text{ for } 0 \le \ell < j,\ X_j = 1\}$$

$$= \mathbb{P}_1\{X_\ell > 0 \text{ for } 0 \le \ell < j,\ X_j = 0\}$$

$$= \mathbb{P}_1\{H = j\}$$

$$\mathbb{P}_2\{H = k,\ T = j\} = \mathbb{P}_1\{H = j\}\mathbb{P}_1\{H = k - j\}$$

$$\mathbb{E}_2[e^{-\alpha H}] = \sum_{k \ge j \ge 0} e^{-\alpha k}\mathbb{P}_2\{H = k, T = j\}$$

$$= \sum_{k \ge j \ge 0} e^{-\alpha(k-j)} e^{-\alpha j}\mathbb{P}_1\{H = k - j\}\,\mathbb{P}_1\{H = j\}$$

$$= \left(\mathbb{E}_1[e^{-\alpha H}]\right)^2$$

In fact the above arguments can be extended to show that $H = T + (H - T)$, where T & $H - T$ are independent & identically distributed.

Thus

$$\phi(\alpha) = e^{-\alpha} q + e^{-\alpha} p \phi^2(\alpha)$$

Hence

$$\phi(\alpha) = \frac{1 \pm \sqrt{1 - 4e^{-2\alpha} p q}}{2e^{-\alpha} p} = \frac{1 \pm \sqrt{1 - 4e^{-2\alpha} p(1-p)}}{2e^{-\alpha} p}$$

- ϕ is continuous on $(0,\infty)$ (dominated convergence)
- $\phi(\alpha) = \mathbb{E}[e^{-\alpha H} \chi_{\{H < \infty\}}]$; by dominated convergence,

$$\lim_{\alpha \to \infty} \phi(\alpha) = 0$$

Thus, we take $-$ root;

$$\phi(\alpha) = \frac{1 - \sqrt{1 - 4e^{-2\alpha} p(1-p)}}{2e^{-\alpha} p}$$

$$\mathbb{P}_1\{H < \infty\} = \lim_{\alpha \downarrow 0} \phi(\alpha) = \frac{1 - \sqrt{1 - 4p(1-p)}}{2p}$$

$$= \frac{1 - \sqrt{1 - 4p + 4p^2}}{2p} = \frac{1 - \sqrt{(1-2p)^2}}{2p}$$

$$= \frac{1 - |1-2p|}{2p} = \begin{cases} \frac{1-(1-2p)}{2p} & \text{if } 2p \le 1 \\ \frac{1-(2p-1)}{2p} & \text{if } 2p > 1 \end{cases}$$

$$= \begin{cases} 1 & \text{if } p \le 1/2 \\ \frac{q}{p} & \text{if } p < 1/2 \end{cases}$$

Embedded Markov Chain

$$T_0 = 0$$

$$T_{n+1} = \inf\{u > T_n : X_u \neq X_{T_n}\}$$

first time X moves after T_n

$$Y_n = X_{T_n} \sim \text{embedded Markov chain}$$

It will be useful to jointly understand T_{n+1} & Y_{n+1}.

Fix $\alpha > 0$ & $f : S \to \mathbb{R}_+$. The by Strong Markov

$$\mathbb{E}[e^{-\alpha(T_{n+1}-T_n)} f(Y_{n+1}) \mid Y_0 = x_0 \sim Y_{n-1} = x_{n-1}, Y_n = x]$$

$$= \mathbb{E}_x[e^{-\alpha T_1} f(X_{T_1}) \chi_{\{T_1 < \infty\}}] = u(x)$$

where u satisfies

$$u = e^{-\alpha} P u$$

$$= \mathbb{P}_i\{X_1 = X_2 = \cdots X_{k-1} = i, X_k = j\}$$ strong MP

$$= P_{ii}^{k-1} P_{ij}$$

Thus,

$$\mathbb{P}\{Y_{n+1} = j \mid Y_0 = i_0, Y_1 = i_1 \cdots Y_{n-1} = i_{n-1}, Y_n = i\}$$

$$= \sum_{k=1}^{\infty} P_{ii}^{k-1} P_{ij} = \frac{P_{ij}}{1-P_{ii}}$$

Thus Y is a Markov process with transition matrix

$$\hat{P}_{ij} = \begin{cases} 0 & \text{if } j = i \\ \dfrac{P_{ij}}{1-P_{ii}} & \text{if } j \neq i \end{cases}$$

Note ☆ says that

$$\mathbb{P}_i\{Y_1 = j, T = k\} = p_{ii}^{k-1} p_{ij}$$

$$\mathbb{P}_i\{Y_1 = j\} = \frac{p_{ij}}{1-p_{ii}} = \mathbb{P}_i\{X_1 = j \mid X_1 \neq i\}$$

check!

$$\mathbb{P}_i\{T = k\} = \sum_{j \neq i} p_{ii}^{k-1} p_{ij} = p_{ii}^{k-1}(1-p_{ii})$$

i.e.

$$\mathbb{P}_i\{Y_1 = j, T = k\} = \frac{p_{ij}}{1-p_{ii}} \, p_{ii}^{k-1}(1-p_{ii})$$

$$= \mathbb{P}_i\{Y_1 = j\}\,\mathbb{P}_i\{T = k\}$$

Y_1 & T are independent;

T is geometric p_{ii}

$\mathbb{P}_i\{Y_1 = j\} = \mathbb{P}\{X_1 = j \mid X_1 \neq i\}$

Shift Operator

Minimal measurable space for Markov chain

$\Omega = S^{\mathbb{Z}_+}$ ← space of sequences

$\mathcal{F} = \mathcal{B}(S^{\mathbb{Z}_+})$ = smallest σ-algebra containing open rectangle sets

Note: probability of any specific sequence is $\emptyset$ (in most cases)

Random variables defining Markov chain are coordinate random variables

$$X_n(\omega) = \omega_n \quad \text{for } \omega = (\omega_0, \omega_1, \text{———})$$

Natural shift operators on Ω: $\theta_n : \Omega \to \Omega$ is given by

$$(\theta_n(\omega))_j = \omega_{n+j} \quad \text{for } \omega = (\omega_0, \omega_1, \sim)$$

In other words,

$$\theta_2(5, 7, 9, 6, 1, 4, 3 \ldots)$$

$$= (9, 6, 1, 4, 3 \cdot \text{———})$$

$$(\theta_2(\omega))_4 = 3 = \omega_6 = \omega_{2+4}$$

Then

$$\hat{X}^n_k(\omega) = X_k(\theta_n(\omega)) = (X_k \circ \theta_n)(\omega)$$

Markov property is

$$\mathbb{P}\{\hat{X}^n_0 = j_0 \sim \hat{X}^n_k = j_k \mid X_0 = i_0 \sim X_n = i\}$$

$$= \mathbb{P}_i\{X_0 = j_0 \sim X_k = j_k\}$$

which can be written as

$$\mathbb{P}\{X_0 \circ \theta_n = j_0 \sim X_k \circ \theta_n = j_k \mid X_0 = i_0 \sim X_n = i\}$$

$$= \mathbb{P}_i\{X_0 = j_0 \sim X_k = j_k\}$$

and writing this as expectations, we get that

$$\mathbb{E}[F \circ \theta_n \mid X_0 = i_0, \ldots, X_n = i] = \mathbb{E}_i[F]$$

for $F : \Omega \to \mathbb{R}$ bounded & measurable

Strong Markov property is similar:

$$\mathbb{E}[F \circ \theta_\tau \mid B, X_\tau = i] = \mathbb{E}_i[F]$$

$B \in \mathcal{G}_T$

Value of shift operator is that it simplifies things like hitting times after hitting times.

$$\tau(\omega) = \inf\{n \geq 0 : X_n(\omega) \neq X_0(\omega)\}$$

$$\left.\begin{aligned} T_0 &= 0 \\ T_{n+1}(\omega) &= T_n(\omega) + (\tau \circ \theta_{T_n})(\omega) \end{aligned}\right\} \text{times Markov chain moves}$$