Recurrence & Transience

Note Title 10/11/2005

As usual, X is Markov (λ, P).

<u>Maxim</u> A drunk man always finds his way home.

A lost bird is lost forever.

<u>Definition:</u> A state $i \in S$ is <u>recurrent</u> if

$$\mathbb{P}_i \{ X \text{ returns to } i \text{ infinitely often} \} = 1$$

initial state is i

A state $i \in S$ is <u>transient</u> if

$$\mathbb{P}_i \{ X \text{ returns to } i \text{ infinitely often} \} = 0$$

ie with prob. 1, visits only a finite # of times

We shall see that random walks in dimension ≥ 3 are transient, while those in dimension 1 or 2 are recurrent.

Also, biased random walk on line is transient.

Obvious quantitative alternate characterization based on # of visits:

$$V_i = \sum_{n=0}^{\infty} \chi_{\{X_n = i\}}$$

i transient $\iff \mathbb{P}_i\{V_i < \infty\} = 1 \iff \mathbb{P}_i\{V_i = \infty\} = 0$

i recurrent $\iff \mathbb{P}_i\{V_i = \infty\} = 1 \iff \mathbb{P}_i\{V_i < \infty\} = 0$

Let's break things up based upon return times to i.

$T^i = \inf\{n \geq 1 : X_n = i\}$ ← stopping time

$T_0^i = 0$

$T_{n+1}^i = T_n^i + T_0^i \circ \theta_{T_n} = \inf\{\cancel{t} \geq T_n + 1 : X_n = i\}$

Let's connect transience/recurrence to statistics of T.

Idea: If $\mathbb{P}_i\{T < \infty\} = 1$, it returns. Restart. It returns again. And again - - - -

If $\mathbb{P}_i\{T^i < \infty\} < 1$, there is a probability of X getting "lost". Compare it to a coin flip. $H = \{\text{lost}\}$, $T = \{\text{return}\}$. Sooner or later, it will get lost.

Claim $\mathbb{P}_i\{V_i \geq n\} = (\mathbb{P}_i\{T^i < \infty\})^{n-1}$ for $n = 1, 2 -$

Pf

$V_i \geq n$ means that X has visited i at least n times (including at time 0). This is true iff only if $T_i^1 < \infty$. Thus

$$\mathbb{P}_i\{V_i \geq n\} = \mathbb{P}_i\{T^i_n < \infty\}$$

$$= \mathbb{P}_i\{X^{T^i_{n-1}} \text{ visits } i \mid T^i_{n-1} < \infty,\ X_{T^i_{n-1}} = i\}$$

strong Markov property

$$\times\ \underbrace{\mathbb{P}_i\{T^i_{n-1} < \infty\}}_{=\ V}$$

$$= \mathbb{P}_i\{T^i < \infty\}\, \mathbb{P}_i\{V_i \geq n-1\}.$$

$$\mathbb{P}_i\{V_i \geq n\} = \left(\mathbb{P}_i\{T^i < \infty\}\right)^{n-1} \underbrace{\mathbb{P}_i\{V_i \geq 1\}}_{=1 \text{ since } X_0 = i} \qquad \blacksquare$$

Above comments can now be made precise:

Proposition

$$i \text{ recurrent} \iff \mathbb{P}_i\{T^i < \infty\} = 1$$

$$\iff \sum_{n=0}^{\infty} p^{(n)}_{ii} = \infty$$

$$i \text{ transient} \iff \mathbb{P}_i\{T^i < \infty\} < 1$$

$$\iff \sum_{n=0}^{\infty} p^{(n)}_{ii} < \infty$$

Pf $\quad \mathbb{P}_i\{V_i = \infty\} = \lim_{n \to \infty} \mathbb{P}_i\{V_i \geq n\}$

$$= \lim_{n\to\infty} \left(\mathbb{P}_i\{T^i < \infty\}\right)^{n-1}$$

$$= \begin{cases} 1 & \text{if } \mathbb{P}_i\{T^i < \infty\} = 1 \\ 0 & \text{if } \mathbb{P}_i\{T^i < \infty\} = 0 \end{cases}$$

This gives characterization in terms of statistics of T^i.

Next, by linearity of expectation,

$$\mathbb{E}_i[V_i] = \mathbb{E}_i\left[\sum_{n=0}^{\infty} \chi_{\{X_n = i\}}\right]$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_i\{X_n = i\} = \sum_{n=0}^{\infty} p_{ii}^{(n)}$$

We then have recall trick

$$\mathbb{E}_i[V_i] = \sum_{n=0}^{\infty} \mathbb{P}_i\{V_i > n\} = \sum_{n=1}^{\infty} \mathbb{P}_i\{V_i \geq n\}$$

$$= \sum_{n=1}^{\infty} \left(\mathbb{P}_i\{T^i < \infty\}\right)^{n-1} = \sum_{n=0}^{\infty} \mathbb{P}_i\{T^i < \infty\}^n$$

$$= \begin{cases} \infty & \text{if } \mathbb{P}_i\{T^i < \infty\} = 1 \\ \dfrac{1}{1 - \mathbb{P}_i\{T^i < \infty\}} & \text{if } \mathbb{P}_i\{T^i < \infty\} < 1 \end{cases}$$

Obvious consequence: Since either $\mathbb{P}_i\{T^i<\infty\}=1$ or $\mathbb{P}_i\{T^i<\infty\} \in [0,1)$, each state is either recurrent or transient; must have

$$\mathbb{P}_i\{X \text{ returns to } i \text{ infinitely often}\} \in \{0,1\};$$

cannot have

$$0 < \mathbb{P}_i\{X \text{ returns to } i \text{ infinitely often}\} < 1$$

(Kolmogorov's 0-1 law; excursions from i are independent; infinite return is a tail event)

Recurrence / Transience is a class property

Claim: C a communicating class. All states are either recurrent or transient.

Pf $i \leftrightarrow j$. $P_{ij}^{(n)} > 0$ & $P_{ji}^{(m)} > 0$.

One way to go from i to i in $n+m+k$ steps is to go from i to j in n steps, stay there for k steps, go back to i in m steps;

$$P_{ii}^{(n+m+k)} \geq P_{ij}^{(n)} P_{jj}^{(k)} P_{ji}^{(m)}$$

j recurrent $\Rightarrow \sum_{k=0}^{\infty} P_{jj}^{(k)} = \infty$

$$\Rightarrow \sum_{k=0}^{\infty} P_{ii}^{(n+m+k)} = \infty \Rightarrow i \text{ recurrent.}$$

i transient $\Rightarrow \displaystyle\sum_{k=0}^{\infty} P_{ii}^{(n+m+k)} < \infty$

$$\Rightarrow \sum_{k=0}^{\infty} P_{jj}^{(k)} < \infty \Rightarrow j \text{ transient}$$

Next we prove:

$$\text{Recurrent Class} \underset{\text{if finite}}{\rightleftarrows} \text{Closed Class}$$

Thm Recurrent Class $\Rightarrow$ Closed Class

Pf $i \in C$ ~ Recurrent Class. $i \to j$.

$$1 = \mathbb{P}_i\{X \text{ visits } i \text{ infinitely often}\}$$

$$= \mathbb{P}_i\{X \text{ visits } i \text{ infinitely often}, X_m = j\}$$

$$+ \mathbb{P}_i\{X \text{ visits } i \text{ infinitely often}, X_m \neq j\}$$

$$\le \mathbb{P}_i\{X \text{ visits } i \text{ after } m,\ X_m = j\}$$

$$+ \mathbb{P}_i\{X_m \ne j\} \sim = 1 - \underbrace{\mathbb{P}_i\{X_m = j\}}_{>0} < 1$$

$$= \mathbb{P}_j\{X \text{ visits } i \text{ after } 0\}\underbrace{\mathbb{P}_i\{X_m = j\}}_{\le 1}$$

$$< \mathbb{P}_j\{X \text{ visits } i\} + 1$$

$$\Rightarrow \mathbb{P}_j\{X \text{ visits } i\} > 0. \qquad \bullet$$

<u>Thm</u> Finite closed class is recurrent

<u>Pf</u> Can't escape C. Start in i.

For some j,

$$\mathbb{P}_i\{X \text{ visits } j \text{ infinitely often}\} > 0$$

$$T = \inf\{n : X_n = j\}.$$

$0 < \mathbb{P}_i\{X \text{ visits } j \text{ infinitely often}\}$

$= \mathbb{P}_i\{X \text{ visits } j \text{ infinitely often after } T \mid T<\infty, X_T = j\}$

$\mathbb{P}_i\{T < \infty\}$

$= \mathbb{P}_j\{X \text{ visits } j \text{ infinitely often after } 0\}\mathbb{P}_i\{T<\infty\}$

$\Rightarrow \mathbb{P}_j\{X \text{ visits } j \text{ infinitely often after } 0\} > 0.$

Finally

Thm If C is a recurrent class,

$\mathbb{P}_i\{\tau_j < \infty\} = 1$ for all $i,j \in C$.

Pf C a recurrent class. Fix $i,j \in C$.

Note: By definition of communicating classes, we know that $\mathbb{P}_i\{T_j < \infty\} > 0$.
We want to show that in fact $\mathbb{P}_i\{T_j < \infty\} = 1$

By definition of communicating classes, $P_{ij}^{(m)} > 0$.
Since i is recurrent,

$$1 = \mathbb{P}_j\{X = j \text{ infinitely often}\}$$

$$= \mathbb{P}_j\{X_n = j \text{ for infinitely many } n \geq m\}$$

$$= \sum_k \mathbb{P}_j\{X_n = j \text{ for infinitely many } n \geq m, X_m = k\}$$

$$= \sum_k \mathbb{P}_k\{X_n = j \text{ for infinitely many } n \geq 0\} P_{ik}^{(m)}$$

$$\leq \sum_k \mathbb{P}_k\{T_j < \infty\} P_{ik}^{(m)}.$$

Since $\sum_k P^{(n)}_{ik} = 1$ & $P^{(n)}_{ij} > 0$,

$$\mathbb{P}_i \{T_j < \infty\} = 1.$$

(If $1 = \sum_k \alpha_k \lambda_k$, where $\lambda_k \geq 0$ & $\sum_k \lambda_k = 1$, then $\alpha_k = 1$ for all k such that $\lambda_k > 0$; check this) ●