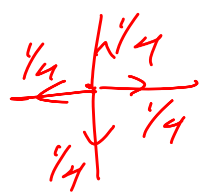
Recurrence & Transience of RW's (6)

Note Title 10/18/2005

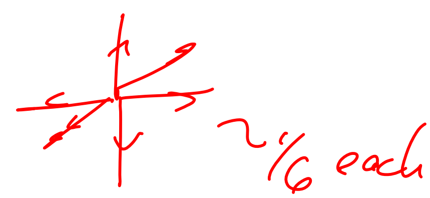
Here we want to determine recurrence or transience of random walks (RW's)

(1) RW on $\mathbb{Z}$ (possibly biased)

(2) Symmetric RW on $\mathbb{Z}^d$



$\mathbb{Z}^2$



$\mathbb{Z}^3$

Note: In these examples ($p \in (0,1)$ for RW on $\mathbb{Z}$), all of $\mathbb{Z}^d$ is a communicating class. Since recurrence/transience is a <u>class</u> property, it suffices to check recurrence/transience of origin

Need to compute

$$\sum_{n=0}^{\infty} P_{00}^{(n)}$$

finite $\Rightarrow$ transience

infinite $\Rightarrow$ recurrence

Let's pass to the Fourier domain. If $f: \mathbb{Z}^d \to \mathbb{R}$, we define

$$\hat{f}(\theta) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}^d} f(k) \exp\left[i \langle k, \theta \rangle \right]$$

$\theta \in \mathbb{R}^d$

Then

$$f(k) = \frac{1}{(2\pi)^d} \int_{\theta \in [0, 2\pi)^d} \hat{f}(\theta) \exp\left[-i \langle k, \theta \rangle \right] d\theta$$

indeed,

$$\frac{1}{(2\pi)^d} \int_{0 \le \theta_i < 2\pi} \sum_{j \in \mathbb{Z}^d} f(j) \exp[\langle k - j, \theta \rangle] \, d\theta = f(k)$$

Here, we set

$$f(k) = P_{0k}^{(n)}$$

$$\hat{f}(\theta) = \sum_{k \in \mathbb{Z}^d} P_{0k}^{(n)} \exp(i \langle k, \theta \rangle)$$

$$= \mathbb{E}[\exp[i \langle \theta, X_n \rangle]]$$

$$P^{(n)}_{00} = \frac{1}{(2\pi)^d} \int_{0 \le \theta_i < 2\pi} \hat{f}(\theta) \exp[-i\langle 0, \theta \rangle] \, d\theta$$

$$= \frac{1}{(2\pi)^d} \int_{0 < \theta_i < 2\pi} \hat{f}(\theta) \, d\theta$$

$$\boxed{P^{(n)}_{00} = \frac{1}{(2\pi)^d} \int_{0 < \theta_i < 2\pi} \mathbb{E}[\exp(i\langle \theta, X_n \rangle)] \, d\theta}$$

note that $X_n = \sum_{1 \le j \le n} \xi_j$ where ξ_j's are iid

$$\mathbb{E}\left[\exp\left[i\langle\Theta, X_n\rangle\right]\right] = \left(\mathbb{E}\left[\exp\left[i\langle\Theta, \xi_1\rangle\right]\right]\right)^n$$

Consider first symmetric RW on $\mathbb{Z}^d$.

$$\mathbb{P}\{\xi = e_i\} = \mathbb{P}\{\xi = -e_i\} = \frac{1}{2d}$$

$$\mathbb{E}\left[\exp\left(i\langle\Theta, \xi\rangle\right)\right] = \frac{1}{2d}\sum_{i=1}^{d}\left\{e^{i\Theta_i} + e^{-i\Theta_i}\right\}$$

$$= \frac{1}{d}\sum_{i=1}^{d}\cos\Theta_i$$

$$\mathbb{E}\left[\exp\left[i\langle\Theta, \xi\rangle\right]\right]^n$$

$$= \left(\frac{1}{d}\sum_{i=1}^{d}\cos\Theta_i\right)^n$$

$$P_{00}^{(n)} = \frac{1}{(2\pi)^d} \int_{\Theta} \underbrace{\left(\frac{1}{d} \sum_{i=1}^{d} \cos \theta_i \right)^n}_{g(\theta)} d\theta$$

$$g(0) = 1$$

$$g\left(\pi/2, \pi/2 \sim \pi/2\right) = -1$$

Fix $\alpha > 0$

$$A = \left\{ \theta : |\theta_1| < \frac{1}{n^\alpha} \right\}$$

$$B = \left\{ \theta : |\theta_1 - \pi/2| < 1/n^\alpha \right\}$$

~ n large enough that $A \cap B = \phi$

$$C = [0, 2\pi)^d \setminus (A \cup B)$$

On A,

$$g(\theta) = \exp[\ln g(\theta)]$$

$$g(\theta) = 1 - \frac{1}{2\alpha}\sum_{j=1}^{d} \theta_j^2 + \mathcal{E}_3(\theta)$$

$$\ln g(\theta) = -\frac{\|\theta\|^2}{2d} + \mathcal{E}_3(\theta)$$

$$g(\theta) = \exp\left[-\frac{1}{2d}\|\theta\|^2 + \mathcal{E}_3(\theta)\right]$$

$$\int_A g^n(\theta)\,d\theta = \int_{|\theta_i| \le \frac{1}{n^\alpha}} \exp\left[-\frac{n\|\theta\|^2}{2d} + n\,\mathcal{E}_3(\theta)\right] d\theta$$

$z = \sqrt{n}\,\theta$

$$= \frac{1}{n^{d/2}} \int_{|z_1| \le n^{1/2-\alpha}} \exp\left(-\frac{\|z\|^2}{2d} + n\,\mathcal{E}_3\left(\frac{z}{\sqrt{n}}\right)\right) dz$$

$n \; n^{-3\alpha} = n^{1-3\alpha}$

want $n^{1/2-\alpha} \to \infty$

$1/2 - \alpha > 0; \quad \alpha < 1/2$

want $n^{1-3\alpha} \to 0$

$3\alpha > 1; \quad \alpha > 1/3$

$1/2 < \alpha < 1/3$

$$\simeq \frac{1}{n^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\|z\|^2}{2d}\right) dz$$

$$\approx \frac{(2\pi d)^{d/2}}{n^{d/2}}$$

Similarly, on $\mathcal{B}$,

$$g(\theta) = -\exp\left(\ln\{-g(\theta)\}\right)$$

$$g(\theta) = -1 + \frac{1}{2d}\sum_{j=1}^{d}\theta_j^2 + \mathcal{E}_3(\theta)$$

$$g(\theta) \approx -\exp\left[-\frac{1}{2d}\|\theta\|^2 + \mathcal{E}_3(\theta)\right]$$

$$\int_{\mathcal{B}} g^n(\theta)\,d\theta \approx (-1)^n \frac{(2\pi d)^{d/2}}{n^{d/2}}$$

Finally, on C, for at least one i,

$$|\theta_i| > \frac{1}{n^\alpha} \qquad |\theta_i - \pi/2| > \frac{1}{n^\alpha};$$

Thus $|\cos\theta_i| < 1 - \frac{K}{n^{2\alpha}}$. Thus

$$|g(\theta)| < 1 - \frac{K}{d n^{2\alpha}}$$

Thus

$$\int_C |g(\theta)|^n d\theta \leq \int_C \left(1 - \frac{K}{d n^{2\alpha}}\right)^n d\theta$$

$$e^{-\alpha} = \sum_j \frac{(-\alpha)^j}{j!} \geq 1-\alpha$$

$$\int_C |g(\theta)|^n d\theta \le \int_C \exp\Big(-\frac{K}{d}\underbrace{n^{1-2\alpha}}\Big)d\theta$$

$\to \infty$ since

$\alpha < 1/2$

$$\le (2\alpha)^d \exp\Big(-\frac{K}{d}n^{1-2\alpha}\Big)$$

Thus

$$P_{0,0}^{(n)} \approx \frac{K}{n^{d/2}} \quad \text{for } n \text{ large}$$

If $d/2 > 1$, then $\sum_{n=0}^{\infty} P_{0,0}^{(n)} < \infty$ so transient

If $d/2 \le 1$, the $\sum_{n=0}^{\infty} P_{0,0}^{(n)} = \infty$ so recurrent