Measure Theory

Note Title 9/23/2005

Consider a simple example

$\Omega = \mathbb{R}$

$\mathcal{F} = \mathcal{B}(\mathbb{R}) = \bigcap \mathcal{S}$ check that this is a σ-algebra

$\mathcal{S}$ a σ-algebra of subsets of $\mathbb{R}$
$\mathcal{S}$ contains open sets

Then $(\Omega, \mathcal{F})$ is a measurable space; we can put a measure on it. Note that there is not only one measure we can put on $(\Omega, \mathcal{F})$. Here are four measures:

$m_1(A) \overset{\text{df}}{=} \chi_A(0)$ this is a clever way of saying that $m_1 = \delta_0$; check it

$m_2(A) = \lambda(A)$ Lebesgue measure Lebesgue measure

$m_3(A) = \lambda(A \cap [0,1])$ Uniform $[0,1]$ measure

$$m_3(A) = \int_A \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt \Leftarrow \text{Gaussian measure}$$

Note that m_1, m_3, & m_4 are probability measures.

Next, let's define a <u>coordinate random variable</u> $X : \Omega \to \mathbb{R}$, by setting $X(\omega) = \omega$.

Under m_1, $X \equiv 0$ almost surely

Under m_3, X is Uniform $[0,1]$

Under m_4, X is $\mathcal{N}(0,1)$.

Note: When you start learning probability, you usually start by assuming a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and then defining random variables. The above thoughts are backward. We have first constructed a measurable space $(\Omega, \mathcal{F})$, then a random variable, and then a probability measure. The above thoughts make clear that, if you want to change the distribution of X, you change only $\mathbb{P}$, not $(\Omega, \mathcal{F})$, or X.

Now let's stretch a bit to consider Markov chains. The "minimal" way to construct a real-valued random variable is as above; $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $X(\omega) = \omega$ for $\omega \in \mathbb{R}$.

The "minimal" place to construct a Markov chain is on the space of S-valued sequences (S being the state space). Namely, set

$$\Omega = S^{\mathbb{N}} = S \times S \times S \text{———}$$

$$= \{ (i_0, i_1, \ldots) \; i_n \in S \}$$

$\mathcal{F} = \mathcal{B}(S^{\mathbb{N}})$. This means the following: $\mathcal{B}(S^{\mathbb{N}})$ is the smallest σ-algebra of subsets of Ω containing all subsets of Ω of the form

$$\underbrace{A_0 \times A_1 \times A_2 \cdots A_n}_{\text{finite-dimensional rectangle}} \times \mathbb{R} \times \mathbb{R} \times \text{——}$$

More accurately, $\mathcal{B}(S^{\mathbb{N}})$ is the Borel σ-algebra generated by the Tikhonov topology on $S^{\mathbb{N}}$

Here the stochastic process is

$$X_n(\omega) = \omega_n$$

Above, we constructed a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to give $X(\omega)=\omega$ a desired distribution.

Here we can do exactly the same thing to construct a Markov(λ, P) process

For any rectangle set as in 🚩, we define

$$\mathbb{P}(A_0 \times A_1 \cdots A_n \times \mathbb{R} \times \mathbb{R} \cdots)$$

$$= \sum_{\substack{i_0 \in A_0, i_1 \in A_1 \\ \cdots i_n \in A_n}} \lambda_{i_0} P_{i_0 i_1} \cdots P_{i_{n-1} i_n}$$

One has to use Kolmogorov's Extension theorem to show that $\mathbb{P}$ can be extended from rectangles to all sets in $\mathcal{F}$.