Consider a simple example

\[ \mathcal{L} = \mathbb{R} \]

\[ \mathcal{F} = \mathcal{B}(\mathbb{R}) = \bigcap \mathcal{I} \]

check that this is a σ-algebra

\[ \sigma \text{-algebra of subsets of } \mathbb{R} \]

\[ \mathcal{I} \text{ contains open sets} \]

Then \((\mathbb{R}, \mathcal{F})\) is a measurable space; we can put a measure on it. Note that there is not only one measure we can put on \((\mathbb{R}, \mathcal{F})\). Here are four measures:

\[ m_1(A) = 0 \quad \chi_A(0) \quad \text{this is a clever way of saying that } m_1 = \delta_0; \text{ check it} \]

Lebesgue measure

\[ m_2(A) = \mathcal{L}(A) \quad \text{Lebesgue measure} \]

\[ m_3(A) = \mathcal{L}(A \cap (0, 1)) \quad \text{Uniform } [0, 1] \text{ measure} \]
\[ m_3(A) = \int_A e^{-t^2/2} \, dt \quad \text{Gaussian measure} \]

Note that \( m, m_3, \) and \( m_4 \) are probability measures.

Next, let's define a coordinate random variable \( X: \mathbb{N} \to \mathbb{R} \), by setting \( X(0) = \infty \).

Under \( m \), \( X = 0 \) almost surely
Under \( m_3 \), \( X \) is Uniform [0,1]
Under \( m_4 \), \( X \) is \( N(0,1) \).

Note: When you start learning probability, you usually start by assuming a probability triple \((\Omega, \mathcal{F}, P)\) and then defining random variables. The above thoughts are backward. We have first constructed a measurable space \((\Omega, \mathcal{F})\), then a random variable, and then a probability measure. The above thoughts make clear
that if you want to change the distribution of $X$, you change only $P$, not $(P, F)$, or $X$. 
Now let's stretch a bit to consider Markov chains. The "minimal" way to construct a real-valued random variable is as above: \( (\Omega, \mathcal{F}) = (\mathbb{R}, B(\mathbb{R})) \), \( X(\omega) = \omega \) for \( \omega \in \mathbb{R} \).

The "minimal" place to construct a Markov chain is on the space of \( S \)-valued sequences (\( S \) being the state space). Namely, set
\[
\Omega = S^\omega = S \times S \times S \rightarrow = \{ (\omega_0, \omega_1, \cdots ) : \omega_n \in S \}
\]
\( \mathcal{F} = B(S^\omega) \). This means the following: \( B(S^\omega) \) is the smallest \( \sigma \)-algebra of subsets of \( \Omega \) containing all subsets of \( \Omega \) of the form
\[
A_0 \times A_1 \times A_2 \cdots \times A_n \times \mathbb{R} \times \mathbb{R} \times \cdots
\]

finite-dimensional rectangles.
More accurately, $\mathcal{B}(S^{\infty})$ is the Borel $\sigma$-algebra generated by the Tikhonov topology on $\mathbb{S}^{\infty}$

Here, the stochastic process is 

$X_{\omega}(\omega) = \omega$

Above, we constructed a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to give $X(\omega) = \omega$ a desired distribution. Here, we can do exactly the same thing to construct a Markov $(\mathbb{R}, \mathcal{P})$ process

For any rectangle set as in $\mathbb{R}$, we define

$P(A_0 \times A_1, A_2 \times \mathbb{R} \times \mathbb{R})$

$= \sum_{l_0 \in A_0, l_2 \in A_2} P(\Pi l_0, l_2 \sim \Pi l_0 l_2)$
One has to use Kolmogorov's Extension Theorem to show that \( P \) can be extended from rectangles to all sets in \( \mathcal{F} \).