Martingale Problem

Note Title 11/5/2006

Let's ask a PDE question.

Consider the recursions

$$\frac{\left\{u_n\left(\frac{j+1}{n}, \frac{k}{\sqrt{n}}\right) - u_n\left(\frac{j}{n}, \frac{k}{\sqrt{n}}\right)\right\}}{1/n} \approx \frac{\partial u}{\partial t}$$

$$= \frac{\left\{u_n\left(\frac{j}{n}, \frac{k+1}{\sqrt{n}}\right) + u_n\left(\frac{j}{n}, \frac{k-1}{\sqrt{n}}\right) - 2u_n\left(\frac{j}{n}, \frac{k}{\sqrt{n}}\right)\right\}}{2/n} \approx \frac{\partial^2 u}{\partial x^2}(\Delta x)$$

$$u_n\left(0, k/\sqrt{n}\right) = f\left(k/\sqrt{n}\right)$$

Set

$$\tilde{u}_n(t,x) := u_n\left(\frac{\lfloor tn \rfloor}{n}, \frac{\lfloor x\sqrt{n} \rfloor}{\sqrt{n}}\right)$$

ie $\quad \tilde{u}_n(t,x) = u_n\left(\frac{j}{n}, \frac{k}{\sqrt{n}}\right)$

if $\quad \frac{j}{n} \le t < \frac{j+1}{n}$,

$\frac{k}{\sqrt{n}} \le x < \frac{k+1}{\sqrt{n}}$

We should have that $\tilde{u}_n \to u$, where

u solves $\quad \frac{\partial u}{\partial t} = \Delta u$

$u(0,\cdot) = f$

How?

Weakly: For any $\varphi \in \underbrace{C_b}_{\text{bounded \& continuous}}(\mathbb{R}_+ \times \mathbb{R})$,

$$\int_{t=0}^{\infty}\int_{x\in\mathbb{R}} \tilde{u}_n(t,x)\varphi(t,x)\,dt\,dx \longrightarrow \int_{t=0}^{\infty}\int_{x\in\mathbb{R}} u(t,x)\varphi(t,x)\,dt\,dx$$

Can be a bit clever about this: for any $t \geq 0$,

$$\int_{x\in\mathbb{R}} \tilde{u}_n(t,x)\varphi(x)\,dx - \int_{x\in\mathbb{R}} \tilde{u}_n(0,x)\varphi(x)\,dx \to \int f(x)\varphi(x)\,dx$$

$$= \sum_{j=0}^{\lfloor tn\rfloor -1} \sum_{k=-\infty}^{\infty} \left\{ u_n\left(\frac{j+1}{n},\frac{k}{\sqrt{n}}\right) - u_n\left(\frac{j}{n},\frac{k}{\sqrt{n}}\right)\right\} \int_{x=k/\sqrt{n}}^{\frac{k+1}{\sqrt{n}}} \varphi(z)\,dz$$

$$= \sum_{j=0}^{\lfloor tn\rfloor} \sum_{k=-\infty}^{\infty} \left\{ u_n\left(\frac{j}{n},\frac{k+1}{\sqrt{n}}\right) + u_n\left(\frac{j}{n},\frac{k-1}{\sqrt{n}}\right) - 2u_n\left(\frac{j}{n},\frac{k}{\sqrt{n}}\right)\right\} \int_{x=k/\sqrt{n}}^{\frac{k+1}{\sqrt{n}}} \varphi(z)\,dz$$

$$= \sum_{j=0}^{\lfloor tn\rfloor} \sum_{k=-\infty}^{\infty} u_n\left(\frac{j}{n},\frac{k}{\sqrt{n}}\right) \int_{\frac{k}{\sqrt{n}}}^{\frac{k+1}{\sqrt{n}}} \left\{\varphi\left(z-\frac{1}{\sqrt{n}}\right) + \varphi\left(z+\frac{1}{\sqrt{n}}\right) - 2\varphi(z)\right\} dz$$

$$\approx \sum_{j=0}^{\lfloor tn\rfloor} \sum_{k=-\infty}^{\infty} u_n\left(\frac{j}{n},\frac{k}{\sqrt{n}}\right) \int_{k/\sqrt{n}}^{\frac{k+1}{\sqrt{n}}} \Delta\varphi(z)\,dz\,\frac{1}{n}$$

$$\approx \int_0^t \int_{\mathbb{R}} \tilde{u}_n(s,z)\,\Delta\varphi(z)\,dz\,ds$$

What we just have shown is:

a qualitatively different calculation

Assume $\{\tilde{u}_n\}_{n=1}^{\infty}$ is equicontinuous in $C([0,\infty)\times\mathbb{R})$.

Then any subsequence has a subsequence which converges; let $\tilde{u}_*$ be the limit. Then

$$\int_{x\in\mathbb{R}} \tilde{u}_*(t,x)\,\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx$$

$$\int_{s=0}^{t}\int_{x\in\mathbb{R}} \tilde{u}_*(s,x)\,\Delta\varphi(x)\,dx\,ds$$

for any $\varphi \in C_b^2(\mathbb{R})$. There is only one such $\tilde{u}_*$; the solution of

$$\frac{\partial u_*}{\partial t} = \Delta u_*$$

$$u_*(0,\cdot) = f$$

Note that we can rephrase ⚑ as follows: for any $\varphi \in C_b^2(\mathbb{R})$,

$$M_T^{\varphi} \overset{def}{=} \int_{\mathbb{R}} \varphi(t,x)\, u_*(t,x)\, dx - \int_0^t \int_{x\in\mathbb{R}} \Delta\varphi(s,x)\, u_*(s,x)\, dx\, ds$$

is constant.

We want to do something similar for Markov processes. For Markov processes we will generalize "constant" processes to be "constant mean" processes, or more accurately "having conditionally zero mean increments"

Let's start with a generalization of constant-mean processes

<u>Definition</u>: An $\mathbb{R}$-valued stochastic process M is a <u>martingale</u> with respect to the filtration generated by X if, for discrete time,

1. M_n is measurable $\overbrace{\sigma\{X_0 \cdots X_n\}}^{\mathcal{F}_n^X}$
2. $\underbrace{\mathbb{E}[M_{n+1} \mid \mathcal{F}_n^X]}_{\text{requires that } \mathbb{E}[|M_{n+1}|] < \infty} = M_n$

or, for continuous time,

1. M_t is measurable $\overbrace{\sigma\{X_s ;\, s \le t\}}^{\mathcal{F}_t^X}$
2. $\underbrace{\mathbb{E}(M_{t+s} \mid \mathcal{F}_t^X)}_{\text{requires that } \mathbb{E}(|M_t|) < \infty} = M_t$

Thus

$$\mathbb{E}[M_T] = \mathbb{E}[\mathbb{E}[M_T \mid \mathcal{F}_0]] = \mathbb{E}[M_0] \sim \text{constant mean}$$

Also,

$$\mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n^X] = \mathbb{E}[M_{n+1} \mid \mathcal{F}_n^X] - M_n = 0$$ ~ martingale increment

Let X be Markov (λ, P). For $\varphi : I \to \mathbb{R}$,

$$\varphi(X_n) - \varphi(X_0) = \sum_{k=0}^{n-1} \varphi(X_{k+1}) - \varphi(X_k)$$

$$\varphi(X_{k+1}) - \varphi(X_k) = \varphi(X_{k+1}) - \mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k] + \mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k] - \varphi(X_k)$$

Note that

$$\mathbb{E}[\{\varphi(X_{k+1}) - \mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k]\} \mid \mathcal{F}_k]$$

$$= \mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k] - \mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k] = 0$$

$$M_n^\varphi = \sum_{k=0}^{n-1} \{\varphi(X_{k+1}) - \mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k]\} + \varphi(X_0)$$

is a martingale

$$\mathbb{E}[\varphi(X_{k+1}) \mid \mathcal{F}_k] = \mathbb{E}[\varphi(X_{k+1}) \mid X_k]$$

$$= \sum_j P_{X_k j}\, \varphi(j) = (P\varphi)(X_k)$$

$$\varphi(X_n) = \sum_{k=0}^{n-1} \{P\varphi(X_k) - \varphi(X_k)\} + M_n$$

$$M_n^\varphi = \varphi(X_n) - \sum_{k=0}^{n-1} (P\varphi - \varphi)(X_k)$$

is a martingale.

<u>Note</u>: If $\hat{X}$ is any process such that

$$\tilde{M}_n^\varphi \overset{\text{def}}{=} \varphi(\hat{X}_n) - \sum_{k=0}^{n-1} (P\varphi - \varphi)(\tilde{X}_k)$$

is a martingale with respect to the filtration generated by $\hat{X}$, then

$$\tilde{M}_n^\varphi - \hat{\tilde{M}}_{n-1}^\varphi = \varphi(\hat{X}_n) - \varphi(\hat{X}_{n-1}) - (P\varphi)(\hat{X}_{n-1})$$

$$+ \varphi(\hat{X}_{n-1})$$

$$= \varphi(\hat{X}_n) - (P\varphi)(\tilde{X}_{n-1})$$

and Thus

$$0 = \mathbb{E}\left[\tilde{m}^{\varphi}_{n} - \tilde{m}^{\varphi}_{n-1} \mid \mathcal{F}^{\tilde{X}}_{n-1}\right]$$

$$= \mathbb{E}\left[\varphi(\hat{X}_n) - (P\varphi)(\hat{X}_{n-1}) \mid \hat{X}_0 \dots \hat{X}_{n-1}\right]$$

$$= \mathbb{E}\left[\varphi(\hat{X}_n) \mid \hat{X}_0 \dots \hat{X}_{n-1}\right) - (P\varphi)(\hat{X}_{n-1})$$

$$\mathbb{E}\left[\varphi(\hat{X}_n) \mid \hat{X}_0 \dots \hat{X}_{n-1}\right] = (P\varphi)(\hat{X}_{n-1})$$

If $\varphi = \chi_{i^*}$, then

fixed $i^* \in I$

$$\mathbb{P}\{\hat{X}_n = i \mid \hat{X}_0 \dots \hat{X}_{n-1}\} = (P\chi_{i^*})(\hat{X}_{n-1})$$

$$= P_{\hat{X}_{n-1} i^*}$$

Thus $\hat{X}$ is Markov P.

Let's use the Martingale Problem to make the transition to jump processes. As usual, Q is an infinitesimal generator, and $P_N = I + \frac{1}{N} Q_N$. $\tilde{X}^{(N)}$ is Markov (Ω, P_N) and $X_t^{(N)} \stackrel{\text{def}}{=} \tilde{X}^{(N)}_{\lfloor tN \rfloor}$. Then

$$\tilde{M}_n^{(N)} \stackrel{\text{def}}{=} \varphi(\tilde{X}_n^{(N)}) - \sum_{k=0}^{n-1} \underbrace{(P_N\varphi - \varphi)(X_k)}_{\frac{1}{N}(Q\varphi)(X_k)}$$

is a martingale.

$$\varphi(X_t^{(N)}) = \frac{1}{N} \sum_{k=0}^{\lfloor tN \rfloor} (Q\varphi)(\tilde{X}_k^{(N)}) + \tilde{M}_{\lfloor tN \rfloor}^{(N)}$$

$$= \underbrace{\frac{1}{N} \sum_{k=0}^{\lfloor tN \rfloor} (Q\varphi)(X_{k/N}^{(N)})}_{\approx \int_0^t (Q\varphi)(X_s^{(N)})\,ds} + M_t^{(N)}$$

Jump Process:

$$\varphi(X_t) = \int_0^t (Q\varphi)(X_s)\,ds + M_t$$

where M is a continuous-time martingale.

Note:

$$\sim \quad q_{ii} + \sum_{j \neq i} q_{ij} = 0$$

$$(Q\varphi)(i) = \sum_j q_{ij}\varphi(j) + q_{ii}\varphi(i)$$

$$= \sum_{j \neq i} q_{ij}\varphi(j) - \sum_{j \neq i} q_{ij}\varphi(i)$$

$$= \sum_{j \neq i} q_{ij}\{\varphi(j) - \varphi(i)\}$$

$$= \bar{q}_i \sum_j \frac{q_{ij}}{\bar{q}_i}\{\varphi(j) - \varphi(i)\}$$

$$= \bar{q}_i \sum_j \Pi_{ij}\{\varphi(j) - \varphi(i)\}$$

Note: If $\mathbb{P}\{X_T = i\} = g_i(t)$ ~ law of X_t

then

$$\mathbb{E}[\varphi(X_t)] = \overset{\text{row vector}}{g(t)}\,\underset{\text{column vector}}{\varphi}$$

$$\varphi(X_t) = \int_0^t (Q\varphi)(X_s)\,ds + M_t^\varphi$$

note: $M_0^\varphi = \varphi(X_0)$

$$\mathbb{E}[\varphi(X_t)] = \int_0^t \mathbb{E}[Q\varphi(X_s)]\,ds + \mathbb{E}[\varphi(X_0)]$$

$$g(t)\varphi = \int_0^t g(s)Q\varphi\,ds + g(0)\varphi$$

$$\dot{g}(t) = g(t)Q$$

$$g(t) = e^{Qt}g(0)$$

As with discrete-time processes martingale characterization gives uniqueness

Claim: If X is a stochastic process such that

$$M_t^\varphi = \varphi(X_t) - \int_0^t (Q\varphi)(X_s)\,ds$$

is a martingale, then X is Markov(Q)

Pf: Fix $\tilde{\varphi}(t,x) = e^{-\lambda t}\varphi(x)$. Then

$$\tilde{\varphi}(t, X_t) = \tilde{\varphi}(0, X_0) + \sum_j \tilde{\varphi}(t_{j+1}, X_{t_{j+1}}) - \tilde{\varphi}(t_j, X_{t_j})$$

$$= \tilde{\varphi}(0, X_0) + \sum_j \frac{\partial \tilde{\varphi}}{\partial t}(t_j, X_{t_j})(t_{j+1} - t_j)$$

$$+ \sum_j \underbrace{\tilde{\varphi}(t_j, X_{t_{j+1}}) - \tilde{\varphi}(t_j, X_{t_j})}_{\simeq (Q\tilde{\varphi})(t_j, X_{t_j}) + \text{martingale increment}}$$

$$= \int_0^t \left\{ \frac{\partial \varphi}{\partial s}(s, X_s) + (Q\varphi)(s, X_s) \right\} ds + \underbrace{M_t}_{\text{martingale}}$$

Fix $t \geq 0$. Set $\tau \overset{\text{def}}{=} \inf\{s \geq t : X_s \neq X_t\}$.

$\tau_N \overset{\text{def}}{=} \min\{\tau, N\}$.

<u>Fact</u>: If M is a martingale and $\sigma \leq \tau$ where σ and τ are bounded stopping times, then

$$\mathbb{E}(M_\tau | \mathcal{F}_\sigma) = M_\sigma \quad \text{(optional sampling)}$$

Use $\tau_N \geq t$.

$$\mathbb{E}[M^\varphi_{\tau_N} | \mathcal{F}_t] = M^\varphi_t$$

$$\mathbb{E}[M^\varphi_{\tau_N} - M^\varphi_t \mid \mathcal{F}_t] = 0$$

$$\mathbb{E}\Big[e^{-\lambda \tau_N} \varphi(X_{\tau_N}) - e^{-\lambda t} \varphi(X_t)$$

$$- \int_t^{\tau_N} \{ -\lambda \varphi(X_s) - (Q\varphi)(X_s) \} e^{-\lambda s} ds \,\Big|\, \mathcal{F}_t \Big] = 0$$

$$\mathbb{E}\left(e^{-\lambda \tau_N} \varphi(X_{\tau_N}) \mid \mathcal{F}_t \right) = e^{-\lambda t} \varphi(X_t)$$

$$+ \left\{ -\lambda \varphi(X_t) + (Q\varphi)(X_t) \right\} \left\{ \frac{e^{-\lambda t} - \mathbb{E}\left(e^{-\lambda \tau_N} \mid \mathcal{F}_t\right)}{\lambda} \right\}$$

$N \longrightarrow \infty$

$$\mathbb{E}\left[e^{-\lambda \tau} \varphi(X_\tau) \mid \mathcal{F}_t \right] = e^{-\lambda t} \varphi(X_t)$$

$$+ \left\{ -\varphi(X_t) + \frac{(Q\varphi)(X_t)}{\lambda} \right\} \left\{ e^{-\lambda t} - \mathbb{E}\left[e^{-\lambda \tau} \mid \mathcal{F}_t \right] \right\}$$

$$\mathbb{E}\left[e^{-\lambda(\tau - t)} \varphi(X_\tau) \mid \mathcal{F}_t \right] = \varphi(X_t)$$

$$+ \left\{ -\varphi(X_t) + \frac{(Q\varphi)(X_t)}{\lambda} \right\} \left\{ 1 - \mathbb{E}\left[e^{-\lambda(\tau - t)} \mid \mathcal{F}_t \right] \right\}$$

$$= \mathbb{E}\left[e^{-\lambda(\tau - t)} \varphi(X_t) \mid \mathcal{F}_t \right)$$

$$+ \frac{(Q\varphi)(X_t)}{\lambda} \left\{ 1 - \mathbb{E}\left(e^{-\lambda(\tau - t)} \mid \mathcal{F}_t \right) \right\}$$

$$\mathbb{E}[e^{-\lambda(T-t)}(\varphi(X_T)-\varphi(X_t))|\mathcal{F}_t]$$

$$= \frac{(Q\varphi)(X_t)}{\lambda}\{1-\mathbb{E}[e^{-\lambda(T-t)}|\mathcal{F}_t]\}$$

$$\mathbb{E}[e^{-\lambda(T-t)}(\varphi(X_T)-\varphi(X_t))|X_{s_0}=i_0 \sim X_{s_n}=i_n, X_s=i]$$

$$= \frac{(Q\varphi)(i)}{\lambda}\{1-\mathbb{E}[e^{-\lambda(T-t)}|X_{s_0}=i_0\sim X_{s_n}=i_n, X_s=i]\}$$

$\varphi(x)=\chi_A(x)$ where $x\notin A$.

$$\mathbb{E}[e^{-\lambda(T-t)}\chi_A(X_T)|X_{s_0}=i_0 \sim X_{s_n}=i_n, X_s=i)$$

$$= \frac{\bar{g}_i}{\lambda}\sum_{j\in A}\Pi_{ij}\{1-\mathbb{E}[e^{-\lambda(T-t)}|X_{s_0}=i_0\sim X_{s_n}=i_n, X_s=i]\}$$

1)

$A = I\setminus\{i\}$

$$\mathbb{E}[e^{-\lambda(T-t)}|X_{s_0}=i_0 \sim X_{s_n}=i_n, X_s=i)$$

$$= \frac{\bar{g}_i/\lambda}{1+\bar{g}_i/\lambda} = \frac{\bar{g}_i}{\lambda+\bar{g}_i}$$

$\Rightarrow$ $\tau - t$ is $\exp(\bar{q}_i)$

2) $A \subset I \setminus \{i\}$

$$\mathbb{E}[e^{-\lambda(\tau - t)} \chi_A(X_\tau) \mid X_{s_0} = i_0 \sim X_{s_n} = i_n, X_s = i]$$

$$= \frac{\bar{q}_i}{\lambda + \bar{q}_i} \sum_{j \in A} \Pi_{ij}$$

$\lambda \to 0$

$$\mathbb{E}[\chi_A(X_\tau) \mid X_{s_0} = i_0 \sim X_{s_n} = i_n, X_s = i]$$

$$= \sum_{j \in A} \Pi_{ij}$$

$\Rightarrow$ X_τ is dist. according to Π_i

3)

$$\mathbb{E}[e^{-\lambda(\tau - t)} \chi_A(X_\tau) \mid X_{s_0} = i_0 \sim X_{s_n} = i_n, X_s = i)$$

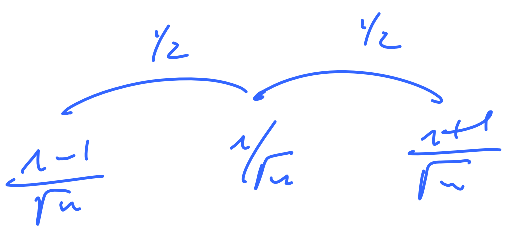
$$= \mathbb{E}[e^{-\lambda(\tau - t)} \mid X_{s_0} = i_0 \sim X_{s_n} = i_n, X_s = i)$$

$$\times \mathbb{E}[\chi_A(X_\tau) \mid X_{s_0} = i_0 \sim X_{s_n} = i_n, X_s = i]$$

$\Rightarrow$ X_T & $T-t$ are indep.

Let's now do another scaling.

$\tilde{X}^{(n)}$ is $P_{x/\sqrt{n},\, \frac{x\pm1}{\sqrt{n}}} = \frac{1}{2}$.



$$X_t^{(n)} = \tilde{X}^{(n)}_{\lfloor tn \rfloor}$$

$$\varphi(X_t^{(n)}) = \sum_{j=0}^{\lfloor tn\rfloor - 1} (P_n\varphi - \varphi)(\tilde{X}_j^{(n)}) + \tilde{M}^{(n)}_{\lfloor tn \rfloor}$$

$$(P_n\varphi)(x/\sqrt{n}) - \varphi(x/\sqrt{n})$$

$$= \frac{1}{2}\varphi\left(\frac{x+1}{\sqrt{n}}\right) + \frac{1}{2}\varphi\left(\frac{x-1}{\sqrt{n}}\right) - \varphi\left(\frac{x}{\sqrt{n}}\right)$$

$$\approx \frac{1}{2n}\varphi''(x/\sqrt{n})$$

$$\varphi(X_T^{(n)}) = \sum_{j=0}^{\lfloor tn \rfloor - 1} \frac{1}{2} \varphi''(\tilde{X}_j^{(n)}) \frac{1}{n} + \tilde{M}^{(n)}_{\lfloor tn \rfloor} + O\left(\frac{n}{n^{3/2}}\right)$$

$$= \sum_{j=0}^{\lfloor tn \rfloor - 1} \frac{1}{2} \varphi''(X^{(n)}_{j/n}) \frac{1}{n} + \tilde{M}^{(n)}_{\lfloor tn \rfloor} + O(\sqrt{n})$$

$$\approx \int_0^t \frac{1}{2} \varphi''(X_s)\, ds + M_t^{(n)}$$

Brownian Motion!

$$\varphi(X_t) = \int_0^t \frac{1}{2} \varphi''(X_s)\, ds + M_t \sim \text{martingale}$$

$$\varphi(X_t) - \varphi(X_s) - \frac{1}{2}\int_s^t \varphi''(X_r)\, dr = M_t - M_s$$

$$\varphi_\theta(x) = e^{i\theta x}$$

$$\mathbb{E}\left[e^{i\theta X_t} - e^{i\theta X_s} + \frac{\theta^2}{2} \int_s^t e^{i\theta X_r}\, dr \,\middle|\, \mathcal{F}_s \right] = 0$$

$$\mathbb{E}\left[e^{i\theta X_t} \mid \mathcal{F}_s\right] - e^{i\theta X_s} + \frac{\theta^2}{2}\int_s^t \mathbb{E}\left[e^{i\theta X_r} \mid \mathcal{F}_s\right] dr = 0$$

$$\mathbb{E}\left[e^{i\theta X_t} \mid \mathcal{F}_s\right] = e^{i\theta X_s} \exp\left[-\frac{\theta^2}{2}(t-s)\right]$$

$$\mathbb{E}\left[e^{i\theta (X_t - X_s)} \mid \mathcal{F}_s\right] = e^{-\frac{\theta^2}{2}(t-s)}$$

$X_t - X_s$ is $\mathcal{N}(0, t-s)$ & indep of $\mathcal{F}_s$.