Jump Processes

Note Title 11/10/2005

We want to start to look at continuous-time processes.

Let's start with the fact that jump times for discrete-time Markov processes are geometric

Q: How can we go from discrete to continuous with respect to the jump times.

ξ is geometric(p); ξ is position of first heads, where $\mathbb{P}\{H\} = p$.

$$\mathbb{P}\{\xi = k\} = (1-p)^{k-1} p \qquad k \in \{1, 2, \sim\}$$

Note that

$$\mathbb{E}(e^{i\theta\xi}) = \sum_{k=1}^{\infty} (1-p)^{k-1} e^{i\theta k} p$$

$$= e^{i\theta} p \sum_{k=0}^{\infty} \{(1-p)e^{i\theta}\}^{k}$$

$$= \frac{e^{i\theta} p}{1-(1-p)e^{i\theta}} \approx \frac{p(1+i\theta)}{1-(1-p)(1+i\theta)}$$

$$\approx \frac{p(1+i\theta)}{\cancel{1}-\cancel{1}+p-(1-p)i\theta} = \frac{1+i\theta}{1-\frac{1-p}{p}i\theta}$$

$$\approx (1+i\theta)\left(1+\frac{1-p}{p}i\theta\right)$$

$$\approx 1 + i\theta\left(1+\frac{1}{p}-1\right) = 1+\frac{i\theta}{p}$$

$$\mathbb{E}(i\xi) = \frac{d}{d\theta}\mathbb{E}(e^{i\theta\xi})\Big|_{\theta=0} = \frac{i}{p}$$

$$\Rightarrow \mathbb{E}(\xi) = 1/p$$

This means that if $\mathbb{P}(H) = p$, we expect to see a heads at about $1/p$ flips.

Now let's bias the coin. If $\mathbb{P}(H) = 1/60$, we expect to see a heads after 60 flips. If we flip the coin once every minute, we expect to see a heads after 1 hour. If $\mathbb{P}(H) = 1/3600$, and we flip the coin once every second, we expect to see a heads after $\frac{3600 \text{ flips}}{\frac{3600 \text{ flips}}{\text{hour}}} = 1$ hour.

In general, fix $\lambda > 0$, and for each $n \geq 0$, let ξ_n be geometric(λ/n); ξ_n is the # of flips needed for a coin (where $\mathbb{P}(H) = \lambda/n$) to come up heads. If we flip the coin n times per hour, the total time until heads is

$$\xi_n \overset{\text{def}}{=} \frac{\hat{\xi}_n \text{ flips}}{n \text{ flips}/\text{hour}} = \frac{\hat{\xi}_n}{n} \text{ hours}$$

Note that

$$\mathbb{E}(\xi_n) = \frac{1}{n}\mathbb{E}(\hat{\xi}_n) = \frac{1}{n}\frac{1}{(\lambda/n)} = \lambda.$$

More generally,

$$\mathbb{E}(e^{i\theta\xi_n}) = \mathbb{E}\left(e^{i\theta/n\,\hat{\xi}_n}\right)$$

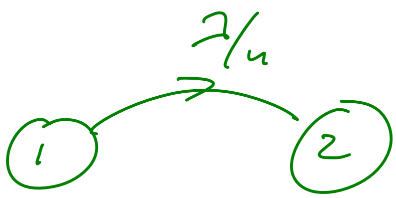
$$= \frac{e^{i\theta/n}\,\lambda/n}{1-(1-\lambda/n)e^{i\theta/n}}$$

$$\approx \frac{1+i\theta/n}{n\{1-(1-\lambda/n)(1+i\theta/n)\}}$$

$$\approx \frac{1+i\theta/n}{n\{\lambda/n - i\theta/n\}} \xrightarrow{n\to\infty} \frac{1}{\lambda - i\theta}$$

$$= \int_0^\infty e^{i\theta t} \underbrace{\lambda e^{-\lambda t} dt}_{\text{exponential}(\lambda)}$$

Let's look at this from a Markov process perspective. Consider



ie $\quad P_n = \begin{pmatrix} 1-\lambda/n & \lambda/n \\ 0 & 1 \end{pmatrix}.$

Set $\quad \tau = \inf\{ n \geq 0 : X_n \neq X_0 \}.$

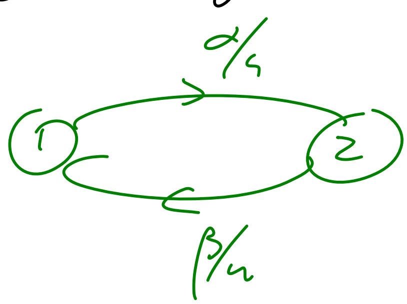
Then under $\mathbb{P}_1$ (ie start in 1), τ is geometric(λ/n); $\mathbb{E}_1(\tau) = \frac{n}{\lambda}$. In other words, we need to iterate the Markov chain n/λ times before it jumps to 2. If we speed up the

Markov chain and run it once every $1/n$th hour, we expect it to jump to ② at time

$$\frac{E_1[\tau]}{n} = \frac{(n/\lambda) \text{ iterations}}{1/n \text{ iterations/hour}} = \frac{1}{\lambda} \text{ hours.}$$

Let $\tilde{X}^{(n)}$ be Markov(δ_1, P_n), where P_n is as in $\star$. Set $X^{(n)}_t \overset{\text{def}}{=} \tilde{X}^{(n)}_{\lfloor tn \rfloor}$. Then in time $t=1$ hour, we will have run $\tilde{X}^{(n)}$ for n iterations; ie $X^{(n)}$ should jump after about 1 hour.

Let's now go back to our favorite Markov chain



$$P_n = \begin{pmatrix} 1-\alpha/n & \alpha/n \\ \beta/n & 1-\beta/n \end{pmatrix}$$

$$= I + \frac{1}{n} \underbrace{\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}}_{Q}$$

$\tilde{X}^n$ is Markov P_n

$$X^n_t = \tilde{X}^n_{\lfloor tn \rfloor}$$

$$\mathbb{P}_i\{X^n_t = j\} = \left(P_n^{\lfloor tn \rfloor}\right)_{ij}$$

$$= \left\{\left(I + \frac{1}{n}Q\right)^{\lfloor tn \rfloor}\right\}_{ij}$$

$$= \left(\Big\{ \underbrace{\left(I + \frac{1}{n}Q\right)^{n}}_{\approx e^{Q}} \Big\}^{\frac{\lfloor tn \rfloor}{n}} \right)_{ij}, \qquad \frac{\lfloor tn \rfloor}{n} \approx t$$

$$\xrightarrow{n\to\infty} \left(e^{Qt}\right)_{ij}$$

Recall that $$e^{Qt} = \underbrace{\sum_{m=0}^{\infty} \frac{Q^m t^m}{m!}}$$

converges; radius of convergence for Taylor expansion of e^x is ∞.

$$\text{Note: } \underbrace{\left(e^{Q(t+s)}\right)_{ij} = \left(e^{Qt}e^{Qs}\right)_{ij} = \sum_k \left(e^{Qt}\right)_{ik}\left(e^{Qs}\right)_{kj}}_{\text{Chapman Kolmogorov}}$$

Infinitesimally, we have

$$P_i\{X_T = j\} = \left(e^{Qt}\right)_{ij} \approx \left(I + Qt + o(t)\right)_{ij}$$

$$= \delta_{ij} + q_{ij}t + o(t)$$

$$\lim_{t\to 0} \frac{o(t)}{t} = 0$$

Also, note properties of $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$

$q_{ii} < 0$

$q_{ij} > 0$ if $j \neq i$

$\sum_j q_{ij} = 0$ for all i

Let's make some general definitions

Definition A matrix $Q = (q_{ij} : i, j \in S)$ is the infinitesimal generator of a jump Markov chain if

- $q_{ii} < 0$ for all i and $q_{ij} \geq 0$ for $j \neq i$
- $\sum_j q_{ij} = 0$ for all i

Definition An S-valued stochastic process X is said to be a jump Markov process governed by Q if

- For each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous with left-hand limits
- For $0 < t_1 \cdots t_n < t$ and $i_0, i_1 \cdots i_n, i$ in S and $j \neq i$,

$$\mathbb{P}\{X_{t+\delta} = j \mid X_0 = i_0 \cdots X_{t_n} = i_n, X_t = i\}$$

$$= q_{ij}\,\delta + o(\delta)$$

where $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$

While this infinitesimal definition is a bit unsettling, note that the ODE $\dot{x}_t = f(x_t)$ means exactly that

$$X_{t+\delta} = X_t + f(X_t)\delta + o(\delta)$$

Note also that if $|S| < \infty$,

$$\mathbb{P}\{X_{t+\delta} = i \mid X_0 = i_0 \sim X_t = i\}$$

$$= 1 - \sum_{j \neq i} \mathbb{P}\{X_{t+\delta} = j \mid X_0 = i_0 \sim X_{t_n} = i_n, X_t = i\}$$

$$= 1 - \sum_{j \neq i} q_{ij} t + o(t)$$

$$q_{ii} + \sum_{j \neq i} q_{ij} = 0$$

$$= 1 + q_{ii} t + o(t)$$

$$q_{ii} < 0$$

Secondly, this definition is enough to pin down the law of X.

Claim If $|S|<\infty$, then for $0<t_1<t_2 \dots t_n$ and $i_1 \dots i_n \in S$,

$$\mathbb{P}_i\{X_{t_1}=i_1 \dots X_{t_n}=i_n\} = (e^{Qt_1})_{i i_1} (e^{Q(t_2-t_1)})_{i_1 i_2} \dots (e^{Q(t_n-t_{n-1})})_{i_{n-1} i_n} \quad \#\#$$

Pf Fix $0<t_1 \dots t_n$ and $i_1 \dots i_n \in S$. For $j \in S$ and $t > t_n$, define

$$f_j(t) = \mathbb{P}_i\{X_t = j \mid X_{t_1}=i_1 \dots X_{t_n}=i_n\}$$

Then

$$f_j(t+\delta) = \sum_{\ell} \mathbb{P}_i\{X_{t+\delta}=j \mid X_t=\ell, X_{t_1}=i_1 \dots X_{t_n}=i_n\} \underbrace{\mathbb{P}_i\{X_t=\ell \mid X_{t_1}=i_1 \dots X_{t_n}=i_n\}}_{f_\ell(t)}$$

$$= f_j(t) + \sum_i f_i(t)\, q_{ij}\, \delta + o(\delta)$$

(use $\star$)

Thus

$$\dot{f}_j(t) = \sum_j f_i(t)\, q_{ij}$$

$$f_j(t_n^+) = \delta_{i_n j}$$ (right-continuity)

Thus

$$f_j(t) = \left(e^{Q(t-t_n)}\right)_{i_n j}$$

ie

$$\mathbb{P}\{X_t = j \,/\, X_{t_1} = i_1, \sim X_{t_n} = i_n\} = \left(e^{Q(t-t_n)}\right)_{i_n j}$$

Repeated use of this result leads to the claim. ●

How to compute e^{Qt}

Explicitly compute e^{Qt} for our two-state chain

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

eigenvalues again

idea: $Q = U \Lambda U^{-1}$

diagonal matrix of eigenvalues

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$e^{Qt} = \underbrace{U \underbrace{e^{\Lambda t}}_{= \begin{pmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{pmatrix}} U^{-1}}_{\text{linear sums of } e^{\lambda_1 t},\ e^{\lambda_2 t}}$$

$$0 = \det\begin{pmatrix}\lambda+\alpha & -\alpha\\ -\beta & \lambda+\beta\end{pmatrix} = (\lambda+\alpha)(\lambda+\beta)-\alpha\beta$$

$$= \lambda^2 + (\alpha+\beta)\lambda = \lambda(\lambda+\alpha+\beta)$$

$$\lambda = 0, \qquad \lambda = -(\alpha+\beta)$$

$Q\mathbb{1} = 0 \Rightarrow 0$ is an eigenvalue

$$g_{11}(t) = \left(e^{Qt}\right)_{11} = c + d e^{-(\alpha+\beta)t}$$

$$g_{11}(0) = (I)_{11} = 1 \quad ; \quad g_{11}(0) = c + d$$

$$\dot g_{11}(0) = Q_{11} = -\alpha \, ; \quad \dot g_{11}(0) = c - (\alpha+\beta)d$$

$\longrightarrow$ like $n=1$ calculation for discrete time

$$1 = c + d$$
$$-\alpha = c - (\alpha+\beta)d$$

$$\Longrightarrow \quad c = \frac{\beta}{1+\alpha+\beta}$$

$$d = \frac{1+\alpha}{1+\alpha+\beta}$$

$$g_{11}(t) = \frac{\beta}{1+\alpha+\beta} + \frac{1+\alpha}{1+\alpha+\beta} e^{-(\alpha+\beta)t}$$

$$g_{12}(t) = 1 - g_{11}(t) = \frac{1+\alpha}{1+\alpha+\beta} - \frac{1+\alpha}{1+\alpha+\beta} e^{-(\alpha+\beta)t}$$

Similarly,

$$g_{22}(t) = \frac{\alpha}{1+\alpha+\beta} + \frac{1+\beta}{1+\alpha+\beta} e^{-(\alpha+\beta)t}$$

$$g_{21}(t) = \frac{1+\beta}{1+\alpha+\beta} - \frac{1+\beta}{1+\alpha+\beta} e^{-(\alpha+\beta)t}$$

$$e^{Qt} = \frac{1}{1+\alpha+\beta} \begin{pmatrix} \beta & 1+\alpha \\ 1+\beta & \alpha \end{pmatrix}$$

$$+ \frac{1}{1+\alpha+\beta} \begin{pmatrix} 1+\alpha & -(1+\alpha) \\ -(1+\beta) & 1+\beta \end{pmatrix} e^{-(\alpha+\beta)t}$$

Example: p. 65

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}$$

Compute $(e^{Qt})_{11}$,

$$0 = \det(\lambda I - Q) = \det \begin{pmatrix} \lambda+2 & -1 & -1 \\ -1 & \lambda+1 & 0 \\ -2 & -1 & \lambda+3 \end{pmatrix}$$

$$= \lambda(\lambda+2)(\lambda+4)$$

$$g_{11}(t) = (e^{Qt})_{11} = a + be^{-2t} + ce^{-4t}$$

Also know that
$$g_{11}'(t) = (Qe^{Qt})_{11}$$
$$g_{11}''(t) = (Q^2 e^{Qt})_{11}$$

$g_{11}(0) = I_{11} = 1\ ;\quad g_{11}(0) = a + b + c$

$g_{11}'(0) = Q_{11} = -2\ ;\quad g_{11}'(0) = -2b - 4c$

$g_{11}''(0) = (Q^2)_{11} = 7\ ;\quad g_{11}''(0) = 4b - 16c$

$$\Longrightarrow\quad g_{11}(t) = \frac{3}{8} + \frac{1}{4}e^{-2t} + \frac{3}{8}e^{-4t}$$

Our next calculation generalizes the approximation we used for the two-state jump Markov process

Define $\quad \bar{q}_i = -q_{ii} = \sum_{j \neq i} q_{ij}$.

<u>Claim</u> Assume $|S| < \infty$. For $N > \max_i \bar{q}_i$, let $\tilde{X}^{(N)}$ be Markov P_N, where $P_N = I + Q/N$. Set $X^{(N)}_t = \tilde{X}^{(N)}_{\lfloor tN \rfloor}$; then the law of $X^{(N)}$ converges to

<u>Pf</u> Note that

$$(P_N)_{ij} = \delta_{ij} + q_{ij}/N = \begin{cases} 1 + q_{ii}/N & \text{if } j = i \\ q_{ij}/N & \text{if } j \neq i \end{cases}$$

$$= \begin{cases} \dfrac{N - \bar{q}_i}{N} & \text{if } j = i \\ q_{ij}/N & \text{if } j \neq i \end{cases}$$

The requirement that $n > \sup_i \bar{q}_i$ ensures that P_n is indeed a transition matrix (note that $\mathbb{P}_N \mathbb{1} = I\mathbb{1} + \frac{Q\mathbb{1}}{N} = I\mathbb{1} = \mathbb{1}$). Thus

$$\mathbb{P}_i \{ X^{(N)}_{t_1} = i_1, X^{(N)}_{t_2} = i_2 \cdots X^{(N)}_{t_n} = i_n \}$$

$$= \mathbb{P}_i \{ \tilde{X}^{(N)}_{\lfloor N t_1 \rfloor} = i_1, \tilde{X}^{(N)}_{\lfloor N t_2 \rfloor} = i_2 \cdots X^{(N)}_{\lfloor N t_n \rfloor} = i_n \}$$

$$= \left(P_N^{\lfloor N t_1 \rfloor} \right)_{i i_1} \left(P_N^{\lfloor N t_2 \rfloor - \lfloor N t_1 \rfloor} \right)_{i_1 i_2}$$

$$\cdots \left(P_N^{\lfloor N t_n \rfloor - \lfloor N t_{n-1} \rfloor} \right)_{i_{n-1} i_n}$$

$$= \left(\left(I + Q/N \right)^{\lfloor N t_1 \rfloor} \right)_{i i_1} \left(\left(I + Q/N \right)^{\lfloor N t_2 \rfloor - \lfloor N t_1 \rfloor} \right)_{i_1 i_2}$$

$$\cdots \left(I + Q/N \right)^{\lfloor N t_n \rfloor - \lfloor N t_{n-1} \rfloor}_{i_{n-1} i_n}$$

$$\xrightarrow{N\to\infty} \left(e^{Qt_1}\right)_{i i_1} \left(e^{Q(t_2-t_1)}\right)_{i_1 i_2} \cdots \left(e^{Q(t_n-t_{n-1})}\right)_{i_{n-1} i_n} \quad \bullet$$

Let's now look at $X^{(n)}$ in its own right.

Set $\tilde\tau_n = \inf\{k \geq 0 : \tilde X^{(n)}_k \neq \tilde X^{(n)}_0\}$, and

$$\tau_n = \inf\{t \geq 0 : X^n_t \neq X_0\} = \inf\left\{\frac{k}{n} : X^n_{k/n} \neq X_0\right\}$$

$$= \inf\left\{\frac{k}{n} : \tilde X^{(n)}_k \neq \tilde X^{(n)}_0\right\} = \frac{\tilde\tau_n}{n}$$

We recall that $\tilde\tau_n$ & $\tilde X^{(n)}_{\tilde\tau_n}$ are independent, and

$\tilde\tau_n$ is geometric$(1-(P_n)_{ii})$ and $\mathbb{P}_i\{\tilde X^{(n)}_{\tilde\tau_n} = \underset{j \neq i}{j}\} = \dfrac{(P_n)_{ij}}{1-(P_n)_{ii}}$

Since $\underbrace{(P_n)_{ii} = 1 + q_{ii}/n}$ and $(P_n)_{ij} = q_{ij}/n$,

$$1-(P_n)_{ii} = -q_{ii}/n = \bar q_i/n$$

$\tilde{T}_n$ is geometric $\bar{q}_i/n$ and $\mathbb{P}_i\{\tilde{X}^{(n)}_{\tilde{T}_n} = j\} = \dfrac{q_{ij}}{\bar{q}_i}$.

Since $T_n = \dfrac{\tilde{T}_n}{n}$, $T_n \to \text{exponential}(\bar{q}_i)$. Set

$$\Pi_{ij} = \begin{cases} q_{ij}/\bar{q}_i & , \text{if } j \neq i,\ \bar{q}_i \neq 0 \\ 0 & , \text{if } j = i,\ \bar{q}_i \neq 0 \\ 0 & , \text{if } j \neq i,\ \bar{q}_i = 0 \\ 1 & , \text{if } j = i,\ \bar{q}_i = 0 \end{cases}$$

Jump Matrix

consistent; if $\bar{q}_i = 0$, $E[\text{waiting time}] = \infty$; Process stays at i

<u>Lemma</u> Let $\{\xi_1, \xi_2 \,\text{——}\}$ be iid $\exp(1)$.
Let $\{Y_n\}$ be Markov(λ, Π). Set

$$\rho_0 = 0$$

$$\rho_{n+1} = \rho_n + \frac{1}{q_{Y_n}} \xi_{n+1}$$

Set

$$X_t \overset{def}{=} Y_n \quad \text{for } \rho_n \le t < \rho_{n+1}.$$

If $\sup_n \bar{q}_n < \infty$, then $X^{(n)} \to X$.

Pf Note that

$$\mathbb{P}\{\rho_{n+1} - \rho_n \ge t \,/\, Y_n = n\} = \mathbb{P}\{\tfrac{1}{q}\xi_{n+1} \ge t\}\Big|_{q=\bar{q}_n}$$

$$= \mathbb{P}\{\xi_{n+1} \ge tq\}\Big|_{q=\bar{q}_n} = \int_{t\bar{q}_n}^{\infty} e^{-s}\,ds = e^{-t\bar{q}_n};$$

thus, conditional on $\{Y_n = n\}$, $T_{n+1} - T_n$ is $\exp(\bar{q}_n)$.

We sort of know that $X^n \to X$, but let's be more precise. Fix $n \ge \sup_n \bar{q}_n$. Define $\rho_0^{(n)} = 0$ and

$$\rho_{j+1}^{(n)} = \inf\{s \ge \rho_j^{(n)} ;\ X_s^{(n)} \ne X_{\rho_j^{(n)}}^{(n)}\}$$

$$\mathbb{E}\Big[\Big(\prod_{j=1}^{J} \chi_{\{X^{(n)}_{\rho^{(n)}_j} = \imath_j\}} \exp\big[\imath\Theta_j \underbrace{(\rho^{(n)}_j - \rho^{(n)}_{j-1})}_{\text{jump times}}\big]\Big)\chi_{\{X^{(n)}_0 = \imath_0\}}\Big]$$

$$= \mathbb{E}\Big[\mathbb{E}\Big[\chi_{\{X^{(n)}_{\rho^{(n)}_J} = \imath_J\}} \exp\big[\imath\Theta_J(\rho^{(n)}_J - \rho^{(n)}_{J-1})\big] \,\Big|\, \mathcal{F}_{\rho^{(n)}_{J-1}}\Big]$$

$$\times\ \mathbb{E}\Big[\prod_{j=0}^{J-1} \chi_{\{X^{(n)}_{\rho^{(n)}_j} = \imath_j\}} \exp\big[\imath\Theta_j(\rho^{(n)}_j - \rho^{(n)}_{j-1})\big]\chi_{\{X^{(n)}_0 = \imath_0\}}\Big]\Big]$$

$$= \mathbb{E}\Big[\ \Pi_{\rho^{(n)}_{J-1},\, \imath_J}\ \frac{\exp\big[\imath\Theta_J/n\big]\,\bar{\delta}\, X^{(n)}_{\rho^{(n)}_{J-1}}}{1 - \Big(1 - \frac{\bar{\delta}\, X^{(n)}_{\rho^{(n)}_{J-1}}}{n}\Big)e^{\imath\Theta_J/n}}$$

$$\times\ \mathbb{E}\Big[\prod_{j=0}^{J-1} \chi_{\{X^{(n)}_{\rho^{(n)}_j} = \imath_j\}} \exp\big[\imath\Theta_j(\rho^{(n)}_j - \rho^{(n)}_{j-1})\big]\chi_{\{X^{(n)}_0 = \imath_0\}}\Big]\Big]$$

$$= \Pi_{i_0 i_1} \Pi_{i_1 i_2} \cdots \Pi_{i_{J-1} i_J} \prod_{j=0}^{J-1} \frac{\exp\left(i\theta_j/n\right)\bar{q}_{i_j}}{1-\left(1-\bar{q}_{i_j}/n\right)e^{i\theta_j/n}}$$

$$\xrightarrow{n\to\infty} \Pi_{i_0 i_1} \cdots \Pi_{i_{J-1}, i_J} \prod_{j=0}^{J-1} \frac{\bar{q}_{i_j}}{\bar{q}_{i_j} - i\theta_j}$$

$$= \mathbb{E}\Big[\prod_{j=1}^{J}\Big(\chi_{\{X_{\rho_j}=i_j\}} \exp\left[i\theta_j(\rho_j-\rho_{j-1})\right]\Big) \chi_{\{X_0=i_0\}}\Big) \quad \bullet$$

Note: This construction does not require that $|S|<\infty$ or that $\sup_i \bar{q}_i < \infty$. Note, however, that we have not proved that X, thus constructed, satisfies the infinitesimal statement of $\star$. In general, our approximation of characterization theorems work primarily when $|S| < \infty$.

When $|S| = \infty$, explosion can occur. If $\sup_n \bar{q}_n = \infty$, then we can have that

$$\zeta \overset{\text{def}}{=} \lim_{n\to\infty} \rho_n < \infty$$ ← Explosion time

which means that X hops more and more frequently as time approaches ρ_∞. Thus $\lim_{t \nearrow \rho_\infty} X_t$ may fail to exist. Since our theory of Markov processes requires that the Markov process be defined on all time & not just a random subset, we can attach a new state $\circledast$ to S & define

$$\hat{X}_T = \begin{cases} X_t & \text{if } t < \zeta \\ \circledast & \text{if } t \geq \zeta \end{cases}$$

This is then a Markov process on $S \cup \circledast$

Now consider linear equations

$$u(x) = \mathbb{E}_x\Big[\sum_{0 \le j < \tilde{H}_n^A} \alpha^j c(\tilde{X}_j^{(n)}) + \alpha^{\tilde{H}_n^A} f(\tilde{X}_{\tilde{H}_n^A}^{(n)}) \Big]$$

$$\tilde{H}_n^A = \inf\{ j \ge 0 : \tilde{X}_j^{(n)} \in A \}$$

$$= \mathbb{E}_x\Big[\sum_{0 \le j < H^A} \alpha^j c(X_{j/n}^{(n)}) + \alpha^{H^A} f(X_{\frac{H^A}{n}}^{(n)}) \Big]$$

$$H_n^A = \inf\{ t \ge 0 : X_t^{(n)} \in A \}$$

$$= \inf\{ \tfrac{k}{n} \ge 0 : X_{k/n}^{(n)} \in A \}$$

$$= \frac{1}{n} \inf\{ k \ge 0 : \tilde{X}_k^{(n)} \in A \}$$

$$= \frac{\tilde{H}_n^A}{n}$$

satisfies

$$u = \alpha P_n u + c \quad \text{on } S \setminus A$$

$$u = f \quad \text{on } A.$$

Let's rescale; set $\alpha = e^{\beta/n}$; on $S \setminus A$

$$u_n(x) = \mathbb{E}_x \left[\sum_{0 \le j < H^A} e^{\beta j/n} c(X^{(n)}_{j/n}) \frac{1}{n} + e^{\beta \frac{H^A}{n}} f(X^{(n)}_{\frac{H^A}{n}}) \right]$$

$$\approx \underbrace{\mathbb{E}_x \left(\int_0^{H^A} e^{\beta s} c(X_s)\, ds + e^{\beta H} f(X_H) \right)}_{= u(x)}$$

We also have that on $S \setminus A$,

$$u_n = e^{\beta/n} (I + Q/n) u_n + c/n$$

$$u_n \approx (1 + \beta/n)(I + Q/n) u_n + c/n$$

$$u_n \approx u_n + \frac{1}{n} \{ Q + \beta \} u_n + c/n$$

we thus expect that

$$Qu + \beta u + c = 0 \qquad \text{on } S \setminus A$$

$$u = f \qquad \text{on } A$$

Other calculations.

1. Jump Processes are Strong Markov

2. Communicating classes: $i \longrightarrow j$ iff

$$q_{i i_1} q_{i_1 i_2} \cdots q_{i_n j} > 0 \quad \text{some } i_1 \sim i_n$$

(look at jumps)

3. Recurrence, transience, etc.

$$T = \inf\{t > \hat{T} : X_t = X_0\}$$

$$\hat{T} = \inf\{t > 0 : X_t \neq X_0\}$$

Stationary Distributions

A distribution λ is stationary if $\mathbb{P}\{X_t = i\} = \lambda_i$ when

X is $\underbrace{\text{Markov}(\lambda, Q)}$

ie initial distribution λ, infinitesimal generator Q

$\iff \lambda e^{Qt} = \lambda$ all $t \geq 0$

$\iff \lambda Q = 0$ (by differentiation)

Definition Distribution λ is stationary if $\lambda Q = 0$

Claim λ is stationary iff $\mu_i = \dfrac{\lambda_i \bar{q}_i}{\sum_i \lambda_i q_i}$ is

stationary for jump matrix Π

Pf Set $\bar{\mu}_i = \lambda_i \bar{q}_i$

$$\sum_i \bar{\mu}_i \underbrace{\Pi_{ij}}_{0 \text{ if } i=j} = \sum_{i \neq j} \lambda_i \bar{q}_i \frac{q_{ij}}{\bar{q}_i}$$

$$= \sum_{i \neq j} \lambda_i q_{ij} = (\lambda Q)_j - \lambda_j q_{jj}$$

$$= \lambda_j \bar{q}_j = \bar{\mu}_j \qquad \bullet$$