Ergodic Theory

Note Title 11/29/2006

Claim: Suppose X is Markov (λP), irreducible & positive recurrent. Then for every bounded $f: I \to \mathbb{R}$,

$$\mathbb{P}\left\{ \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(X_j) = \underbrace{\pi f}_{\sum_i \pi_i f_i} \right\} = 1$$

Pf: Fix $i \in I$.

$$T_0^i = \inf\{k \geq 0 : X_k = i\}$$

T_n^i = time of n^{th} visit to i

$$T_{n+1}^i \overset{def}{=} \inf\{k \geq T_n^i : X_k = i\}$$

$$= T_n^i + \widetilde{T}_0 \circ \Theta_{T_n^i}$$

$\inf\{k \geq 1 : X_k = X_0\}$

Set $S_n^i \overset{def}{=} T_{n+1}^i - T_n^i$ for $n \geq 1$ $\sim$ iid

$$\mathbb{E}[S_n^i] = \underbrace{\mathbb{E}_i[\widetilde{T}]}_{m_i = \frac{1}{\pi_i}} < \infty \text{ by pos. recurrence}$$

By SLLN,

$$\mathbb{P}\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} S_j^1 = m_1\right\}=1$$

Now set

$$\underbrace{V_1(n) \overset{\text{def}}{=} \sum_{j=1}^{n}\chi_{\{X_j=1\}}}_{\#\text{ of visits to }1}$$

$$\frac{1}{n}V_1(n) = \frac{1}{n}\sum_{j=1}^{n}\delta_1(X_j)$$

δ is Kronecker δ

If $V_1(n)=k$, then X has made k visits to 1 by time n. Thus $T_k^1 \le n < T_{k+1}^1$

$$\sum_{j=1}^{k-1} S_1^j + T_1^1 = T_k^1 \le n$$

$$\sum_{j=1}^{k} S_1^j + T_1^1 = T_{k+1}^1 > n$$

In other words,

$$\sum_{j=1}^{V_1(u)-1} S_1^j + T_1^1 \le u$$

$$\sum_{j=1}^{V_1(u)} S_1^j + T_1^1 > u$$

$$\frac{V_1(u)-1}{V_1(u)} \quad \frac{\sum_{j=1}^{V_1(u)-1} S_1^j}{V_1(u)} + \frac{T_1^1}{V_1(u)} \le u$$

$$< \frac{\sum_{j=1}^{V_1(u)} S_1^j}{V_1(u)} + \frac{T_1^1}{V_1(u)}$$

By recurrence,

$$\mathbb{P}\left\{ \sum_{j=1}^{\infty} \chi_{\{X_j = 1\}} = \infty \right\} = 1$$

$$\mathbb{P}\left\{ \lim_{u \to \infty} V_1(u) = \infty \right\} = 1$$

Since both of ⚑ are sets of full measure, their intersection is also of full measure. Hence

$$\mathbb{P}\left\{ \lim_{n\to\infty} \frac{n}{V_i(n)} = m_i \right\} = 1$$

Since I is countable & $m_i < \infty$

$$\mathbb{P}\left\{ \overline{\lim_{n\to\infty}} \left| \frac{V_i(n)}{n} - \pi_i \right| > 0 \right\} = 0$$

Then

$$\frac{1}{n} \sum_{j=1}^{n} f(X_j) - \pi f$$

$$= \sum_{i \in I} f_i \left\{ \frac{1}{n} \sum_{j=1}^{n} \chi_{\{X_j = i\}} - \pi_i \right\}$$

$$= \sum_{i \in I} f_i \left\{ \frac{U_i(\omega)}{n} - \pi_i \right\}$$

$$\left| \frac{1}{n} \sum_{j=1}^{n} f(X_j) - \pi f \right|$$

$$\leq \|f\| \sum_{i \in I} \left| \frac{U_i(\omega)}{n} - \pi_i \right|$$

Fix $J \subset I$ with $|J| < \infty$. Then

$$\sum_{i \in I} \left| \frac{U_i(\omega)}{n} - \pi_i \right| = \sum_{i \in J} \left| \frac{U_i(\omega)}{n} - \pi_i \right|$$

$$+ \sum_{i \notin J} \left| \frac{U_i(\omega)}{n} - \pi_i \right|.$$

$$\sum_{i \notin J} \left| \frac{U_i(\omega)}{n} - \pi_i \right| \leq \sum_{i \notin J} \frac{U_i(\omega)}{n} + \pi_i$$

$$= \sum_{i \notin J} \left\{ \frac{U_i(\omega)}{n} - \pi_i \right\} + 2 \sum_{i \notin J} \pi_i$$

Note that

$$\sum_{i \in I} \frac{V_i(n)}{n} = 1$$

$$\sum_{i \in I} \pi_i = 1$$

Thus

$$\sum_{i \notin J} \left\{ \frac{V_i(n)}{n} - \pi_i \right\} = - \sum_{i \in J} \left\{ \frac{V_i(n)}{n} - \pi_i \right\}$$

$$\leq \sum_{i \in J} \left| \frac{V_i(n)}{n} - \pi_i \right|.$$

Thus

$$\sum_{i \in I} \left| \frac{V_i(n)}{n} - \pi_i \right| \leq 2 \sum_{i \in J} \left| \frac{V_i(n)}{n} - \pi_i \right|$$

$$+ 2 \sum_{i \notin J} \pi_i$$

Then

$$\left| \frac{1}{n} \sum_{j=1}^{n} f(X_j) - \pi f \right|$$

$$\leq 2\|f\| \sum_{i \in J} \left| \frac{V_i(n)}{n} - \pi_i \right| + 2\|f\| \sum_{i \notin J} \pi_i$$

Taking J large enough, we have that for any $\varepsilon > 0$,

$$\mathbb{P}\left\{ \lim_{n \to \infty} \left| \frac{1}{n} \sum_{j=1}^{n} f(X_j) - \pi f \right| \geq \varepsilon \right\} = 0.$$