Class Structure

Note Title 9/3/2005

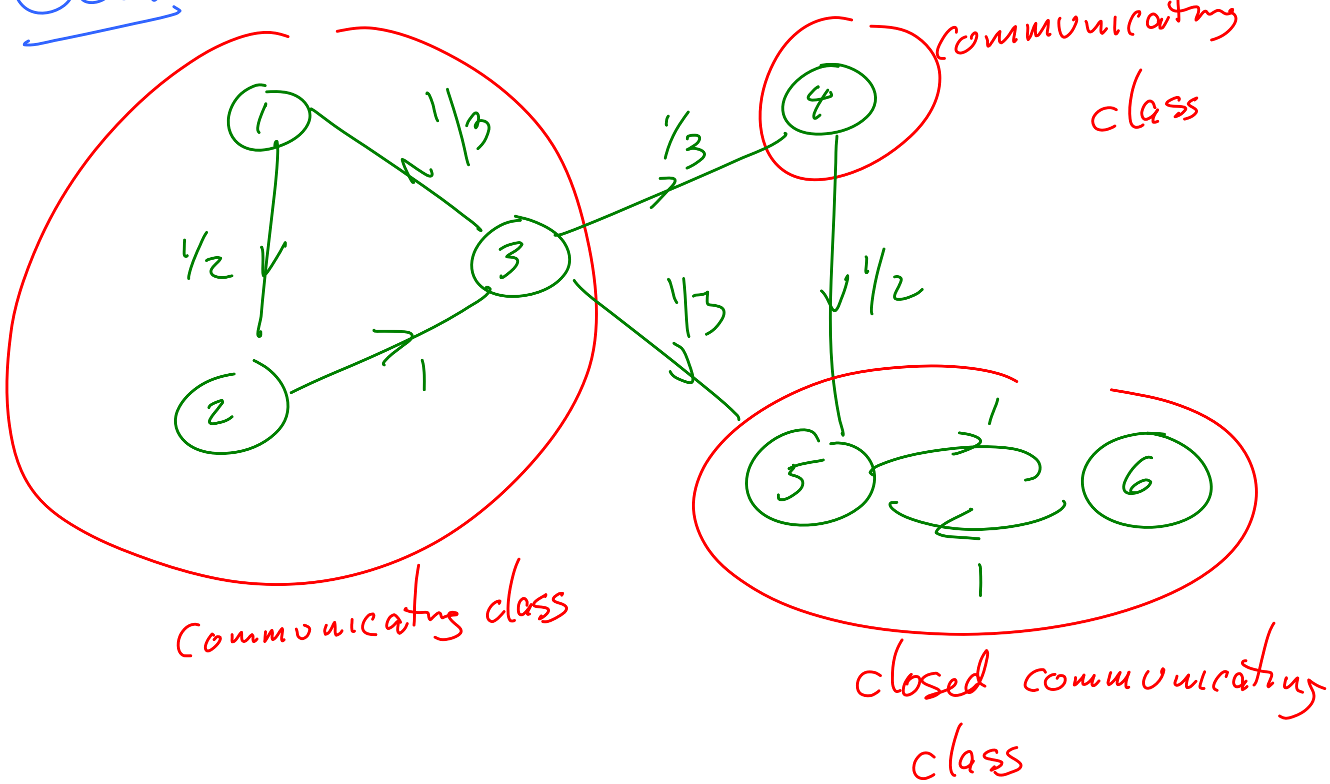
$(X_n)_0^\infty$ is Markov (λ, P) chain with state space I

Definition: $i \rightarrow j$ if $\mathbb{P}_i\{X_n = j \text{ some } n \geq 0\} > 0$

$i \leftrightarrow j$ if $i \rightarrow j$ and $j \rightarrow i$.

i & j communicate

Goal:



Claim $i \to j$

$\iff p_{ii_1} p_{i_2 i_3} \cdots p_{i_{n-1}j}$ some $i_1, i_2 \sim i_{n-1} \in I$

$\iff p_{ij}^{(n)} > 0$ some $n \geq 0$

Pf Fix $n \geq 0$

$$p_{ij}^{(n)} = \mathbb{P}_i\{X_n = j\} \leq \mathbb{P}_i\{X_m = j \text{ some } m \geq 0\}$$

$$= \mathbb{P}_i\left(\bigcup_{m=0}^{\infty} \{X_m = j\}\right) \leq \sum_{m=0}^{\infty} \mathbb{P}_i\{X_m = j\}$$

$$= \sum_{m=0}^{\infty} p_{ij}^{(m)}$$

Thus, $p_{ij}^{(n)} > 0 \Rightarrow i \to j \Rightarrow p_{ij}^{(m)} > 0$ some m

Secondly,

$$p_{ij}^{(n)} = \sum_{i_1, i_2 \sim i_{n-1} \in I} p_{ii_1} p_{i_1 i_2} \sim p_{i_{n-1} j}$$

so

$$p_{ij}^{(n)} > 0 \iff p_{ii_1} p_{i_1 i_2} \sim p_{i_{n-1} j} > 0 \quad \text{some } i_1, i_2 \sim$$

Note $i \to j$ & $j \to k \Rightarrow i \to k$

$j \to i$ & $k \to j \Rightarrow k \to j$

$i \to i$

$P^{(0)}_{ii} > 0$

Thus $\leftrightarrow$ is an equivalence relation.

transitivity, identity

Thus, we can break I into equivalence classes;

Let's call these equivalence classes communicating classes; ie

Definition $C \subset I$ is a communicating class if, for any $i \in C$, $I = \{j \in I : i \leftrightarrow j\}$

Definition If I is a single communicating class, ie $i \leftrightarrow j$ for all i & j in I, P is said to be irreducible

Definition A communicating class C is closed if $i \in C$, $i \to j$ implies that $j \in C$. (ie no escape)

Definition A state i is absorbing if $\{i\}$ is a closed communicating class.