Absorption Probabilities/Hitting Times ③

Note Title 9/11/2005

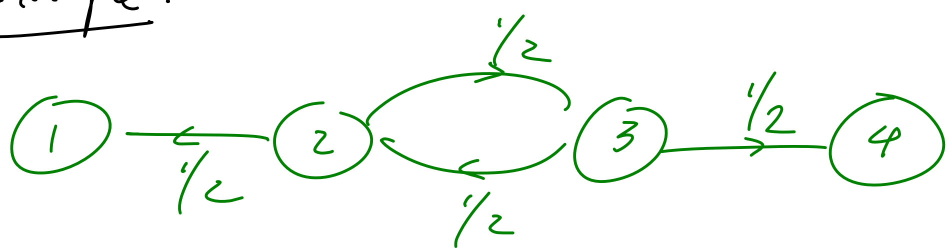
Definition $A \subset I$ (state space)

$$H^A \overset{def}{=} \inf\{n \geq 0 : X_n \in A\}$$

$\inf \emptyset \overset{def}{=} \infty$

Hitting time; a random variable

Example:



Communicating classes; $\{1\}, \{4\}, \{2,3\}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$A = \{4\}$. If $X_0(\omega) = 2$, $X_1(\omega) = 3$, $X_2(\omega) = 4$ (X hits $\{4\}$ first at time 2)

$H^{\{4\}}(\omega) = 2$

$h_1 = \mathbb{P}_1\{H^{\{4\}} < \infty\}$ ($h_1 = 0$; calculation is not trivial)

$k_1 = \mathbb{E}_1[H^{\{4\}}]$ ($k_1 = \infty$)

Calculation

$h_1 = 0, \quad h_4 = 1,$

$\mathbb{P}(A \cap B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \mathbb{P}(B) = \mathbb{P}(A | B)\mathbb{P}(B)$

$h_2 = \mathbb{P}_2\{H < \infty\} = \mathbb{P}_2\{H < \infty, X_1 = 1\} + \mathbb{P}_2\{H < \infty, X_1 = 3\}$

$= \mathbb{P}_2\{H < \infty | X_1 = 1\} \mathbb{P}_2\{X_1 = 1\} + \mathbb{P}_2\{H < \infty | X_1 = 3\} \mathbb{P}_2\{X_1 = 3\}$

($\mathbb{P}_2\{X_1 = 1\} = p_{21}$, $\mathbb{P}_2\{X_1 = 3\} = p_{23}$)

(see lemma below)

$= h_1 p_{21} + h_2 p_{23}$

Similarly

$h_3 = h_2 p_{32} + h_4 p_{34}$

$$h_1 = 0$$
$$h_4 = 1$$
$$h_2 = \frac{1}{2} h_1 + \frac{1}{2} h_3$$
$$h_3 = \frac{1}{2} h_2 + \frac{1}{2} h_4$$

$\Rightarrow$

$$h_1 = 0$$
$$h_4 = 1$$
$$h_2 = 1/3$$
$$h_3 = 2/3$$

Note that

$$h_i = \mathbb{E}_i\left[\chi_{\{H^A<\infty\}}\right]$$

Consider the more general problem:

$$\phi_i = \mathbb{E}_i\left[\sum_{0\le j<H^A} \beta^j c(X_j) + \beta^{H^A} f(X_{H^A})\chi_{\{H^A<\infty\}}\right]$$

$\beta>0$; discount

$f\geq 0$; terminal cost

$c\geq 0$; running cost

if $c\equiv 0$ & $f\equiv 1$, $\beta=1$, $\phi_i = \mathbb{P}_i\{H^A<\infty\}$

if $c\equiv 1$ & $f\equiv 0$, $\beta=1$, $\phi_i = \mathbb{E}_i[H^A]$

Theorem:

(1) $\phi\geq 0$ solves (ie $\phi_i\geq 0$ all $i\in I$)

$\phi = f$ on A ie $\phi_i = f(i)$ on A

$\phi = P\phi + c$ on A^c ie $\phi_i = \sum_j p_{ij}\phi_j + c(i)$ for $i\in A^c$

(2) If $\psi\geq 0$ solves

$\psi\geq f$ on A

$\psi \geq P\psi + c$ on A^c

pointwise, like

Then $\psi \geq \phi$

(3) If $P_i\{H^A < \infty\} = 1$ for all $i \in I$, then there is at most one bounded solution of

$$\phi = f \quad \text{on } A$$
$$\phi = P\phi + c \quad \text{on } A^c$$

Note: (2) implies that ϕ is the minimal nonnegative solution of

$$\phi = f \quad \text{on } A$$
$$\phi = P\phi + c \quad \text{on } A^c.$$

In other words, if $\psi \geq 0$ is any solution of

$$\psi = f \quad \text{on } A$$
$$\psi = P\psi + c \quad \text{on } A^c,$$

then $\psi \geq \phi$. If ϕ^* is any other such minimal nonnegative solution, then $\phi^* \geq \phi$ & $\phi \geq \phi^*$, so $\phi = \phi^*$; minimal nonnegative solution is unique

Let's set up some approximations of ϕ. For $n \geq 0$, set

$$\phi_x(n) \overset{\text{def}}{=} E_x\left[\sum_{j=0}^{n} \beta^j c(X_j)\chi_{\{j<H^A\}} + \beta^{H^A} f(X_{H^A})\chi_{\{H^A \leq n\}}\right]$$

Then $\phi_x(n) \nearrow \phi_x$ as $n \to \infty$ (use monotone convergence)

↳ increases

Let's make some simple observations.

If $x \in A$, $H^A = 0$ under P_x;

$$\{0 < H^A\} = \phi$$

$$\beta^{H^A} f(X_{H^A})\chi_{\{H^A \leq n\}} = f(x) \quad \text{all } n \geq 0$$

Thus if $x \in A$, $\phi_x(n) = f(x)$ all $n \geq 0$

If $x \notin A$, then $H^A \geq 1$ under P_x. Thus

$$\beta^0 c(X_0)\chi_{\{0<H^A\}} = c(x)$$

$$\{H^A \leq 0\} = \phi$$

$$\phi_x(0) = c(x)$$

For $n \geq 1$, we also have

$$\beta^{H^A} f(X_{H^A}) \chi_{\{H^A \leq n\}} = \sum_{j=1}^{n} \beta^j f(X_j) \chi_{\{H^A = j\}}$$

$$\phi_x(n) = c(x) + \mathbb{E}_x\left[\sum_{j=1}^{n} \beta^j \left\{ c(X_j) \chi_{\{H^A > j\}} + f(X_j) \chi_{\{H^A = j\}} \right\} \right]$$

$$= c(x) + \sum_{j=1}^{n} \beta^j \left\{ \mathbb{E}_x\left[c(X_j) \chi_{\{H^A > j\}} \right] + \mathbb{E}_x\left[f(X_j) \chi_{\{H^A = j\}} \right] \right\}$$

for $j \geq 1$,

$$\{H^A = j\} = \{H^A > j-1, X_j \in A\}$$
$$= \{X_\ell \notin A \text{ for } 0 \leq \ell \leq j-1\} \cap \{X_j \in A\}$$

$$\{H^A > j\} = \{H^A > j-1, X_j \notin A\}$$
$$= \{X_\ell \notin A \text{ for } 0 \leq \ell \leq j-1\} \cap \{X_j \notin A\}$$

For $i \notin A$, $j \geq 1$

$B \subset I$ fixed

$$\mathbb{E}_i\left[g(X_j)\chi_{\{X_\ell \notin A \text{ for } 0 \leq \ell \leq j-1\}}\chi_{\{X_j \in B\}}\right]$$

$$= \mathbb{E}_i\left(g(X_j)\chi_{\{X_\ell \notin A \text{ for } 1 \leq \ell \leq j-1\}}\chi_{\{X_j \in B\}}\right)$$

$$= \mathbb{E}_i\left(g(X'_{j-1})\chi_{\{X'_\ell \notin A \text{ for } 0 \leq \ell \leq j-2\}}\chi_{\{X'_{j-1} \in B\}}\right]$$

$$= \sum_{k \in I} \mathbb{E}_i\left(g(X'_{j-1})\chi_{\{X'_\ell \notin A \text{ for } 0 \leq \ell \leq j-2\}}\chi_{\{X'_{j-1} \in B\}} \middle| X_1 = k\right] \underbrace{\mathbb{P}_i\{X_1 = k\}}_{P_{ik}}$$

$$= \sum_{k \in I} \mathbb{E}_k\left(g(X_{j-1})\chi_{\{X_\ell \notin A \text{ for } 0 \leq \ell \leq j-2\}}\chi_{\{X_{j-1} \in B\}}\right] P_{ik}$$

$$= \sum_{k \in I} \mathbb{E}_k\left[g(X_{j-1})\chi_{\{H^A \geq j-2,\, X_{j-1} \in B\}}\right] P_{ik}$$

Thus

$$\mathbb{E}_i \left[c(X_j) \chi_{\{j < H^A\}} \right]$$

$$= \sum_{k \in I} p_{ik} \mathbb{E}_k \left[g(X_{j-1}) \chi_{\{H^A > j-1\}} \right]$$

$$\mathbb{E}_i \left[f(X_j) \chi_{\{H^A = j\}} \right)$$

$$= \sum_{k \in I} p_{ik} \mathbb{E}_p \left[f(X_{j-1}) \chi_{\{H^A = j-1\}} \right)$$

$$\phi_i(n) = c(i) + \sum_{j=1}^{n} \sum_{k \in I} \beta^j p_{ik} \Big\{ \mathbb{E}_k \left[c(X_{j-1}) \chi_{\{H^A > j-1\}} \right]$$

$$+ \mathbb{E}_p \left[f(X_{j-1}) \chi_{\{H^A = j-1\}} \right] \Big\}$$

$$= c(i) + \beta \sum_{k \in I} p_{ik} \sum_{j=0}^{n-1} \beta^j \Big\{ \mathbb{E}_p \left[c(X_j) \chi_{\{H^A > j\}} \right]$$

$$+ \mathbb{E}_p \left[f(X_j) \chi_{\{H^A = j\}} \right] \Big\}$$

$$= c(i) + \beta \sum_{k \in I} p_{ik} \phi_k(n-1)$$

In other words,

$$\phi(n) = c + \beta P \phi(n-1) \quad \text{on} \quad I \setminus A$$

Let's collect together what we know thus far:

$$\phi_i(n) \nearrow \phi_i \qquad \text{a)}$$

$$\text{If } i \in A, \quad \phi_i(n) = f(i) \qquad \text{b)}$$

If $i \notin A$,

$$\phi_i(0) = c(i) \qquad \text{c)}$$

$$\phi_i(n) = \beta \sum_j P_{ij} \phi_j(n-1) + c(i) \quad n \geq 1 \qquad \text{d)}$$

Pf of Theorem

① Take n ↗ ∞ in d) and use a)

② Note that

$$\psi_i \geq f(i) = \phi_i(n) \quad \text{if } i \in A$$

$$\psi_i \geq c(i) = \phi_i(0) \quad \text{if } i \notin A \quad (\text{use } \psi \geq 0)$$

Assume now $\psi \geq \phi(n)$. Then

$$\psi_i \geq f(i) = \phi_i(n+1) \quad \text{if } i \in A$$

$$\left.\begin{aligned} \psi_i &\geq \beta \sum_j p_{ij} \psi_j + c(i) \\ &\geq \beta \sum_j p_{ij} \phi_j(n) + c(i) \\ &= \phi_i(n+1) \end{aligned}\right\} \text{ if } i \notin A.$$

Thus $\psi \geq \phi(n+1)$. Take $n \nearrow \infty$ and use a).

(3) Let Φ satisfy

$$\Phi = f \quad \text{on } A$$

$$\Phi = P\Phi + c \quad \text{on } A^c$$

i.e. Φ is bounded

Assume that $|\Phi| \leq M$. Note that

$M = PM$, (i.e. $M_i = \sum_j p_{ij} M_j$, where M is vector of M's)

($\sum_j p_{ij}$ =1 since P is stochastic)

Note what (2) says: If we have a nonnegative supersolution, where c & f are nonnegative, it is bounded from below by the probabilistic solution.

Let's shift Φ up & down so that it becomes nonnegative. Let's find the equation it solves, & then compare it to a probabilistic formula.

Set

$$\Phi^+ = \Phi + M \sim \text{add on homogeneous eqn}$$

Then

$\Phi^+ = f + M \sim$ on A^c

$\Phi^+ = \Phi + M = P\Phi + c + PM = P\Phi^+ + c$ on A

By minimality,

$$\Phi^+_x \geq \mathbb{E}_x\left[\sum_{0 \leq n < H^A} c(X_n) + (f(X_{H^A}) + M)\chi_{\{H^A<\infty\}}\right]$$

$$= \phi_x + M\mathbb{P}_x\{H^A < \infty\}$$

Since we assume $\mathbb{P}_x\{H^A < \infty\} = 1$,

$\Phi^+ \geq \phi + M$; ie $\Phi + M \geq \phi + M$; ie

$$\Phi \geq \phi.$$

Set now

$$\Phi^- = \phi - \Phi + M$$

$$\Phi^- \geq f - \Phi + M \geq M \quad \text{on } A^c$$

$$\left.\begin{aligned} \Phi^- &= \phi - \Phi + M = P\phi + c - P\Phi - c - PM \\ &= P\Phi^- \end{aligned}\right\} \text{on } A$$

By minimality,

$$\Phi^- \geq \mathbb{E}_x\left[M\chi_{\{H^A<\infty\}}\right] = M\mathbb{P}_x\{H^A < \infty\}$$

Since we assume that $\mathbb{P}_i\{H^A < \infty\} = 1$,

$\Phi^- \geq M$; ie $\phi - \Phi + M \geq M$; ie $\phi - \Phi \geq 0$;

ie $\phi \geq \Phi$.

Thus $\Phi \geq \phi$ & $\phi \geq \Phi$. ∎

Back to example: minimal nonnegative solution of

$$\left.\begin{aligned} h_4 &= 1 \\ h_3 &= \tfrac{1}{2} h_2 + \tfrac{1}{2} h_4 \\ h_2 &= \tfrac{1}{2} h_1 + \tfrac{1}{2} h_3 \\ h_1 &= h_1 \end{aligned}\right\}$$

$$h_3 = \tfrac{1}{2}\left(\tfrac{1}{2} h_1 + \tfrac{1}{2} h_3\right) + \tfrac{1}{2}$$

$$\tfrac{3}{4} h_3 = \tfrac{1}{4} h_1 + \tfrac{1}{2}$$

$$h_3 = \tfrac{1}{3} h_1 + \tfrac{2}{3}$$

$$h_2 = \tfrac{1}{2} h_1 + \tfrac{1}{2}\left(\tfrac{1}{3} h_1 + \tfrac{2}{3}\right)$$

$$= \tfrac{2}{3} h_1 + \tfrac{1}{3}$$

Solution space

$$h_4 = 1$$
$$h_2 = \tfrac{2}{3}\alpha + \tfrac{1}{3}$$
$$h_3 = \tfrac{1}{3}\alpha + \tfrac{2}{3}$$
$$h_1 = \alpha$$

For nonnegativity, need $\alpha \geq 0$. Let's set $\alpha = 0$;

ie
$$h_4 = 1$$
$$h_2 = \tfrac{1}{3}$$
$$h_3 = \tfrac{2}{3}$$
$$h_1 = 0$$

If we have any other nonnegative solution $\hat{h}$, of ★, then

$$\tilde{h}_4 = 1 \geq 1 = h_4$$
$$\hat{h}_2 = \tfrac{2}{3}\alpha + \tfrac{1}{3} \geq \tfrac{1}{3} = h_2$$
$$\hat{h}_3 = \tfrac{1}{3}\alpha + \tfrac{2}{3} \geq \tfrac{2}{3} = h_3$$
$$\tilde{h}_1 = \alpha \geq 0 = h_1$$

Example: Expected hitting times for same problem

$k_i = \mathbb{E}_i(H^{\{4\}})$; minimal nonnegative solution of

$$k_4 = 0$$

$$k_3 = \frac{k_2}{2} + \frac{k_4}{2} + 1 \quad \rightarrow \quad k_3 = \infty$$

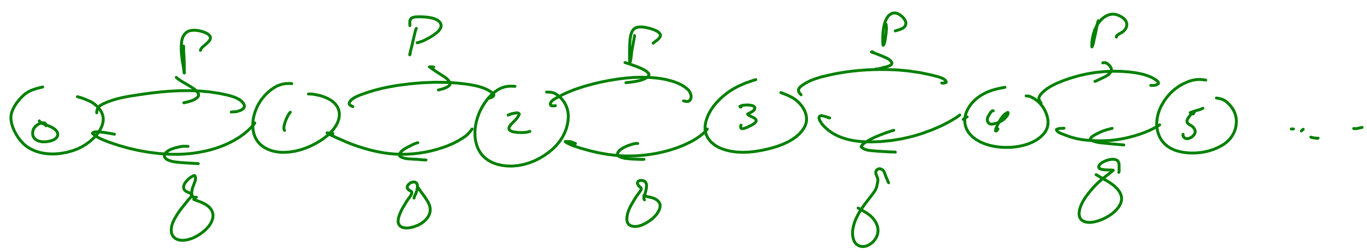
$$k_2 = \frac{k_1}{2} + \frac{k_3}{2} + 1 \quad \rightarrow \quad k_2 = \infty$$

$$k_1 = k_1 + 1 \quad \rightarrow \quad k_1 = \infty$$

Example Same process, $A = \{1,4\}$

$$\left.\begin{aligned} k_1 &= k_4 = 0 \\ k_2 &= \frac{k_3 + k_1}{2} + 1 \\ k_3 &= \frac{k_2 + k_4}{2} + 1 \end{aligned}\right\} \Rightarrow \begin{aligned} k_1 &= k_4 = 0 \\ k_2 &= k_3 = 2 \end{aligned}$$

Example (Gambler's Ruin) wealth in casino



$(p+q=1)$

$A=\{0\}$. Compute $\mathbb{P}_n\{H^A<\infty\}$

$$h_0 = 1$$

$$h_n = p\,h_{n+1} + q\,h_{n-1} \qquad n\in\{1,2,\ldots\}$$

Recursion on $\mathbb{Z}_+$

Search for $h_n = \lambda^n$

$$\lambda^n = p\lambda^{n+1} + q\lambda^{n-1} \Rightarrow 1 = p\lambda + \frac{q}{\lambda}$$

$$\Rightarrow \lambda = p\lambda^2 + q \Rightarrow p\lambda^2 - \lambda + q = 0$$

$$\Rightarrow \lambda = \frac{1\pm\sqrt{1-4pq}}{2p} = \frac{1\pm\sqrt{1-4p(1-p)}}{2p}$$

$$= \frac{1 \pm \sqrt{1+4p^2-4p}}{2p} = \frac{1 \pm \sqrt{(1-2p)^2}}{2p}$$

$$= \left\{ \frac{1+1-2p}{2p}, \frac{1-1+2p}{2p} \right\} = \left\{ \frac{1-p}{p}, 1 \right\} = \left\{ 1, \frac{q}{p} \right\}$$

a) If $p \neq q$,

$h_i = A + B\left(\frac{q}{p}\right)^i \qquad 1 = h_0 = A + B;$

If $q > p$ (casino's advantage), $\frac{q}{p} > 1$,

If $B < 0$, then $\lim_{i \to \infty} h_i = -\infty$. Must have $B \geq 0$. We can write

$$h_i = 1 - B + B\left(q/p\right)^i = 1 + B\left\{\left(q/p\right)^i - 1\right\}$$

Set $B = 0$; $h_i = 1$. Then if k satisfies

☆, then there is a $\beta \geq 0$ such that

$k_i = 1 + \beta\left\{\left(q/p\right)^i - 1\right\}$ all i. Thus $k_i \geq h_i$ all i.

$h_i = 1$ is minimal solution

If $q < p$ (your advantage), $q/p < 1$.

If $A < 0$,

$$\lim_{i \to \infty} h_i = A < 0.$$

Must have $A \geq 0$. We can write

$$h_i = A + (1-A)(q/p)^i = (q/p)^i + A\{1-(q/p)^i\}$$

Set $A = 0$; $h_i = (q/p)^i$. If k satisfies

☆, then there is an $\alpha \geq 0$ such that

$$k_i = (q/p)^i + \alpha\{1-(q/p)^i\}$$

all i. Thus $k_i \geq h_i$ all i.

$h_i = (q/p)^i$ is minimal solution

Nonnegativity implies $A \geq 0$

Minimality implies $A = 0$

$$h_i = \left(\frac{q}{p}\right)^i$$

b) If $p = q$, $\lambda = \{1\}$; repeated eigenvalue

$h_n = A + Bn$

$1 = h_0 = A$;

$h_n = 1 + Bn$

Nonnegativity implies $B \geq 0$

Minimality implies $B = 0$.

<u>Example</u> In this last problem ($p = q = 1/2$), compute $\mathbb{E}_n[H^{\{0\}}]$.
Even though $\mathbb{P}_n\{H < \infty\} = 1$, this does not imply that
$\mathbb{E}_n[H] < \infty$.

Recursion:

$k_0 = 0$
$k_n = \frac{k_{n+1}}{2} + \frac{k_{n-1}}{2} + 1$

Homogeneous problem: $\lambda = \{1\}$; repeated eigenvalue. Homogeneous problem is solved by

$1^i(A+Bi)$.

Recursion for k is

$$k_i = \frac{k_{i+1}}{2} + \frac{k_{i-1}}{2} + 1 1^i$$

general form is: see below

$$k_i = 1^i(A+Bi+Ci^2) = A+Bi+Ci^2$$

indeed

$$\frac{k_{i+1}}{2} + \frac{k_{i-1}}{2} = \frac{A+B(i+1)+C\overbrace{(i+1)^2}^{i^2+2i+1}}{2} + \frac{A+B(i-1)+C\overbrace{(i-1)^2}^{i^2-2i+1}}{2}$$

$$= \frac{2A + 2Bi + 2Ci^2 + 2C}{2}$$

$$= k_i + C$$

need

$$k_i = \frac{k_{i+1}}{2} + \frac{k_{i-1}}{2} + 1$$

$$k_i = k_i + C + 1 \Rightarrow C = -1$$

$0 = k_0 = A;$

$k_i = B_i - i^2$

For any B, $\lim_{i \to \infty} k_i < 0$; must set $B = \infty$;

$E_i[H] = k_i = \infty$