Stochastic Processes

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CHAPTER 1

Brownian Motion

Let's start by constructing Brownian motion. Let's fix a time horizon $T > 0$, and construct Brownian motion on $[0,T]$. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $L_2[0,T]$; e.g.

$$\phi_0(t) = \frac{1}{\sqrt{T}}$$
$$\phi_{2n}(t) = \sqrt{\frac{2}{T}}\cos\left(\frac{2\pi n t}{T}\right)$$
$$\phi_{2n+1}(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{2\pi n t}{T}\right)$$

Let's also fix an i.i.d. collection $\{\xi_n\}_{n=0}^N$ of i.i.d. $\mathfrak{N}(0,1)$ random variables. Define

$$W_t^N \overset{\text{def}}{=} \sum_{n=0}^N \xi_n \int_{s=0}^t \phi_n(s)ds = \sum_{n=0}^N \xi_n \left\langle \chi_{[0,t]}, \phi_n \right\rangle_{L^2}.$$

for each $t \in [0,T]$ and $n \in \{0,1,\dots\}$. Since W_t^N is a linear combination of random variables, $W_t^N \in L^2[0,T]$

Lemma 0.1. *For each $t \in [0,T]$, $W_t^\infty \overset{def}{=} \lim_{N\to\infty} W_t^N$ exists as a limit in $L^2(\Omega)$.*

Proof. For integers N_1 and N_2 such that $N_1 \le N_2$,

$$\mathbb{E}\left[|W_t^{N_2} - W_t^{N_1}|^2\right] = \sum_{n=N_1+1}^{N_2} \left\langle \chi_{[0,t]}, \phi_n \right\rangle_{L^2}^2; \tag{1}$$

the claim follows from Bessel's inequality. □

Let's next show that W has independent Gaussian increments.

Lemma 0.2. *Fix $0 = t_0 \le t_0 < t_1 \le t_J = T$, $\{W_{t_1}^\infty - W_{t_0}^\infty, W_{t_2}^\infty - W_{t_1}^\infty \dots W_{t_J}^\infty - W_{t_{J-1}}^\infty\}$ are independent random variables and $W_t^\infty - W_s^\infty$ is $\mathfrak{N}(0,t-s)$.*

Proof. Fix $\{\theta_j\}_{j=1}^J \subset \mathbb{R}$. It suffices to show that

$$\mathbb{E}\left[\exp\left[\sum_{j=1}^J \sqrt{-1}\theta_j(W_{t_j}^\infty - W_{t_{j-1}}^\infty)\right]\right] = \exp\left[-\frac{1}{2}\sum_{j=1}^n \theta_j^2(t_j - t_{j-1})\right]. \tag{2}$$

To see this, note that

$$\sum_{j=1}^J \sqrt{-1}\theta_j(W_{t_j}^\infty - W_{t_{j-1}}^\infty) = \lim_{N\to\infty}\sum_{j=1}^n \sqrt{-1}\theta_j(W_{t_j}^N - W_{t_{j-1}}^N) = \sqrt{-1}\sum_{n=0}^N \xi_n \left\langle f, \phi_n\right\rangle_{L^2},$$

this limit being in L^2, where

$$f = \sum_{j=1}^J \theta_j\left(\chi_{[0,t_j]} - \chi_{[0,t_{j-1}]}\right) = \sum_{j=1}^n \theta_j \chi_{(t_{j-1},t_j]}.$$

Thus

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left[\sqrt{-1}\sum_{j=1}^J \sqrt{-1}\theta_j(W^\infty_{t_j}-W^\infty_{t_{j-1}})\right]\right] = \lim_{N\to\infty}\mathbb{E}\left[\exp\left[\sqrt{-1}\sum_{n=0}^N \xi_n\left\langle f,\phi_n\right\rangle_{L^2}\right]\right] \\
&= \lim_{N\to\infty}\exp\left[-\sum_{n=0}^N \left\langle f,\phi_n\right\rangle^2_{L^2}\right] = \exp\left[-\frac{1}{2}\|f\|^2_{L^2}\right] = \exp\left[-\frac{1}{2}\sum_{j=1}^J \theta_j^2(t_j-t_{j-1})\right]
\end{aligned} \tag{3}
$$

□

Let's also understand *continuity*.

LEMMA 0.3. *There is a process $\{W_t;\, 0\le t\le T\}$ which is $\mathbb{P}$-a.s. continuous such that $\mathbb{P}\{W_t=W^\infty_t\}=1$ for all $t\in[0,T]$. In fact, for each $\gamma\in(0,1/2)$, there is a $\mathbb{P}$-a.s. finite random variable Ξ_γ such that*

$$|W_t(\omega)-W_s(\omega)|\le \Xi_\gamma(\omega)|t-s|^\gamma$$

for all s and t in $[0,T]$ and all $\omega\in\Omega$.

PROOF. Let's begin by showing that W^∞ can't vary too much over adjacent dyadic rationals. Fix

$$p>\frac{1}{\frac{1}{2}-\gamma} \tag{4}$$

(thus $p>2$) and note that

$$\mathbb{E}[|W_t-W_s|^p]=K_p|t-s|^{p/2}$$

for all s and t in $[0,T]$, where

$$K_p \overset{\text{def}}{=} \int_{z\in\mathbb{R}}|z|^p\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz.$$

For each $n\in\mathbb{N}$, define

$$A_n \overset{\text{def}}{=} \left\{\max_{0\le j\le 2^n-1}|W^\infty_{(j+1)/2^n}-W^\infty_{j/2^n}|\ge\frac{1}{2^{n\gamma}}\right\}.$$

Note that

$$
\begin{aligned}
\mathbb{P}(A_n)&\le\sum_{j=0}^{2^n-1}\mathbb{P}\left\{\left|W^\infty_{(j+1)/2^n}-W^\infty_{j/2^n}\right|\ge\frac{1}{2^{n\gamma}}\right\}\le\sum_{j=0}^{2^n-1}2^{n\gamma p}\mathbb{E}\left[\left|W^\infty_{(j+1)/2^n}-W^\infty_{j/2^n}\right|^p\right]\\
&\le K_p 2^{n(1+\gamma p)}\left(\frac{1}{2^{n/2}}\right)^p\le\frac{K_p}{2^{n(p(1/2-\gamma)-1)}}.
\end{aligned}
$$

By our choice (4) of p, we have that $p(\frac{1}{2}-\gamma)-1>0$, so

$$\sum_{n=1}^\infty\mathbb{P}(A_n)<\infty.$$

By Borel-Cantelli,

$$\mathbb{P}\left(\bigcap_{N=1}^\infty\bigcup_{n=N}^\infty A_n\right)=0.$$

Define

$$\Omega_\circ\overset{\text{def}}{=}\Omega\setminus\left(\bigcap_{N=1}^\infty\bigcup_{n=N}^\infty A_n\right)=\bigcup_{N=1}^\infty\bigcap_{n=N}^\infty A_n^c.$$

For $\omega\in\Omega_\circ$, there is an $N_\omega\in\mathbb{N}$ such that

$$\sup_{n\ge N_\omega}2^{n\gamma}\max_{0\le j\le 2^n-1}\left|W^\infty_{(j+1)/2^n}-W^\infty_{j/2^n}\right|\le 1$$

Since

$$\sup_{n<N_\omega}2^{n\gamma}\max_{0\le j\le 2^n-1}\left|W^\infty_{(j+1)/2^n}-W^\infty_{j/2^n}\right|$$

is also finite for each $\omega \in \Omega_\circ$,

$$\Xi^\circ_\gamma(\omega) \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} 2^{n\gamma} \max_{0\le j\le 2^n-1} \left|W^\infty_{(j+1)/2^n} - W^\infty_{j/2^n}\right|$$

is finite; i.e.,

$$\max_{0\le j\le 2^n-1} \left|W^\infty_{(j+1)/2^n} - W^\infty_{j/2^n}\right| \le \frac{\Xi^\circ_\gamma(\omega)}{2^{n\gamma}}$$

for all $\omega \in \Omega_\circ$.

For each $t\in[0,T]$ and $n\in\mathbb{N}$, define $\tau_n(t) \stackrel{\text{def}}{=} \lfloor t2^n\rfloor/2^n$.

Fix $t\in[0,T]$. We first want to compare $\tau_n(t)$ and $\tau_{n+1}(t)$ (both of which can be written as dyadic rationals of level n). Assume that $\tau_n(t) = k/2^n$; i.e.,

$$\frac{k}{2^n} \le t < \frac{k+1}{2^n}.$$

If $t < (k+\frac{1}{2})/2^n$, then

$$\frac{2k}{2^{n+1}} \le t < \frac{2k+1}{2^{n+1}}$$

so $\tau_{n+1}(t) = 2k/2^{n+1}$. Thus

$$W^\infty_{\tau_n(t)} - W^\infty_{\tau_{n+1}(t)} = W^\infty_{(2k)/2^{n+1}} - W^\infty_{(2k)/2^{n+1}} = 0.$$

If $t \ge (k+\frac{1}{2})/2^n$, then

$$\frac{2k+1}{2^{n+1}} \le t < \frac{2k+2}{2^{n+1}}$$

and $\tau_{n+1}(t) = (2k+1)/2^{n+1}$. In this case

$$\left|W^\infty_{\tau_n(t)} - W^\infty_{\tau_{n+1}(t)}\right| = \left|W^\infty_{(2k)/2^{n+1}} - W^\infty_{(2k+1)/2^{n+1}}\right| \le \frac{\Xi^\circ_\gamma(\omega)}{2^{(n+1)\gamma}}.$$

Thus for any n_1 and n_2 in $\mathbb{N}$ with $n_1 < n_2$,

$$|W^\infty_{\tau_{n_2}(t)} - W^\infty_{\tau_{n_1}(t)}| \le \sum_{n'=n_1}^{n_2-1} \left|W^\infty_{\tau_{n+1}(t)} - W^\infty_{\tau_n(t)}\right| \le \sum_{n'=n_1}^{n_2-1} \frac{\Xi^\circ(\omega)}{2^{(n'+1)\gamma}}. \tag{5}$$

Thus

$$\lim_{n_1,n_2\to\infty} |W^\infty_{\tau_{n_1}(t)} - W^\infty_{\tau_{n_2}(t)}| = 0$$

and since $\mathbb{R}$ is complete, $\lim_{n\to\infty} W^\infty_{\tau_n(t)}$ is well-defined for each $\omega\in\Omega_\circ$. Let's then define

$$W_t \stackrel{\text{def}}{=} \begin{cases} \lim_{n\to\infty} W^\infty_{\tau_n(t)}(\omega) & \text{if } \omega\in\Omega_\circ \\ 0 & \omega \in A. \end{cases}$$

We now claim that W is continuous. Clearly W is continuous on A; we really only need consider $\omega\in\Omega_\circ$. Fix s and t in $[0,T]$ and assume that $t-s>0$. Let $N\in\mathbb{N}$ be such that

$$\frac{1}{2^{N+1}} \le t-s < \frac{1}{2^N}.$$

Assume that $\tau_N(s) = k/2^N$; i.e.,

$$\frac{k}{2^N} \le s < \frac{k+1}{2^N}.$$

If $t < (k+1)/2^n$, then

$$\frac{k}{2^n} \le s \le t < \frac{k+1}{2^n}$$

and $\tau_n(t) = k/2^n$. In this case

$$W^\infty_{\tau_n(t)} - W^\infty_{\tau_n(s)} = W^\infty_{k/2^n} - W^\infty_{k/2^n} = 0.$$

If $t \ge (k+1)/2^n$, then

$$\frac{k+1}{2^n} \le t \le s + \frac{1}{2^N} \le \frac{k+2}{2^N}.$$

and $\tau_n(t) = (k+1)/2^n$; we here get that

$$\left|W^\infty_{\tau_n(t)} - W^\infty_{\tau_n(s)}\right| = \left|W^\infty_{(k+1)/2^n} - W^\infty_{(k+1)/2^n}\right| \le \frac{\Xi^\circ_\gamma(\omega)}{2^{n\gamma}}.$$

Thus for any $n \ge N$, we have that

$$\begin{aligned}
|W^\infty_{\tau_n(t)} - W^\infty_{\tau_n(s)}| &\le |W^\infty_{\tau_n(t)} - W^\infty_{\tau_N(t)}| + |W^\infty_{\tau_N(t)} - W^\infty_{\tau_N(s)}| + |W^\infty_{\tau_n(s)} - W^\infty_{\tau_N(s)}| \\
&\le 2\sum_{n'=N}^{n} \frac{\Xi^\circ_\gamma(\omega)}{2^{\gamma n'}} + \frac{\Xi^\circ_\gamma(\omega)}{2^{N\gamma}} \le 2\Xi^\circ_\gamma(\omega)\sum_{n'=N}^{\infty}\left(\frac{1}{2^\gamma}\right)^{n'} + \Xi^\circ_\gamma(\omega)\left(\frac{1}{2^N}\right)^\gamma \\
&\le \Xi^\circ_\gamma(\omega)\left\{\frac{2}{1-2^{-\gamma}} + 1\right\}\left(\frac{1}{2^N}\right)^\gamma \\
&\le \frac{\Xi^\circ_\gamma(\omega)}{2^\gamma}\left\{\frac{2}{1-2^{-\gamma}} + 1\right\}\left(\frac{1}{2^{N-1}}\right)^\gamma
\end{aligned}$$

Letting $n \nearrow \infty$, we get that

$$|W_t - W_s| \le \frac{\Xi^\circ_\gamma(\omega)}{2^\gamma}\left\{\frac{2}{1-2^{-\gamma}} + 1\right\}|t-s|^\gamma$$

Finally, note that

$$\begin{aligned}
\mathbb{E}[|W_t - W^\infty_t| \wedge 1] &= \lim_{n\to\infty}\mathbb{E}[|W^\infty_{\tau_n(t)} - W^\infty_t| \wedge 1] \le \lim_{n\to\infty}\mathbb{E}[|W^\infty_{\tau_n(t)} - W^\infty_t|] \\
&\le \lim_{n\to\infty}\sqrt{\mathbb{E}[|W^\infty_{\tau_n(t)} - W^\infty_t|^2]} = \lim_{n\to\infty}\sqrt{\tau_n(t) - t} = 0.
\end{aligned}$$

□

This calculation is essentially due to Kolmogorov.

Exercises

(1) Prove (1).

(2) In the notation of (3), prove that

$$\mathbb{E}\left[\exp\left[\sqrt{-1}\sum_{n=0}^{N}\xi_n\langle f,\phi_n\rangle_{L^2}\right]\right] = \exp\left[-\frac{1}{2}\sum_{n=0}^{N}\langle f,\phi_n\rangle^2_{L^2}\right]$$

(3) Note that W^N is differentiable. Define

$$\mathcal{W}^N(\theta) \overset{\text{def}}{=} \int_{t=0}^{T}\dot W^N(t)\exp\left[\sqrt{-1}\theta t\right]dt$$

for all $\theta \in \mathbb{R}$ and $N \in \mathbb{N}$. Compute

$$\lim_{N\to\infty}\mathbb{E}\left[\left|\mathcal{W}^N(\theta)\right|^2\right]$$

for all $\theta \in \mathbb{R}$.

CHAPTER 2

Martingales

Let W be a standard Brownian motion. Define

$$\mathscr{W}_t \stackrel{\text{def}}{=} \bigcap_{s>t} \sigma\{W_r;\, r \le s\}. \tag{6}$$

Then $\{\mathscr{W}_t\}_{t\ge 0}$ is a right-continuous filtration and that W is adapted to $\{\mathscr{W}_t\}_{t\ge 0}$. Fix $0 \le s < t$ and $A \in \mathscr{W}_t$. For any $s' \in (s,t)$, $A \in \sigma\{W_r; r \le s'\}$, so

$$\mathbb{E}[W_t \chi_A] = \mathbb{E}[(W_t - W_{s'})\chi_A] + \mathbb{E}[W_{s'}\chi_A] = \mathbb{E}[W_{s'}\chi_A]$$

since $W_t - W_{s'}$ is independent of A. Since $\lim_{s'\searrow t} \mathbb{E}[|W_{s'} - W_s|] = 0$,

$$\mathbb{E}[W_t \chi_A] = \mathbb{E}[W_s \chi_A],$$

so $\mathbb{E}[W_t|\mathscr{W}_s] = W_s$. Thus W is a *continuous square integrable martingale with respect to a right-continuous filtration.*

Now that we have one continuous square-integrable martingale with a right-continuous filtration, let's generalize. Let M be a square-integrable continuous process and let $\{\mathscr{F}_t\}_{t\ge 0}$ be a right-continuous filtration and suppose that M is a martingale with respect to $\{\mathscr{F}_t\}_{t>0}$.

Let's next understand *quadratic variation.* Fix $0 \le s < t$ and $s' \in (s,t]$. For any $A \in \sigma\{W_r;\, r \le s'\}$,

$$\begin{aligned}\mathbb{E}[(W_t^2 - t)\chi_A] &= \mathbb{E}\left[\left\{(W_t - W_{s'} + W_{s'})^2 - t\right\}\chi_A\right] = \mathbb{E}\left[\left\{(W_t - W_{s'})^2 + 2(W_t - W_{s'})W_{s'} + W_{s'}^2 - t\right\}\chi_A\right] \\ &= \mathbb{E}\left[\left\{t - s' + W_{s'}^2 - t\right\}\chi_A\right] = \mathbb{E}[(W_{s'}^2 - s')\chi_A]\end{aligned}$$

where we have again used the fact that $W_t - W_{s'}$ is independent of A. Since $\lim_{s'\searrow s} \mathbb{E}[|W_{s'} - W_s|^2] = 0$,

$$\mathbb{E}[(W_t^2 - t)\chi_A] = \mathbb{E}[(W_s^2 - s)\chi_A]$$

so in fact $\mathbb{E}[W_t^2 - t|\mathscr{W}_s] = W_s^2 - s$; i.e., $W_t^2 - t$ is a martingale. We also note that the process $\{t;\, t \ge 0\}$ is continuous, adapted, and non-decreasing, and that it vanishes at 0.

Definition 0.4. Let M be a square-integrable martingale with respect to a filtration $\{\mathscr{F}_t\}_{t>0}$. A continuous and adapted non-decreasing process $\langle M\rangle$ for which $\langle M\rangle_0 = 0$ is called the *quadratic variation* of M if $\{M_t^2 - \langle M\rangle_t\,;\, t > 0\}$ is a martingale (with respect to $\{\mathscr{F}_t\}_{t>0}$).

In fact, we could be more general (and require slightly less than continuity of $\langle M\rangle$), but at the cost of a number of technicalities. See [**?**]. It also turns out that $\langle M\rangle$ must be unique. Note that if $M_0 = 0$, then

$$\mathbb{E}[M_t^2] = \mathbb{E}[M_t^2 - \langle M\rangle_t] + \mathbb{E}[\langle M\rangle_t] = \mathbb{E}[M_0^2 - \langle M\rangle_0] + \mathbb{E}[\langle M\rangle_t] = \mathbb{E}[\langle M\rangle_t] \tag{7}$$

(which in the case of Brownian motion, recovers the fact that $\mathbb{E}[W_t^2] = t$).

Assume now that M is a continuous square-integrable martingale with respect to a right-continuous filtration $\{\mathscr{F}_t\}_{t\ge 0}$. A number of calculations work only if we assume that M is bounded. Let's understand that we can localize to achieve this. Fix $L > 0$ and define

$$\tau \stackrel{\text{def}}{=} \inf\left\{t \ge 0 : |M_t| > L \text{ or } \langle M\rangle_t > L\right\}.$$

Then τ is a stopping time. Define $\tilde{M}_t \stackrel{\text{def}}{=} M_{\tau\wedge t}$ for all $t \ge 0$. We claim that $\tilde{M}$ is also a continuous martingale with respect to $\{\mathscr{F}_t\}_{t\ge 0}$ (and of course it is bounded). We also claim that $\left\langle \tilde{M}\right\rangle_t = \langle M\rangle_{\tau\wedge t}$. Clearly $\tilde{M}$ is square integrable. Secondly, define $\tau_N \stackrel{\text{def}}{=} \frac{\lceil \tau N\rceil}{N}$. Then the τ_N's are stopping times and $\tau_N \searrow \tau$.

Assume for a moment that X is a continuous process which is adapted to $\{\mathscr{F}_t\}_{t\ge 0}$. For any $t \ge 0$ and $A \in \mathscr{B}(\mathbb{R})$,

$$\{X_{\tau_N\wedge t} \in A\} = (\{\tau_N > t\} \cap \{X_t \in A\}) \cup (\{\tau_N \le t\} \cap \{X_{\tau_N} \in A\}).$$

Since τ_N is a stopping time and X is adapted, $\{\tau_N > t\}$ and $\{X_t \in A\}$ are both in $\mathscr{F}_t$. Secondly,

$$\{\tau_N \le t\} \cap \{X_{\tau_N} \in A\} = \bigcup_{\substack{n\in\mathbb{N}\\ n/N\le t}} \left\{\tau_N = \frac{n}{N}\right\} \cap \{X_{n/N} \in A\}.$$

If $n/N \le t$, then $\{\tau_N = n/N\}$ and $\{X_{n/N} \in A\}$ are both in $\mathscr{F}_{n/N} \subset \mathscr{F}_t$. Collecting things together, we get that $X_{\tau_N\wedge t}$ is $\mathscr{F}_t$-measurable. Since $X_{\tau\wedge t} = \lim_{N\to\infty} X_{\tau_N\wedge t}$, $X_{\tau\wedge t}$ is the limit of $\mathscr{F}_t$-measurable random variables; thus $X_{\tau\wedge t}$ is also $\mathscr{F}_t$-measurable. Hence both $\{M_{\tau_N\wedge t};\, t\ge 0\}$ and $\{\langle M\rangle_{\tau_N\wedge t};\, t \ge 0\}$ are adapted to $\{\mathscr{F}_t\}_{t\ge 0}$. Let's now moreover assume that X is a square-integrable martingale with respect to $\{\mathscr{F}_t\}_{t\ge 0}$. Fix $0 \le s \le t$ and $A \in \mathscr{F}_s$. Then

$$\begin{aligned}\mathbb{E}[X_{\tau_N\wedge t}\chi_A] &= \mathbb{E}\left[X_{\tau_N\wedge t}\chi_{A\cap\{\tau_N>s\}}\right] + \mathbb{E}\left[X_{\tau_N\wedge t}\chi_{A\cap\{\tau_N\le s\}}\right]\\ &= \mathbb{E}\left[X_{(\tau_N\wedge t)\vee s}\chi_{A\cap\{\tau_N>s\}}\right] + \mathbb{E}\left[X_{\tau_N}\chi_{A\cap\{\tau_N\le s\}}\right]\\ &= \mathbb{E}\left[X_s\chi_{A\cap\{\tau_N>s\}}\right] + \mathbb{E}\left[X_{\tau_N}\chi_{A\cap\{\tau_N\le s\}}\right]\\ &= \mathbb{E}\left[X_{\tau_N\wedge s}\chi_A\right]\end{aligned} \tag{8}$$

Note that $(\tau_N \wedge t) \vee s$ and s are both finite-valued stopping times and $(\tau_N \wedge t) \vee s \ge s$; we have used optional sampling in (8). We now let $N \to \infty$. From Doob's inequality, we have that

$$\mathbb{E}\left[\sup_{0\le s\le t} X_s^2\right] \le 4\mathbb{E}[X_t^2] < \infty.$$

Using this, we can show that

$$\lim_{N\to\infty} \mathbb{E}[X_{\tau_N\wedge r}\chi_A] = \mathbb{E}[X_{\tau\wedge r}\chi_A]$$

for all $r \in [0, t]$. Taking limits in (8), we get that indeed $\mathbb{E}[X_{\tau\wedge t}|\mathscr{F}_s] = X_{\tau\wedge s}$. Hence both $\{N_{\tau_N\wedge t};\, t \ge 0\}$ and $\{N_{\tau_N\wedge t} - \langle N\rangle_{\tau_N\wedge t};\, t \ge 0\}$ are martingales, so $\left\langle \tilde{M}\right\rangle_t = \langle M\rangle_{\tau_N\wedge t}$.

Let's look at the variation of martingales. Again assume that M is a continuous and square-integrable martingale with respect to $\{\mathscr{F}_t\}_{t\ge 0}$; assume that $\langle M\rangle$ is the quadratic variation of M. For $0 \le s \le t$,

$$(M_t - M_s)^2 - (\langle M\rangle_t - \langle M\rangle_s) = \left(M_t^2 - \langle M\rangle_t\right) - \left(M_s^2 - \langle M\rangle_s\right) - 2(M_t - M_s)M_s,$$

we have that

$$\mathbb{E}\left[(M_t - M_s)^2 - (\langle M\rangle_t - \langle M\rangle_s)\middle|\mathscr{F}_s\right] = \mathbb{E}\left[\left(M_t^2 - \langle M\rangle_t\right) - \left(M_s^2 - \langle M\rangle_s\right) - 2(M_t - M_s)M_s\middle|\mathscr{F}_s\right] = 0. \tag{9}$$

Rearranging this, we get that

$$\mathbb{E}\left[(M_t - M_s)^2\middle|\mathscr{F}_s\right] = \mathbb{E}\left[\langle M\rangle_t - \langle M\rangle_s\middle|\mathscr{F}_s\right] \tag{10}$$

and taking expectations, we have that

$$\mathbb{E}\left[(M_t - M_s)^2\right] = \mathbb{E}\left[\langle M\rangle_t - \langle M\rangle_s\right]$$

In other words, the expected increments of $\langle M\rangle$ control the expected variance of increments of M. This will be useful.

Fix $t \ge 0$. For each $N \in \mathbb{N}$, define

$$s_n^{(N)} \overset{\text{def}}{=} \frac{tn}{N}. \tag{11}$$

Define next

$$V^{(N)} \overset{\text{def}}{=} \sum_{n=0}^{N-1} \left(M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}\right)^2. \tag{12}$$

Let's also assume that there is a $K > 0$ such that

$$\sup_{0\le s\le t} |M_s| \le K \qquad \text{and} \qquad \sup_{0\le s\le t} |\langle M\rangle_s| \le K.$$

We first claim that

$$\mathbb{E}\left[\left(V^{(N)}\right)^2\right] \le 4K^4 + 8K^3. \tag{13}$$

Indeed,

$$\mathbb{E}\left[\left(V^{(N)}\right)^2\right] = 2\sum_{0\le n<m\le N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{m+1}} - M_{s^{(N)}_m}\right)^2 \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] + \sum_{n=0}^{N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^4\right]$$

We first note that

$$\begin{aligned}
&\sum_{0\le n<m\le N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{m+1}} - M_{s^{(N)}_m}\right)^2 \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \\
&= \sum_{n=0}^{N-1}\sum_{m=n+1}^{N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{m+1}} - M_{s^{(N)}_m}\right)^2 \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \\
&= \sum_{n=0}^{N-1}\sum_{m=n+1}^{N-1} \mathbb{E}\left[\mathbb{E}\left[\left(M_{s^{(N)}_{m+1}} - M_{s^{(N)}_m}\right)^2 \middle| \mathscr{F}_{s^{(N)}_j}\right] \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \\
&= \sum_{n=0}^{N-1}\sum_{m=n+1}^{N-1} \mathbb{E}\left[\mathbb{E}\left[\langle M\rangle_{s^{(N)}_{m+1}} - \langle M\rangle_{s^{(N)}_m} \middle| \mathscr{F}_{s^{(N)}_j}\right] \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \\
&= \sum_{n=0}^{N-1} \mathbb{E}\left[\mathbb{E}\left[\langle M\rangle_t - \langle M\rangle_{s^{(N)}_n} \middle| \mathscr{F}_{s^{(N)}_j}\right] \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \\
&\le 2K\sum_{n=0}^{N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \le 2K\sum_{n=0}^{N-1} \mathbb{E}\left[\langle M\rangle_{s^{(N)}_{n+1}} - \langle M\rangle_{s^{(N)}_n}\right] \\
&\le 2K\mathbb{E}[\langle M\rangle_T - \langle M\rangle_0] \le 4K^2.
\end{aligned}$$

Secondly

$$\begin{aligned}
\sum_{n=0}^{N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^4\right] &\le 4K^2\sum_{n=0}^{N-1} \mathbb{E}\left[\left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2\right] \le 4K^2\sum_{n=0}^{N-1} \mathbb{E}\left[\langle M\rangle_{s^{(N)}_{n+1}} - \langle M\rangle_{s^{(N)}_n}\right] \\
&\le 4K^2\mathbb{E}\left[\langle M\rangle_t - \langle M\rangle_0\right] \le 4K^2\mathbb{E}\left[\langle M\rangle_t - \langle M\rangle_0\right] = 8K^3.
\end{aligned}$$

Combining things together, we get (13).

We next claim that

$$\lim_{N\to\infty} \mathbb{E}\left[\left(V^{(N)} - \langle M\rangle_t - \langle M\rangle_0\right)^2\right] = 0; \tag{14}$$

i.e., $V^{(N)}$ converges to $\langle M\rangle_t - \langle M\rangle_0$ in L^2. For convenience, define

$$D^{(N)}_n \overset{\text{def}}{=} \left(M_{s^{(N)}_{n+1}} - M_{s^{(N)}_n}\right)^2 - \left(\langle M\rangle_{s^{(N)}_{n+1}} - \langle M\rangle_{s^{(N)}_n}\right). \tag{15}$$

Then

$$\begin{aligned}
\mathbb{E}\left[\left|V^{(N)} - (\langle M\rangle_t - \langle M\rangle_0)\right|^2\right] &= \mathbb{E}\left[\left(\sum_{n=0}^{N-1} D^{(N)}_n\right)^2\right] \quad = 2\sum_{0\le n<m\le N-1} \mathbb{E}\left[D^{(N)}_m D^{(N)}_n\right] + \mathbb{E}\left[\sum_{n=0}^{N-1} |D^{(N)}_n|^2\right] \\
&= \mathbb{E}\left[\sum_{n=0}^{N-1} |D^{(N)}_n|^2\right]
\end{aligned}$$

We have used here (9) to see that $\mathbb{E}[D_n^{(N)}|\mathscr{F}_n] = 0$. Note next that

$$
\begin{aligned}
|D_n^{(N)}|^2 &= \left\{ \left(M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}} \right)^2 - \left(\langle M \rangle_{s_{n+1}^{(N)}} - \langle M \rangle_{s_n^{(N)}} \right) \right\}^2 \\
&\le 2 \left(M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}} \right)^4 + 2 \left(\langle M \rangle_{s_{n+1}^{(N)}} - \langle M \rangle_{s_n^{(N)}} \right)^2 \\
&\le 2|\xi_N^a|^2 \left(M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}} \right)^2 + 2|\xi_N^b| \left\{ \langle M \rangle_{s_{n+1}^{(N)}} - \langle M \rangle_{s_n^{(N)}} \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_N^a &\overset{\text{def}}{=} \max_{0 \le n \le N-1} |M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}| \\
\xi_N^b &\overset{\text{def}}{=} \max_{0 \le n \le N-1} \left\{ \left| \langle M \rangle_{s_{n+1}^{(N)}} - \langle M \rangle_{s_n^{(N)}} \right| \right\}
\end{aligned}
\tag{16}
$$

Note that $|\xi_N^a| \le 2K$ and $|\xi_N^b| \le 2K$ and that by continuity, $\lim_{N\to\infty} \xi_N^a = 0$ and $\lim_{N\to\infty} \xi_N^b = 0$ $\mathbb{P}$-a.s. Thus

$$
\begin{aligned}
\mathbb{E}\left[\sum_{n=0}^{N-1} |D_n^{(N)}|^2 \right] &\le 2\mathbb{E}\left| |\xi_N^a|^2 V_2^{(N)} \right] + 2\mathbb{E}\left[|\xi_N^b|^2 \left\{ \langle M \rangle_t - \langle M \rangle_0 \right\} \right] \\
&\le 2 \left\{ \mathbb{E}\left[|\xi_N^b|^4 \right] \mathbb{E}\left[\left(V^{(N)} \right)^2 \right] \right\}^{1/2} + 4K\mathbb{E}[|\xi_N^a|^2].
\end{aligned}
$$

Thus

$$
\lim_{N\to\infty} \mathbb{E}\left[\sum_{n=0}^{N-1} |D_n^{(N)}|^2 \right] = 0;
\tag{17}
$$

Using this and combining things together, we indeed get (14).

Exercises

(1) Assume that W is a Brownian motion.
  (a) Show that $\lim_{s\to t} \mathbb{E}[|W_s - W_t|] = 0$ for all $t \ge 0$.
  (b) Show that $\lim_{s\to t} \mathbb{E}[W_t \chi_A] = \mathbb{E}[W_s \chi_A]$ for all $t \ge 0$ and $A \in \mathscr{F}$.

(2) One reason we are interested in right-continuous filtrations is as follows. Suppose that X is a continuous stochastic process which is adapted to a right-continuous filtration $\{\mathscr{F}_t\}_{t\ge 0}$ and which takes values in a topological space S. Fixing an open subset O of S, define $\tau \overset{\text{def}}{=} \inf\{t \ge 0 : X_t \in O\}$. Show that

$$
\{\tau \le t\} = \bigcap_{\substack{s>t \\ s\in\mathbb{Q}}} \left(\bigcup_{\substack{r<s \\ r\in\mathbb{Q}}} \{X_r \in O\} \right)
$$

and that thus $\{\tau \le t\} \in \mathscr{F}_t$; i.e., τ is a stopping time.

(3) Suppose that τ is a stopping time. Define $\tau_N \overset{\text{def}}{=} \frac{\lceil \tau N \rceil}{N}$ for all $N \in \{1, 2 \dots\}$.
  - Show that each τ_N is also a stopping time.
  - Show that $\{\tau_N = n/N\} \in \mathscr{F}_{n/N}$ for all $n \in \mathbb{N}$.

(4) Suppose that τ and σ are two stopping times (with respect to a reference filtration $\{\mathscr{F}_t\}_{t\ge 0}$). Show that $\tau \wedge \sigma$ is a stopping time with respect to $\{\mathscr{F}_t\}_{t\ge 0}$.

(5) Suppose that $\lim_{n\to\infty} X_n = 0$ $\mathbb{P}$-a.s. and that $\sup_{n\in\mathbb{N}} \mathbb{E}[X_n^2] < \infty$. Show that $\lim_{n\to\infty} \mathbb{E}[|X_n|] = 0$.

(6) Suppose that M is a continuous square-integrable martingale with respect to a filtration $\{\mathscr{F}_t\}_{t\ge 0}$ and that $\langle M \rangle$ is its quadratic variation. Let τ be a bounded stopping time, and define $\tilde{M}_t \overset{\text{def}}{=} M_{t\wedge\tau}$. Show that $\langle M \rangle_{t\wedge\tau}$ is the quadratic variation of $\tilde{M}$.

CHAPTER 3

Stochastic Integration

We want to understand a notion of stochastic integration which focusses on *martingale transforms*, by which we can make new martingales out of old ones.

Let's first define a functional space. If ξ is a continuous process on an interval containing $[0,T]$, define

$$\|\xi\|_{H(T)} \stackrel{\text{def}}{=} \sqrt{\mathbb{E}\left[\sup_{0\le t\le T}|\xi_s|^2\right]}.$$

Let H_T be the collection of continuous and adapted (with respect to $\{\mathscr{F}_t\}_{t\ge 0}$) processes on $[0,T]$ for which $\|f\|_{H(T)}<\infty$. Then H_T is a Banach space with norm $\|\cdot\|_{H(T)}$. Note that if $\xi\in H_T$ then $\|\xi\|_{H(T')}\le\|\xi\|_{H(T)}$ if $T'\le T$.

Define a *predictable simple function* f as a map $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ of the form

$$f_t(\omega)\stackrel{\text{def}}{=}\sum_{j=0}^{J-1}\xi_j(\omega)\chi_{(t_j,t_{j+1}]}(t)\qquad t\ge 0$$

for some $0=t_0<t_1\dots t_J\le T$, where ξ_j is a bounded $\mathscr{F}_{t_j}$-measurable random variable. We then define

$$\tilde{M}_t\stackrel{\text{def}}{=}\int_{s=0}^t f_s dM_s=\sum_{j=0}^{n-1}\xi_j(M_{t_j\wedge t}-M_{t_{j-1}\wedge t})$$

for all $0\le t\le T$.

In the problems, you will see that $\tilde{M}$ is a square-integrable martingale with quadratic variation

$$\begin{aligned}\left\langle\tilde{M}\right\rangle_t&=\int_{s=0}^t\left(\sum_{j=0}^{J-1}\xi_j^2\chi_{(t_j,t_{j+1}]}(s)\right)^2 d\langle M\rangle_s=\int_{s=0}^t f_s^2 d\langle M\rangle_s\\&=\sum_{j=0}^{J-1}\xi_j^2\left(\langle M\rangle_{t_{j+1}\wedge t}-\langle M\rangle_{t_j\wedge t}\right),\end{aligned}$$

To proceed, note that since $\langle M\rangle$ is by assumption nondecreasing and continuous, we can pathwise define the measure $d\langle M\rangle_t$; i.e, for any $0\le s<t$,

$$\left(\int_{r\in(s,t]}d\langle M\rangle_r\right)(\omega)=(d\langle M\rangle(\omega))(s,t]=\langle M\rangle_t(\omega)-\langle M\rangle_s(\omega).$$

Note that thus (by (7)),

$$\mathbb{E}[\tilde{M}_t^2]=\mathbb{E}\left[\int_{s=0}^t f_s^2 d\langle M\rangle_s\right]. \tag{18}$$

This is the *Ito isometry.* It says that the $L^2(\Omega)$ norm of a random variable (given by stochastic integration) is equal to the $L^2(\Omega\times d\langle M\rangle)$ integral of the integrand. Note that $\tilde{M}\in\mathsf{H}(T)$ and that, by Doob's inequality,

$$\|\tilde{M}\|_{H(T)}\le 2\sqrt{\mathbb{E}\left[\int_{s=0}^t f_s^2 d\langle M\rangle_s\right]}. \tag{19}$$

We want to close things up.

Remark 0.5. What we are about to do is a bit subtle. Let's consider a toy problem which will provide some exemplary geometry.

The set $\mathbb{Q}\times\{0\}$ is a subset of $\mathbb{R}^2$ (obviously). For $(q,0)\in\mathbb{Q}\times\{0\}$, define

$$\mathcal{T}(q,0)\overset{\text{def}}{=}(2q,3q,q)\in\mathbb{R}^3.$$

Thus $T:\mathbb{Q}\times\{0\}\to\mathbb{R}^3$. Of course $\mathcal{T}$ is linear. It is also continuous;

$$\|\mathcal{T}(q_1,0)-\mathcal{T}(q_2,0)\|_{\mathbb{R}^3}\le\sqrt{14}\|(q_1,0)-(q_2,0)\|_{\mathbb{R}^2} \tag{20}$$

for all $(q_1,0)$ and $(q_2,0)$ in $\mathbb{Q}\times\{0\}$ (we are using the standard norms on $\mathbb{R}^2$and $\mathbb{R}^3$ here).

We should be able to "extend" $\mathcal{T}$; we should be able to (naturally and uniquely) extend $\mathcal{T}$ to all of $\mathbb{R}\times\{0\}$ (which is the closure of $\mathbb{Q}\times\{0\}$); i.e., the map

$$\mathcal{T}^e(x,0)\overset{\text{def}}{=}(2x,3x,x)\qquad (x,0)\in\mathbb{R}\times\{0\}$$

is extension of T.

The procedure is as follows. Fix $(x,0)\in\mathbb{R}\times\{0\}$. Let $\{(q_n,0)\}_{n\in\mathbb{N}}$ be sequence of points in $\mathbb{Q}\times\{0\}$ which converges to $(x,0)$. Then

$$\varlimsup_{n,n'\to\infty}\|(q_n,0)-(q_{n'},0)\|_{\mathbb{R}^2}\le\varlimsup_{n,n'\to\infty}\{\|(q_n,0)-(x,0)\|_{\mathbb{R}^2}+\|(q_{n'},0)-(x,0)\|_{\mathbb{R}^2}\}=0.$$

Thus by (20),

$$\varlimsup_{n,n'\to\infty}\|\mathcal{T}(q_n,0)-\mathcal{T}(q_{n'},0)\|_{\mathbb{R}^3}\le\sqrt{14}\varlimsup_{n,n'\to\infty}\|(q_n,0)-(q_{n'},0)\|_{\mathbb{R}^2}=0.$$

Since $\mathbb{R}^3$ is complete, there is a point (which we shall label as $\mathcal{T}^e(x,0)$) in $\mathbb{R}^3$ such that

$$\lim_{n\to\infty}\|\mathcal{T}(q_n,0)-\mathcal{T}^e(x,0)\|_{\mathbb{R}^3}=0.$$

In fact (see the problems) $\mathcal{T}^e$ is unique and linear and $\mathcal{T}^e=\mathcal{T}$ on $\mathbb{Q}\times\{0\}$.

Note that we simply have no idea how to define $\mathcal{T}(2,5)$; i.e., how to extend T off of $\mathbb{R}\times\{0\}$.

The general geometry is that we have a subset X' which is a subset of a Banach space X. We have a continuous map $\mathcal{T}$ from X' into another Banach space Y. We can then uniquely extend $\mathcal{T}$ to $\overline{\mathsf{X}'}$, the closure of X' in X.

The Ito isometry allows us to define the Ito integral. Fix $T>0$. For a measurable stochastic process f, define

$$\|f\|_{\mathsf{X}}\overset{\text{def}}{=}\sqrt{\mathbb{E}\left[\int_{s=0}^T f_s^2 d\langle M\rangle_s\right]}.$$

Let X be the collection of measurable processes for which $\|f\|_{\mathsf{X}}<\infty$; then X is a Banach space with norm $\|\cdot\|_{\mathsf{X}}$. Let X' be the collection of simple predictable functions; clearly $\mathsf{X}'\subset\mathsf{X}$. Define a new stochastic process by stochastic integration; define

$$\mathcal{T}_t(f)\overset{\text{def}}{=}\int_{s=0}^t f_s dM_s.$$

Then $\mathcal{T}:\mathsf{X}'\to\mathsf{H}(T)$ and by (19) we in fact have that

$$\|\mathcal{T}(f)\|_{H(T)}\le 2\|f\|_{\mathsf{X}}.$$

Define $\mathsf{P}\overset{\text{def}}{=}\overline{\mathsf{X}'}$; we refer to P as the collection of *predictable* functions; we can then uniquely extend $\mathcal{T}$ to a linear map on $\overline{\mathsf{X}'}$. In other words, if $f\in\mathsf{P}$, then there is a collection $\{f^{(n)}\}_{n\in\mathbb{N}}$ of predictable simple functions such that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_{s=0}^T |f_s^n-f|^2 d\langle M\rangle_s\right]=0,$$

and there is a $\mathcal{T}(f)\in\mathsf{H}(T)$ such that

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\left|\mathcal{T}_t(f)-\int_{s=0}^t f_s^{(n)}dM_s\right|^2\right]=0.$$

For $0 \le s \le t \le T$ and $A \in \mathscr{F}_s$,

$$\mathbb{E}[\mathcal{T}_t(f)\chi_A] = \lim_{n\to\infty} \mathbb{E}\left[\int_{r=0}^t f_r^{(n)} dM_t \chi_A\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_{r=0}^s f_r^{(n)} dM_r \chi_A\right] = \mathbb{E}[\mathcal{T}_s(f)\chi_A]$$

$$\begin{aligned}\mathbb{E}\left[\left\{(\mathcal{T}_t(f))^2 - \int_{r=0}^t f_r^2 d\langle M\rangle_r\right\}\chi_A\right] &= \lim_{n\to\infty} \mathbb{E}\left[\left\{\left(\int_{r=0}^t f_r^{(n)} dM_r\right)^2 - \int_{r=0}^t (f_r^{(n)})^2 d\langle M\rangle_r\right\}\chi_A\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\left\{\left(\int_{r=0}^s f_r^{(n)} dM_r\right)^2 - \int_{r=0}^s (f_r^{(n)})^2 d\langle M\rangle_r\right\}\chi_A\right] \\ &= \mathbb{E}\left[\left\{(\mathcal{T}_s(f))^2 - \int_{r=0}^s f_r^2 d\langle M\rangle_r\right\}\chi_A\right],\end{aligned}$$

Thus $\mathcal{T}(f)$ is a square-integrable continuous martingale with quadratic variation process

$$\int_{s=0}^t f_s^2 d\langle M\rangle_s$$

Exercises

(1) Fix $0 \le t_0 < t_1$. As usual,

$$\chi_{(t_0,t_1]}(r) = \begin{cases} 1 & \text{if } s \in (t_0,t_1] \\ 0 & \text{else.}\end{cases}$$

Show that

$$\int_{s=0}^t \chi_{(t_0,t_1]}(s)ds = (t_1 \wedge t) - (t_0 \wedge t).$$

(2) Fix $0 \le t_1 < t_2$. Let M be an integrable $\{\mathscr{F}\}_{t\ge 0}$-adapted continuous stochastic process. Suppose that

$$\mathbb{E}[M_t|\mathscr{F}_s] = M_s \qquad (21)$$

if either
- $t_1 \le s < t \le t_2$
- $0 \le s < t \le t_1$.

Show that (21) then holds for all $0 \le s \le t \le t_2$.

(3) Fix $0 \le t_1 \le t_2$ and a bounded $\mathscr{F}_{t_1}$-measurable random variable ξ. Define

$$\tilde{M}_t \overset{\text{def}}{=} \xi(M_{t_2\wedge t} - M_{t_1\wedge t}).$$

(a) Show that $\tilde{M}$ is a martingale with respect to $\{\mathscr{F}_t\}_{t\ge 0}$ (remember to show that it is adapted).
(b) Show that $\left\langle \tilde{M}\right\rangle_t = \xi_{t_1}^2\left(\langle M\rangle_{t_2\wedge t} - \langle M\rangle_{t_1\wedge t}\right)$.

(4) Fix $0 \le t_1 < t_2 \le t_3 < t_4$ and two bounded random variables ξ_A and ξ_B, where ξ_A is $\mathscr{F}_{t_1}$-measurable and $\xi_B$is $\mathscr{F}_{t_3}$-measurable. Define

$$\tilde{M}_t^A \overset{\text{def}}{=} \xi_A(M_{t_2\wedge t} - M_{t_1\wedge t}) \qquad \text{and} \qquad \tilde{M}_t^B \overset{\text{def}}{=} \xi_B(M_{t_4\wedge t} - M_{t_3\wedge t}).$$

Show that $\tilde{M}^A\tilde{M}^B$ (i.e, the product of the two processes) is a martingale.

(5) Back to (6). Defining $\mathscr{W}_{t+} \overset{\text{def}}{=} \cap_{s>t}\mathscr{W}_s$ for all $t \ge 0$, Show that $\mathscr{W}_{t+} = \mathscr{W}_t$ for all $t \ge 0$; i.e., $\{\mathscr{W}_t\}_{t\ge 0}$ is right-continuous.

(6) Show that if f is adapted, bounded, and continuous, it is predictable.

(7) Suppose that τ is a stopping time. Again define $\tau_N \overset{\text{def}}{=} \lceil \tau N\rceil/N$.
(a) For each $k \in \mathbb{N}$ and $T > 0$, show that

$$\chi_{\{\tau_N = k/N\}}\chi_{(k/N,T]}$$

is a simple predictable function.
(b) Show that $\chi_{(\tau_N,T]}$ is a simple predictable function.
(c) Show that $\chi_{[0,\tau_N\wedge T]}$ is a simple predictable function.
(d) Show that $\chi_{[0,\tau\wedge T]}$ is a predictable function.

(e) Show that $\chi_{[0,\tau]}$ is a predictable function.

(8) In Remark 0.5,

(a) Show that $T^e(x,0)$ is uniquely defined; if there are two sequences $\{(q_n,0)\}_{n\in\mathbb{N}}$ and $\{(q'_n,0)\}_{n\in\mathbb{N}}$ converging to $(x,0)$ in $\mathbb{R}^2$, then $\lim_{n\to\infty} T(q_n,0) = \lim_{n\to\infty} T(q'_n,0)$.

(b) Show that T^e is linear

(c) Show that $T^e = T$ on $\mathbb{Q}\times\{0\}$.

(9) For each $T>0$, show that H_T is indeed a Banach space.

CHAPTER 4

SDE's

Let's consider the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$
$$X_0 = x_\circ.$$

where W is a standard Brownian motion. This is short for the integral equation

$$X_t = x_\circ + \int_{s=0}^t b(X_s)ds + \int_{s=0}^t \sigma(X_s)dW_s.$$

We will assume that b and σ are Lipshitz with common Lipshitz constant K_L.

We will use a Picard iteration. Fix $T>0$. Define

$$X_t^{(1)} \stackrel{\text{def}}{=} x_\circ$$
$$X_t^{(n+1)} \stackrel{\text{def}}{=} x_\circ + \int_{s=0}^t b(X_s^{(n)})ds + \int_{s=0}^t \sigma(X_s^{(n)})dW_s$$

for $0 \le t \le T$. Let's formalize this as a recursion on $\mathsf{H}(T)$, where $\mathsf{H}(T)$ was introduced in Chapter 2. For $Z \in \mathsf{H}(T)$, let's define

$$\begin{aligned} T_t^A(Z) &\stackrel{\text{def}}{=} \int_{s=0}^t b(Z_s)ds \\ T_t^B(Z) &\stackrel{\text{def}}{=} \int_{s=0}^t \sigma(Z_s)dW_s \\ T_t(Z) &\stackrel{\text{def}}{=} x_\circ + T_t^A(Z) + T_t^B(Z) \end{aligned} \tag{22}$$

for all $t \in [0,T]$. We have defined $X^{(n+1)} = T(X^{(n)})$ and we want this to solve

$$X = T(X)$$

Let's first use the Lipshitz requirement to see that T^A and T^B map $\mathsf{H}(T)$ into itself. Since b and σ are Lipshitz,

$$|b(x)| \le |b(0)| + |b(x)-b(0)| \le |b(0)| + K_L|x| \le K(1+|x|)$$
$$|\sigma(x)| \le |\sigma(0)| + |\sigma(x)-\sigma(0)| \le |\sigma(0)| + K_L|x| \le K(1+|x|)$$

where $K = \max\{|b(0)|, |\sigma(0)|, K_L\}$. Fix $Z \in \mathsf{H}(T)$. Suppose that $K>0$ is such that

$$|b(x)| \le K(1+|x|) \qquad \text{and} \qquad |\sigma(x)| \le K(1+|x|)$$

for all $x \in \mathbb{R}$. For $0 \le t \le T$,

$$\begin{aligned} (T_s^A(Z))^2 &\le \left|\int_{r=0}^s b(Z_r)dr\right|^2 \le \left|\int_{r=0}^s |b(Z_r)|dr\right|^2 \le \left|\int_{r=0}^t |b(Z_r)|dr\right|^2 \\ &\le K\left|\int_{r=0}^t (1+|Z_r|)dr\right|^2 \le Kt\left\{1+\sup_{0\le r\le T}|Z_r|\right\}^2 \le 2Kt\left\{1+\sup_{0\le r\le T}|Z_r|^2\right\} \end{aligned}$$

so

$$\|T^A(Z)\|_{H(T)}^2 \le 2KT(1+\|Z\|_{H(T)}^2).$$

We next use the Ito isometry to see that

$$\|T^B(Z)\|^2_{H(T)} \le 2\mathbb{E}\left[\int_{s=0}^T \sigma^2(Z_s)^2 ds\right] \le K^2\mathbb{E}\left[\int_{s=0}^T (1+|Z_s|)^2 ds\right]$$
$$\le 2K^2\mathbb{E}\left[\int_{s=0}^T \left\{1+|Z_s|^2\right\} ds\right] \le 2K^2 T\left\{1+\|Z\|^2_{H(T)}\right\}$$

Let's next show that the $X^{(n)}$'s converge. Fix $Z^{(1)}$ and $Z^{(2)}$ in $\mathsf{H}(T)$. Fix $0 \le s \le t \le T$. Then

$$|T^A_s(Z^{(1)}) - T^A_s(Z^{(1)})|^2 = \left|\int_{r=0}^s \left\{b(Z^{(1)}_r) - b(Z^{(2)}_r)\right\} dr\right|^2 \le t\int_{r=0}^s \left|b(Z^{(1)}_r) - b(Z^{(2)}_r)\right|^2 dr$$
$$\le t\int_{r=0}^t \left|b(Z^{(1)}_r) - b(Z^{(2)}_r)\right|^2 dr \le K^2 t\int_{r=0}^t \left|Z^{(1)}_r - Z^{(2)}_r\right|^2 dr$$
$$\le K^2 T\int_{r=0}^t \sup_{0\le r'\le r}\left|Z^{(1)}_{r'} - Z^{(2)}_{r'}\right|^2 dr$$

Thus

$$\sup_{0\le s\le t}|T^A_s(Z^{(1)}) - T^A_s(Z^{(1)})|^2 \le Kt\int_{r=0}^t \sup_{0\le r'\le t}\left|Z^{(1)}_r - Z^{(2)}_r\right|^2 dr$$

so taking expectations we get that

$$\|T^A(Z^{(1)}) - T^A(Z^{(1)})\|^2_{H(t)} \le K^2 T\int_{r=0}^t \|Z^{(1)} - Z^{(2)}\|^2_{H(r)} dr.$$

To similarly bound T^B, we use Doob's inequality. For $0 \le t \le T$ and get that

$$\mathbb{E}\left[\sup_{0\le s\le t}|T^B_s(Z^{(1)}) - T^B_s(Z^{(1)})|^2\right] \le 2\mathbb{E}\left[|T^B_t(Z^{(1)}) - T^B_t(Z^{(1)})|^2\right] \le 2\mathbb{E}\left[\int_{r=0}^t \left\{\sigma(Z^{(1)}_r) - \sigma(Z^{(2)}_r)\right\}^2 dr\right]$$
$$\le 2K^2\int_{r=0}^t \mathbb{E}\left[\left|Z^{(1)}_r - Z^{(2)}_r\right|^2\right] dr \le 2K^2\int_{r=0}^t \mathbb{E}\left[\sup_{0\le r'\le r}\left|Z^{(1)}_{r'} - Z^{(2)}_{r'}\right|^2\right] dr.$$

Here we get that

$$\|T^B(Z^{(1)}) - T^B(Z^{(1)})\|^2_{H(t)} \le 2K^2\int_{r=0}^t \|Z^{(1)} - Z^{(2)}\|^2_{H(r)} dr.$$

Combining things together, we get that

$$\|T(Z^{(1)}) - T(Z^{(1)})\|^2_{H(t)} \le 2K^2(T+2)\int_{r=0}^t \|Z^{(1)} - Z^{(2)}\|^2_{H(r)} dr. \tag{23}$$

Thus in particular

$$\mathbb{E}\left[\left|X^{(n+1)}_t - X^{(n)}_t\right|^2\right] \le \frac{\left(2K^2(T+2)t\right)^{n-1}}{(n-1)!}\sup_{0\le s\le T}\mathbb{E}\left[\left|X^{(1)}_s - X^{(0)}_s\right|^2\right]$$

To show convergence, we want to show that

$$\sum_{n=1}^\infty \left\{\mathbb{E}\left[\sup_{0\le t\le T}\left|X^{(n+1)}_t - X^{(n)}_t\right|^2\right]\right\}^{1/2} < \infty. \tag{24}$$

Since

$$\lim_{n\to\infty}\frac{\sqrt{\frac{1}{(n+1)!}\left(2K^2(T+2)\right)^{n+1}T^{n+1}}}{\sqrt{\frac{1}{n!}\left(2K^2(T+2)\right)^n T^n}} = \lim_{n\to\infty}\sqrt{\frac{1}{n+1}2K^2(T+2)T} = 0.$$

The ratio test thus implies that (24) holds. Hence there is an $X \in \mathsf{H}(T)$ such that $\lim_{n\to\infty}\|X^{(n)} - X\|_{H(T)} = 0$.

Collecting things together and using (23), we have that

$$\begin{aligned}\mathbb{E}[|X_t - T_t(X)|^2] &\le 2\mathbb{E}[|X_t - X_t^{(n+1)}|^2] + 2\mathbb{E}[|T_t(X^{(n)}) - T_t(X)|^2] \\ &\le 2\|X - X^{(n)}\|^2_{H(T)} + 2K^2(T+2)T\|X^{(n)} - X\|_{H(T)}.\end{aligned}$$

Letting $n \to \infty$, we indeed get that $X = T(X)$.

Let's finally show uniqueness. Assume that X and Y are two solutions. We then have that

$$\mathbb{E}\left[|X_t - Y_t|^2\right] \le 2K^2(T+2)\int_{s=0}^t \mathbb{E}\left[|X_s - Y_s|^2\right] ds$$

Gronwall's inequality implies that $X - Y = 0$; i.e., $X = Y$.

Exercises

(1) Show that T^A is adapted if $Z \in \mathsf{H}(T)$ of (22) is adapted.

(2) Fix $\lambda \in \mathbb{R}$ and consider the SDE

$$\begin{aligned} dX_t &= \lambda(X_t - \bar{x})dt + \sigma dW_t \qquad t \ge 0 \\ X_0 &= x_\circ \end{aligned} \tag{25}$$

(a) Show

$$X_t = e^{\lambda t}x_\circ + (1 - e^{\lambda t})\bar{x} + \sigma e^{\lambda t}\int_{s=0}^t e^{-\lambda s} dW_s$$

solves (25).

(b) Set $\mu(t) \overset{\text{def}}{=} \mathbb{E}[X_t]$. Show that $\dot{\mu}(t) = \lambda(\mu(t) - \bar{x})$.

(c) Set $R(t) \overset{\text{def}}{=} \mathbb{E}[(X_t - \bar{x})^2]$. Show that

$$\dot{R}(t) = 2\lambda R(t) + \sigma^2$$

CHAPTER 5

Ito's formula

Let M be a martingale with quadratic variation $\langle M\rangle$. Also fix $f \in \mathsf{P}$ and define

$$A_t \overset{\text{def}}{=} \int_{s=0}^t f_s ds$$
$$X_t = A_t + M_t$$

for all $t \in [0,T]$. Fix $\phi \in C_b^\infty(\mathbb{R})$ (the collection of infinitely-differentiable functions all of whose derivatives are bounded). We want to write $\phi(X)$ as a stochastic integral.

We need some bounds. First, let's assume that M and $\langle M\rangle$ are bounded by some constant $K > 0$. Secondly, let's assume that f is bounded by some constant $K_f > 0$. Thirdly, let's assume that ϕ and its first three derivatives are bounded by some constant K_ϕ.

Fix $t > 0$. Fix also a positive integer N (which we will let become large) and define $s_n^{(N)}$ as in (11). We then have that

$$\phi(X_t) - \phi(X_0) = \sum_{j=1}^6 I_j^{(N)}$$

where

$$I_1^{(N)} \overset{\text{def}}{=} \sum_{n=0}^{N-1} \phi'(X_{s_n^{(N)}})\left\{A_{s_{n+1}^{(N)}} - A_{s_n^{(N)}}\right\}$$
$$I_2^{(N)} \overset{\text{def}}{=} \sum_{n=0}^{N-1} \phi'(X_{s_n^{(N)}})\left\{M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}\right\}$$
$$I_3^{(N)} \overset{\text{def}}{=} \frac{1}{2}\sum_{n=0}^{N-1} \phi''(X_{s_n^{(N)}})\left\{M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}\right\}^2$$
$$I_4^{(N)} \overset{\text{def}}{=} \frac{1}{2}\sum_{n=0}^{N-1} \phi''(X_{s_n^{(N)}})\left\{A_{s_{n+1}^{(N)}} - A_{s_n^{(N)}}\right\}^2$$
$$I_5^{(N)} \overset{\text{def}}{=} \sum_{n=0}^{N-1} \phi''(X_{s_n^{(N)}})\left\{M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}\right\}\left\{A_{s_{n+1}^{(N)}} - A_{s_n^{(N)}}\right\}$$
$$I_6^{(N)} \overset{\text{def}}{=} \sum_{n=0}^{N-1} \mathcal{E}\left(X_{s_n^{(N)}}, X_{s_{n+1}^{(N)}} - X_{s_n^{(N)}}\right)$$

where

$$\mathcal{E}(x;y) \overset{\text{def}}{=} \phi(x+y) - \left\{\phi(x) - \phi'(x)y - \frac{1}{2}\phi''(x)y^2\right\} = x^3\int_{\theta=0}^1 (1-s)^2\phi'''(x+\theta y)d\theta.$$

Let's first rewrite some things. Defining

$$\xi_t^{(N)} \overset{\text{def}}{=} \sum_{n=0}^{N-1} \phi'(X_{s_n^{(N)}})\chi_{(s_n^{(N)}, s_{n+1}^{(N)}]}(s) \qquad 0 \le s \le t$$

we have that $\xi^{(N)}$ is bounded and that

$$\lim_{N\to\infty} \xi_t^{(N)} = \phi'(X_t)$$

for all $t \geq 0$. Thus

$$I_1^{(N)} = \int_{s=0}^t \xi_s^{(N)} f_s ds \qquad \text{and} \qquad I_2^{(N)} = \int_{s=0}^t \xi_s^{(N)} dM_s;$$

thus

$$\lim_{N\to\infty} I_1^{(N)} = \int_{s=0}^t \phi'(X_s) f_s ds = \int_{s=0}^t \phi'(X_s) dA_s$$
$$\lim_{N\to\infty} I_2^{(N)} = \int_{s=0}^t \phi'(X_s) dM_s;$$

the former occurs pointwise and the latter occurs in L^2; thus both occur in probability.

Let's next bound $I_4^{(N)}$. Note that

$$|I_4^{(N)}| \leq \tfrac{1}{2} K_\phi K_f^2 t \left(\frac{t}{N}\right);$$

thus $\lim_{N\to\infty} I_4^{(N)} = 0$ $\mathbb{P}$-a.s.

We can also bound $I_5^{(N)}$ fairly easily. We have that

$$|I_5^{(N)}| \leq K_\phi K_f \sup_{0\leq n\leq N-1} |M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}|;$$

thus $\lim_{N\to\infty} I_5^{(N)} = 0$ $\mathbb{P}$-a.s.

To bound the remaining parts, we will need some calculations from Chapter 2. Recall (12) and (16). Let's first bound $I_6^{(N)}$. We have that

$$|I_6^{(N)}| \leq \frac{3}{2} K_\phi \sum_{n=0}^{N-1} \left\{ K_f^3 \left(\frac{t}{N}\right)^3 + |M_{s_{n+1}^{(N)}} - M_{s_n^{(N)}}|^3 \right\} \leq \frac{3}{2} K_\phi \left\{ K_f^3 t \left(\frac{t}{N}\right)^2 + \xi_N^a V^{(N)} \right\}.$$

Thus (using (13)), we have that $\lim_{N\to\infty} \mathbb{E}[|I_6^{(N)}|] = 0$.

Thus Let's next look at $I_3^{(N)}$. Define

$$\bar{I}_3^{(N)} \overset{\text{def}}{=} \frac{1}{2} \sum_{n=0}^{N-1} \phi''(X_{s_n^{(N)}}) \left\{ \langle M \rangle_{s_{n+1}^{(N)}} - \langle M \rangle_{s_n^{(N)}} \right\}.$$

Defining now

$$\bar{\xi}_t^{(N)} \overset{\text{def}}{=} \frac{1}{2} \sum_{n=0}^{N-1} \phi''(X_{s_n^{(N)}}) \chi_{(s_n^{(N)}, s_{n+1}^{(N)}]}(s), \qquad 0 \leq s \leq t$$

we have that

$$\bar{I}_3^{(N)} = \int_{s=0}^t \bar{\xi}_s^{(N)} d \langle M \rangle_s .$$

Since $\lim_{N\to\infty} \bar{\xi}^{(N)} = \frac{1}{2}\phi''(X)$, we have that

$$\lim_{N\to\infty} \bar{I}_3^{(N)} = \frac{1}{2} \int_{s=0}^t \phi''(X_s) d \langle M \rangle_s$$

pointwise, and thus in probability.

We want to show that $I_3^{(N)} - \bar{I}_3^{(N)}$ is in fact small. We will use here (15). Note that

$$I_3^{(N)} - \bar{I}_3^{(N)} = \frac{1}{2} \sum_{n=0}^{N-1} \phi''(X_{s_n^{(N)}}) D_n^{(N)}.$$

Recalling (9) we have that

$$\begin{aligned}\mathbb{E}\left[|I_3^{(N)} - \bar{I}_3^{(N)}|^2\right] &= \frac{1}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left(\phi''(X_{s_n^{(N)}})\right)^2\left(D_j^{(N)}\right)^2\right] \\ &\le \frac{K_\phi}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left(D_n^{(N)}\right)^2\right]\end{aligned}$$

Using (17), we indeed get that $\lim_{N\to\infty}\mathbb{E}[|I_3^{(N)} - \bar{I}_3^{(N)}|^2] = 0$.

Exercises

(1) Fix $t > 0$. Let's again use the definition (11). Show that

$$\lim_{N\to\infty}\sum_{n=0}^{N-1}\left(X_{s_{n+1}^{(N)}} - X_{s_n^{(N)}}\right)^2 = \langle M\rangle_t,$$

this being at least convergence in probability. Do *not* reprove (14).

(2) Fix $b \in \mathbb{R}$ and $\sigma > 0$. Define

$$X_t \overset{\text{def}}{=} x\exp\left[(b - \sigma^2/2)t + \sigma W_t\right].$$

Show that

$$\begin{aligned}dX_t &= bX_t dt + \sigma X_t dW_t \\ X_0 &= x\end{aligned}$$

CHAPTER 6

Feynman-Kac

Lets put some of our machinery to work. Fix functions b and σ which have linear growth and which are Lipshitz-continuous. Define the partial differential operator

$$(\mathscr{L}f)(x) \overset{\text{def}}{=} \frac{1}{2}\sigma^2(x)\ddot{f}(x) + b(x)\dot{f}(x) \qquad x \in \mathbb{R}$$

for $f \in C^\infty(\mathbb{R})$. Consider the parabolic PDE

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \mathscr{L}v(t,x) \qquad t>0,\, x\in\mathbb{R} \\ v(0,x) &= v_\circ. \qquad x \in \mathbb{R} \end{aligned} \tag{26}$$

We want to represent the solution of this via a *path integral*. For each $x \in \mathbb{R}$, let's solve

$$\begin{aligned} dX^x_t &= b(X^x_t)dt + \sigma(X^x_t)dW_t \\ X^x_0 &= x. \end{aligned} \tag{27}$$

Assume that (26) has a solution which is smooth in time and space and such that v and its partial derivatives in time and space are bounded. Fix $T > 0$ and $x \in \mathbb{R}$. Define

$$Z_t \overset{\text{def}}{=} v(T-t, X^x_t) \qquad 0 \le t \le T \tag{28}$$

By Ito's formula, Z is a martingale. Also, $Z_0 = v(T,x)$, and $Z_T = v_\circ(X^x_T)$. Thus

$$v(T,x) = \mathbb{E}[Z_0] = \mathbb{E}[Z_T] = \mathbb{E}[v_\circ(X^x_T)].$$

Let's next impose a boundary condition. Fix $L > 0$. Fix also two functions f_L and f_R in $C[0,\infty)$.

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \mathscr{L}v(t,x) \qquad t>0, 0<x<L \\ v(0,x) &= v_\circ. \qquad 0<x<L \\ v(t,0) &= f_L(t) \qquad t>0 \\ v(t,L) &= f_R(t) \qquad t>0 \end{aligned} \tag{29}$$

Again fix $T > 0$ and $x \in (0,L)$, and again define (27). Here, though, we define the stopping time.

$$\tau \overset{\text{def}}{=} \inf\left\{t>0:\, X^x_t \notin [0,L]\right\}. \tag{30}$$

As in (28), let's define

$$Z_t \overset{\text{def}}{=} v(T - t\wedge\tau, X^x_{t\wedge\tau}) \qquad 0 \le t \le T \tag{31}$$

Again Z is a martingale. Here, however,

$$Z_T = \begin{cases} f_L(T-\tau) & \text{if } \tau < T \text{ and } X^x_\tau = L \\ f_R(T-\tau) & \text{if } \tau < T \text{ and } X^x_\tau = R \\ v_\circ(X^x_T) & \text{if } \tau \ge T \end{cases}$$

Thus we get that

$$\begin{aligned} v(T,x) = \mathbb{E}[Z_T] &= \mathbb{E}\left[f_L(T-\tau)\chi_{\{\tau<T,\, X^x_\tau = L\}}\right] \\ &\quad + \mathbb{E}\left[f_R(T-\tau)\chi_{\{\tau<T,\, X^x_\tau=R\}}\right] + \mathbb{E}\left[v_\circ(X^x_T)\chi_{\{\tau=T\}}\right]. \end{aligned}$$

Note that the *maximum principle* follows. If $v_\circ$, f_L, and f_R are all nonnegative, then v must also be nonnegative.

Thirdly, let's consider the elliptic PDE

(32)
$$\begin{aligned} \mathscr{L}v(t,x) &= f(x) \qquad 0<x<L \\ v(0) &= g_L \\ v(L) &= g_R \end{aligned}$$

for some function f and some constants g_L and g_R. Fix $x \in (0,L)$. We again define τ as in (30), and set

$$Z_t \stackrel{\text{def}}{=} v(X^x_t).$$

Then

$$dZ_t = f(X^x_t)dt + \dot{f}(X^x_t)\sigma(X^x_t)dW_t.$$

Consequently, if $\mathbb{P}\{\tau<\infty\}=1$, then

$$v(x) = \mathbb{E}[Z_0] = \mathbb{E}[Z_\tau] = \mathbb{E}\left[\int_{s=0}^{\tau} f(X^x_s)ds\right] + f_L\mathbb{P}\{X^x_\tau = L\} + f_R\mathbb{P}\{X^x_\tau = R\}.$$

Exercises

(1) Let's use Feynman-Kac to verify the *reflection principle*. Let $\mathfrak{s}(x) \stackrel{\text{def}}{=} \frac{x}{|x|}\chi_{\mathbb{R}\setminus\{0\}}$ be the signum function. For each $t>0$ and $x\in\mathbb{R}$, define

$$u(t,x) = \int_{y\in\mathbb{R}} \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{(x-y)^2}{2t}\right]\mathfrak{s}(y)dy.$$

Fix also $T>0$ and $x^*>0$. Define $\tau \stackrel{\text{def}}{=} \inf\{t\ge 0:\, x^*+W_t<0\}$.

(a) Show that

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) \qquad t>0,\, x\in\mathbb{R} \\ \lim_{t\searrow 0} u(t,x) &= \mathfrak{s}(x) \qquad x\in\mathbb{R} \end{aligned}$$

(b) Show that

$$u(T,x^*) = \mathbb{E}\left[u(T-t,x^*+W_t)\chi_{\{\tau>t\}}\right].$$

for each $t\in(0,T)$.

(c) Show that

$$u(T,x^*) = \mathbb{P}\left\{\tau \ge T\right\}.$$

(d) Show that

$$\mathbb{P}\{\tau<T\} = 2\int_{y=-\infty}^{0} \frac{1}{\sqrt{2\pi T}}\exp\left[-\frac{(x^*-y)^2}{2T}\right]dy = 2\mathbb{P}\{x^*+W_T\le 0\}.$$

CHAPTER 7

CIR Processes

Lets next look at *CIR* (Cox-Ingersoll-Ross) processes; i.e., solutions of the SDE

$$\begin{aligned} d\lambda_t &= -\alpha(\lambda_t - \bar{\lambda})dt + \sigma\sqrt{\lambda_t}dW_t \\ \lambda_0 &= \lambda_\circ \end{aligned} \tag{33}$$

where α, $\bar{\lambda}$, σ and $\lambda_\circ$ are all fixed constants, and W is a Brownian motion. These processes are heavily used in finance. We'll see why.

First, note that (33) only makes sense if λ is nonnegative. To separate the non negativity of λ from the existence of the solution to (33), let's replace (33) with

$$\begin{aligned} d\lambda_t &= -\alpha(\lambda_t - \bar{\lambda})dt + \sigma\sqrt{\lambda_t \vee 0}dW_t \\ \lambda_0 &= \lambda_\circ. \end{aligned} \tag{34}$$

Let's first show that the solution of (34), if it exists, is indeed nonnegative. Then (33) and (34) are equivalent.

First fix $\varphi \in C([0,\infty);[0,1])$ such that $\varphi \le \chi_{(-\infty,0]}$. Define

$$g_\varphi(x) \overset{\text{def}}{=} \chi_{(-\infty,0)}(x)\int_{y=0}^x \int_{z=0}^y \varphi(z)dz\,dy. \qquad x \in \mathbb{R}$$

Note also that

$$\begin{aligned} g'_\varphi(x) &= \chi_{(-\infty,0)}(x)\int_{z=0}^x \varphi(z)dz \\ g''_\varphi(x) &= \chi_{(-\infty,0)}(x)\varphi(x) \end{aligned}$$

Applying Ito's formula, we have that

$$g_\varphi(\lambda_t) = g_\varphi(\lambda_\circ) - \alpha\int_{s=0}^t (\lambda_s - \bar{\lambda})g'_\varphi(\lambda_s)ds + \tfrac{1}{2}\sigma^2\int_{s=0}^t (\lambda_s \vee 0)g''_\varphi(\lambda_s)ds + M_t$$

where M is a martingale. Note that g''_φ has support on $(-\infty,0)$ so $(\lambda\vee 0)g''_\varphi(\lambda) \le 0$. Also note that

$$-\alpha(\lambda - \bar{\lambda})g'_\eta(\lambda) = -\alpha\lambda g'_\varphi(\lambda) + \alpha\bar{\lambda}g'_\varphi(\lambda).$$

We note that $g'_\varphi \le 0$, so $\alpha\bar{\lambda}g'_\varphi(\lambda) \le 0$. Secondly, g'_φ has support in $(-\infty,0)$ and is negative in its support, so $\lambda g'_\varphi(\lambda) \ge 0$, so $-\alpha\lambda g'_\varphi(\lambda) \le 0$. Combining things together, we get that

$$\mathbb{E}[g_\varphi(\lambda_t)] \le 0.$$

Taking a sequence of φ's with $\varphi \nearrow \chi_{(-\infty,0)}$, we get that $g_\varphi(x) \nearrow x^2\chi_{(-\infty,0)}(x)$ (pointwise) and using monotone convergence, we get that $\mathbb{E}\left[\lambda_t^2\chi_{(-\infty,0)}(\lambda_t)\right] = 0$; thus indeed $\lambda_t \ge 0$ $\mathbb{P}$-a.s.

Let's next construct a solution to (33) via relaxation. Fix ε and ε' in $[0,1)$ and assume that λ and λ' satisfy

$$\begin{aligned} \lambda_t &= \lambda_\circ - \alpha\int_{s=0}^t \left\{\lambda_s - \bar{\lambda}\right\}ds + \sigma\int_{s=0}^t \sqrt{\lambda_s \vee \varepsilon}dW_s \\ \lambda'_t &= \lambda_\circ - \alpha\int_{s=0}^t \left\{\lambda'_s - \bar{\lambda}\right\}ds + \sigma\int_{s=0}^t \sqrt{\lambda'_s \vee \varepsilon'}dW_s; \end{aligned} \tag{35}$$

we are assured that λ and λ' exist only for positive ε and ε'; setting ε or ε' to be zero returns us to (34). Define $\nu_t \stackrel{\text{def}}{=} \lambda_t - \lambda'_t$; then

$$\nu_t = -\alpha \int_{s=0}^t \nu_s ds + \sigma \int_{s=0}^t \left\{ \sqrt{\lambda_s \vee \varepsilon} - \sqrt{\lambda'_s \vee \varepsilon'} \right\} dW_s.$$

Fix $\eta \in (0,1)$. Let $\varphi \in C([0,\infty);[0,1])$ be such that $\varphi \le \chi_{[\eta,\eta^{1/2}]}$. Define

$$g_\varphi(x) \stackrel{\text{def}}{=} \frac{2}{\ln \eta^{-1}} \int_{y=0}^{|x|} \int_{z=0}^{y} \frac{1}{z} \varphi(z) dz\, dy.$$

Note that

$$g''_\varphi(x) = \frac{2}{\ln \eta^{-1}} \frac{1}{|x|} \varphi(x) \le \frac{2}{\ln \eta^{-1}} \le \frac{1}{|x|} \chi_{[\eta,\eta^{1/2}]}(x) \le \frac{2}{\ln \eta^{-1}} \min\left\{ \frac{1}{|x|}, \frac{1}{\eta} \right\}.$$

Note also that

$$g'_\varphi(x) = \frac{2}{\ln \eta^{-1}} \int_{z=0}^{|x|} \frac{1}{z} \varphi(z) dz \le \frac{2}{\ln \eta^{-1}} \int_{z=0}^{\infty} \frac{1}{z} \chi_{[\eta,\eta^{1/2}]}(z) dz \le \frac{2}{\ln \eta^{-1}} \left\{ \ln \eta^{1/2} - \ln \eta \right\} = 1. \tag{36}$$

Applying Ito's formula to $g_\varphi(\nu_t)$, we have that

$$\begin{aligned} g_\varphi(\nu_t) &= -\alpha \int_{s=0}^t \nu_s g'_\varphi(\nu'_s)\, ds \\ &\quad + \frac{1}{2} \int_{s=0}^t \left(\sqrt{\lambda_s \vee \varepsilon} - \sqrt{\lambda'_s \vee \varepsilon'} \right)^2 g''_\varphi(\nu_s)\, ds + M_t \end{aligned}$$

Now note that

$$\left| \sqrt{x} - \sqrt{y} \right| \le \sqrt{|x-y|} \tag{37}$$

for all x and y in $\mathbb{R}_+$. Thus

$$\begin{aligned} &\frac{1}{2} \int_{s=0}^t \left(\sqrt{\lambda_s \vee \varepsilon} - \sqrt{\lambda'_s \vee \varepsilon'} \right)^2 g''_\varphi(\nu'_s)\, ds \\ &\quad \le \frac{1}{2} \int_{s=0}^t \left| (\lambda_s \vee \varepsilon) - (\lambda'_s \vee \varepsilon') \right| g''_\varphi(\nu'_s)\, ds \\ &\quad \le \frac{1}{2} \int_{s=0}^t \left\{ \left| (\lambda_s \vee \varepsilon) - (\lambda'_s \vee \varepsilon) \right| + \left| (\lambda'_s \vee \varepsilon) - (\lambda'_s \vee \varepsilon') \right| \right\} g''_\varphi(\nu_s)\, ds \\ &\quad \le \frac{1}{2} \int_{s=0}^t \left| \lambda_s - \lambda'_s \right| g''_\varphi(\nu_s)\, ds + \frac{1}{2} \int_{s=0}^t |\varepsilon - \varepsilon'| g''_\varphi(\nu_s)\, ds \\ &\quad \le \frac{t}{\ln \eta^{-1}} + \frac{|\varepsilon - \varepsilon'| t}{\eta \ln \eta^{-1}} \end{aligned}$$

Using (36), we also have that

$$-\alpha \int_{s=0}^t \nu_s g'_\varphi(\nu_s)\, ds \le \alpha \int_{s=0}^t |\nu_s|\, ds.$$

Combining things together, we get that

$$\mathbb{E}\left[g_\varphi(\lambda_t - \lambda'_t) \right] \le \alpha \int_{s=0}^t \mathbb{E}\left[|\lambda_s - \lambda'_s| \right] ds + \left\{ \frac{1}{\ln \eta^{-1}} + \frac{|\varepsilon - \varepsilon'|}{\eta \ln \eta^{-1}} \right\} t.$$

Letting $\varphi \nearrow \chi_{[\eta,\eta^{1/2}]}$, we get that $g_\varphi(x) \nearrow \bar{g}_\eta(x)$ (pointwise) where

$$\bar{g}_\eta(x) \stackrel{\text{def}}{=} \frac{2}{\ln \eta^{-1}} \int_{y=0}^{|x|} \int_{z=0}^{y} \frac{1}{z} \chi_{[\eta,\eta^{1/2}]}(z) dz\, dy.$$

Thus

$$\mathbb{E}\left[\bar{g}_\eta(\nu_t) \right] \le \alpha \int_{s=0}^t \mathbb{E}\left[|\nu_s| \right] ds + \left\{ \frac{1}{\ln \eta^{-1}} + \frac{|\varepsilon - \varepsilon'|}{\eta \ln \eta^{-1}} \right\} t.$$

Let's now compare $\bar{g}_\eta$ to $x \mapsto |x|$. Since $\bar{g}_\eta$ is even, we only need to do this on $\mathbb{R}_+$. Since

$$\bar{g}'_\eta(x) = \frac{2}{\ln \eta^{-1}} \int_{z=0}^{x} \frac{1}{z} \chi_{[\eta,\eta^{1/2}]}(z) dz$$

on $\mathbb{R}_+$, we see that $\bar{g}'_\eta$ is nonnegative and nondecreasing on $\mathbb{R}_+$. Furthermore $\bar{g}_\eta(0) = 0$, so

$$0 \le \bar{g}'_\eta(x) \le \bar{g}'_\eta(\sqrt{\eta}) = 1$$

for all $x > 0$. (using a calculation like (36)). Furthermore $\bar{g}'_\eta$ is constant on $[\sqrt{\eta}, \infty)$, so in fact $\bar{g}'_\eta(x) = 1$ for $x \in [\sqrt{\eta}, \infty)$. Since $g_\eta(0) = 0$, we in fact have that

$$|x| - g_\eta(x) = \int_{y=0}^{x} \{1 - \bar{g}_\eta(y)\} \, dy \le \int_{y=0}^{\sqrt{\eta}} dy = \sqrt{\eta}$$

for all $x \in \mathbb{R}$. Summing up, we have that

$$|x| \le g_\eta(x) + \sqrt{\eta}$$

for all $x \in \mathbb{R}$.

Thus

$$\mathbb{E}\left[|\nu_t|\right] \le \alpha \int_{s=0}^{t} \mathbb{E}\left[|\nu_s|\right] ds + \frac{t}{\ln \eta^{-1}} + \frac{|\varepsilon - \varepsilon'| t}{\eta \ln \eta^{-1}} + \sqrt{\eta}.$$

Thus

$$\mathbb{E}\left[|\nu_t|\right] \le e^{\alpha t} \left\{ \frac{\sigma^2 t}{\ln \eta^{-1}} + \frac{\sigma^2 |\varepsilon - \varepsilon'| t}{\eta \ln \eta^{-1}} + \sqrt{\eta} \right\}.$$

Letting ε and ε' tend to zero and then letting $\eta \searrow 0$, we get that

(38) $$\lim_{\varepsilon,\varepsilon' \searrow 0} \mathbb{E}\left[|\lambda_s - \lambda'_s|\right] = 0.$$

We also have uniqueness. If $\varepsilon = \varepsilon' = 0$ in (35), then we have that

$$\mathbb{E}\left[|\lambda_t - \lambda'_t|\right] \le e^{\alpha t} \left\{ \frac{\sigma^2 t}{\ln \eta^{-1}} + \sqrt{\eta} \right\}.$$

Letting $\eta \searrow 0$, we have that

$$\mathbb{E}\left[|\lambda_t - \lambda'_t|\right] = 0,$$

ensuring uniqueuenss.

Exercises

(1) Prove (37).

(2) For each $\varepsilon > 0$, show that the map $x \mapsto \sqrt{x \vee \varepsilon}$ is Lipshitz-continuous and has at most linear growth.

(3) Let's complete the calculations.

 (a) From (38), show that

$$\lim_{\varepsilon,\varepsilon' \searrow 0} \mathbb{E}\left[\sup_{0 \le s \le t} |\lambda_s - \lambda'_s|\right] = 0$$

 (as usual, this is needed to ensure that λ has continuous paths, thus implying that λ is predictable).

 (b) Suppose that λ^* is a continuous and adapted process such that

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\sup_{0 \le s \le t} |\lambda_s - \lambda^*_s|\right] = 0.$$

 Show that λ^* satisfies (34).

(4) Define $r(t) \stackrel{\text{def}}{=} \mathbb{E}[|\lambda_t|]$. Show that $\dot{r}(t) = -\alpha(r(t) - \bar{\lambda})$; thus

$$r(t) = \lambda_\circ e^{-\alpha t} + \bar{\lambda}(1 - e^{-\alpha t}).$$

(5) Show that if $2\bar{\lambda}\alpha > \sigma^2$, then a gamma distribution is a stationary distribution for λ; what are the parameters for the gamma distribution?

CHAPTER 8

Girsanov's theorem

Let W be a Brownian motion with respect to $\{\mathscr{F}_t\}_{t\geq 0}$ (which may be right-continuous). Let f be a bounded predictable process. Define

$$\tilde{W}_t \overset{\text{def}}{=} W_t - \int_{s=0}^t f_s ds.$$

Fix $T>0$ and define

$$\tilde{\mathbb{P}}(A) \overset{\text{def}}{=} \mathbb{E}\left[\chi_A \exp\left[\int_{s=0}^T f_s dW_s - \frac{1}{2}\int_{s=0}^T f_s^2 ds\right]\right]. \qquad A\in\mathscr{F}$$

We claim that $\tilde{\mathbb{P}}$ is a probability measure and that, under $\tilde{\mathbb{P}}$, $\{\tilde{W}\}_{0\leq s\leq t}$ is a Brownian motion with respect to $\{\mathscr{F}_t\}_{t\geq 0}$.

We start by defining

$$L_t^\theta \overset{\text{def}}{=} \exp\left[\int_{s=0}^t \left(\sqrt{-1}\theta + f_s\right) dW_s - \frac{1}{2}\int_{s=0}^t \left(\sqrt{-1}\theta + f_s\right)^2 ds\right]$$

for all $\theta\in\mathbb{R}$ and $t\in[0,T]$. Note that

$$L_t^\theta = L_t^0 \exp\left[\sqrt{-1}\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right].$$

By Ito's formula,

$$(39)\quad dL_t^\theta = L_t^\theta\left\{\left(\sqrt{-1}\theta + f_t\right) dW_t - \frac{1}{2}\left(\sqrt{-1}\theta + f_t\right)^2 dt\right\} + \frac{1}{2}L^\theta\left(\sqrt{-1}\theta + f_t\right)^2 dt = L_t^\theta\left(\sqrt{-1}\theta + f_t\right) dW_t.$$

Thus L^θ is a martingale. We also have that L_0^θ, and that

$$\tilde{\mathbb{P}}(A) \overset{\text{def}}{=} \mathbb{E}[\chi_A L_T^0]. \qquad A\in\mathscr{F}$$

We first note that by the martingale property,

$$\tilde{\mathbb{P}}(\Omega) = \mathbb{E}[L_T^0] = \mathbb{E}[L_0^0] = 1;$$

thus $\tilde{\mathbb{P}}$ is indeed a probability measure.

Secondly, let's note that

$$\tilde{\mathbb{E}}\left[\exp\left[\sqrt{-1}\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right]\middle|\mathscr{F}_s\right] = \frac{\mathbb{E}\left[L_T^0\exp\left[\sqrt{-1}\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right]\middle|\mathscr{F}_s\right]}{\mathbb{E}[L_T^0|\mathscr{F}_s]}$$

By the martingale property,

$$\mathbb{E}[L_T^0|\mathscr{F}_s] = L_t^0.$$

We also have that

$$\begin{aligned}\mathbb{E}\left[L_T^0\exp\left[\sqrt{-1}\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right]\middle|\mathscr{F}_s\right] &= \mathbb{E}\left[\mathbb{E}[L_T^0|\mathscr{F}_t]\exp\left[\sqrt{-1}\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right]\middle|\mathscr{F}_s\right]\\ &= \mathbb{E}\left[L_t^0\exp\left[\sqrt{-1}\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right]\middle|\mathscr{F}_s\right]\\ &= \mathbb{E}[L_t^\theta|\mathscr{F}_s] = L_s^\theta = L_s^0\exp\left[\sqrt{-1}\theta\tilde{W}_s + \frac{1}{2}\theta^2 s\right].\end{aligned}$$

Thus

$$\tilde{\mathbb{E}}\left[\exp\left[\sqrt{-1}\theta\tilde{W}_t+\frac{1}{2}\theta^2 t\right]\middle|\mathscr{F}_s\right]=\exp\left[\sqrt{-1}\theta\tilde{W}_s+\frac{1}{2}\theta^2 s\right].$$

Rearranging, we get that

$$\tilde{\mathbb{E}}\left[\exp\left[\sqrt{-1}\theta\left(\tilde{W}_t-\tilde{W}_s\right)\right]\middle|\mathscr{F}_s\right]=\exp\left[-\frac{\theta^2}{2}(t-s)\right].$$

Exercises

(1) Prove (39) by considering real and imaginary parts.

CHAPTER 9

The martingale problem and Levy's characterization of Brownian motion

Suppose that X is a continuous stochastic process which is adapted to a filtration $\{\mathscr{F}_t\}_{t\ge 0}$. Suppose that $X_0=0$ and that for every $f\in C_b^\infty(\mathbb{R})$ (i.e., $f\in C^2(\mathbb{R})$ such that f and its first two derivatives are bounded)

$$f(X_t)=f(X_0)+\frac{1}{2}\int_{r=0}^t \ddot{f}(X_r)dr+M_t \tag{40}$$

where M is a martingale with respect to $\{\mathscr{F}_t\}_{t\ge 0}$ (which depends of course on f). This is an example of the *martingale problem*; see [**?**]. We claim that necessarily X is a Brownian motion with respect to $\{\mathscr{F}_t\}_{t\ge 0}$.

Fix $\theta\in\mathbb{R}$. Define $f_\theta(x)\overset{\text{def}}{=}\exp\left[\sqrt{-1}\theta x\right]$ for all $x\in\mathbb{R}$. Then

$$f_\theta(X_t)=f_\theta(X_0)-\frac{1}{2}\theta^2\int_{r=0}^t f_\theta(X_r)dr+M_t.$$

This in turn implies that

$$f_\theta(X_t)=f_\theta(X_s)-\frac{1}{2}\theta^2\int_{r=s}^t f_\theta(X_r)dr+M_t-M_s.$$

Taking conditional expectations, we get that

$$\mathbb{E}[f_\theta(X_t)|\mathscr{F}_s]=f_\theta(X_s)-\frac{1}{2}\theta^2\int_{r=s}^t \mathbb{E}[f_\theta(X_r)|\mathscr{F}_s]ds. \tag{41}$$

Consequently

$$\mathbb{E}[f_\theta(X_t)|\mathscr{F}_s]=f_\theta(X_s)\exp\left[-\frac{1}{2}\theta^2(t-s)\right]. \tag{42}$$

Rearranging, we get that

$$\mathbb{E}\left[\exp\left[\sqrt{-1}\theta(X_t-X_s)\right]\,\middle|\,\mathscr{F}_s\right]=\exp\left[-\frac{1}{2}\theta^2(t-s)\right].$$

Thus in particular, if M is a martingale with quadratic variation $\langle M\rangle_t=t$, then M must be a Brownian motion.

1. Poisson processes

We can do something similar with jump processes. Let's do this in a general way. Fix a measure π on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$. Define

$$(\mathscr{L}f)(x)=\int_{y\in\mathbb{R}}\{f(y)-f(x)\}\,\pi(dy)\qquad x\in\mathbb{R} \tag{43}$$

for all $f\in B(\mathbb{R})$ (the collection of bounded measurable functions on $\mathbb{R}$). Suppose that X is a right-continuous process such that

$$f(X_t)=f(X_0)+\int_{s=0}^t(\mathscr{L}f)(X_s)ds+M_t \tag{44}$$

where M is a martingale with respect to $\{\mathscr{F}^X_t\}_{t\ge 0}$, where

$$\mathscr{F}^X_t \overset{\text{def}}{=} \sigma\{X_s;\, s\le t\}.$$

By a small amount of work, we can extend (44). For $f\in B(\mathbb{R}_+\times\mathbb{R})$ which is differentiable in the first argument, we get that

$$f(t,X_t)=f(0,X_0)+\int_{s=0}^t\left\{\frac{\partial f}{\partial t}(s,X_s)+(\mathscr{L}f)(s,X_s)\right\}ds+M_t \tag{45}$$

where M is a martingale.

Let's assume that $X_0=x^*$. Define

$$\tau\overset{\text{def}}{=}\inf\left\{t\ge 0:\ X_t\ne x^*\right\}.$$

We are interested in the statistics of (τ,X_τ); i.e., when X jumps, and where it jumps to. We will see that (43) leads to something nice.

Let's also carefully pick f in (45). Namely, fix $\lambda>0$ and fix $\phi\in B(\mathbb{R})$ Let's pick $f(t,x)=e^{-\lambda t}\phi(x)$. For $s\in[0,\tau)$, we have that $\phi(X_s)=\phi(x^*)$. For $t>0$, we optional sampling implies that

$$\begin{aligned}\mathbb{E}\left[\exp\left[-\lambda(\tau\wedge t)\right]\phi(X_{\tau\wedge t})\right]&=\phi(x^*)+\mathbb{E}\left[\int_{s=0}^{\tau\wedge t}e^{-\lambda s}ds\left\{-\lambda\phi(x^*)+\langle\phi,\pi\rangle-\phi(x^*)\right\}\right]\\&=\phi(x^*)+\frac{1}{\lambda}\mathbb{E}\left[1-\exp\left[-\lambda(\tau\wedge t)\right]\right]\left\{\langle\phi,\pi\rangle-(\lambda+\pi(\mathbb{R}\setminus\{x^*\}))\phi(x^*)\right\}\end{aligned} \tag{46}$$

where

$$\langle\phi,\pi\rangle\overset{\text{def}}{=}\int_{x\in\mathbb{R}}\phi(x)\pi(dx).$$

Taking $t\nearrow\infty$, we get that

$$\mathbb{E}\left[\exp\left[-\lambda\tau\right]\phi(X_\tau)\right]=\phi(x^*)+\left\{1-\mathbb{E}\left[\exp\left[-\lambda\tau\right]\right]\right\}\left\{\frac{1}{\lambda}\langle\phi,\pi\rangle-\frac{\lambda+\pi(\mathbb{R}\setminus\{x^*\})}{\lambda}\phi(x^*)\right\} \tag{47}$$

Let's begin by setting $\phi=\chi_{\mathbb{R}\setminus\{x^*\}}$. Then (47) implies that

$$\mathbb{E}\left[\exp\left[-\lambda\tau\right]\right]=\frac{\pi(\mathbb{R}\setminus\{x^*\})}{\lambda}\mathbb{E}\left[1-\exp\left[-\lambda\tau\right]\right].$$

Rearranging this, we get that

$$\mathbb{E}\left[\exp\left[-\lambda\tau\right]\right]=\frac{\pi(\mathbb{R}\setminus\{x^*\})}{\lambda+\pi(\mathbb{R}\setminus\{x^*\})}.$$

Thus τ is exponential with parameter $\pi(\mathbb{R}\setminus\{x^*\})$. Reinserting this back in (47), we get that

$$\begin{aligned}\mathbb{E}\left[\exp\left[-\lambda\tau\right]\phi(X_\tau)\right]&=\phi(x^*)+\frac{\lambda}{\lambda+\pi(\mathbb{R}\setminus\{x^*\})}\left\{\frac{1}{\lambda}\langle\phi,\pi\rangle-\frac{\lambda+1}{\lambda}\phi(x^*)\right\}\\&=\phi(x^*)+\frac{\pi(\mathbb{R}\setminus\{x^*\})}{\lambda+\pi(\mathbb{R}\setminus\{x^*\})}\left\{\langle\phi,\tilde\pi_{x^*}\rangle-\frac{\lambda+\pi(\mathbb{R}\setminus\{x^*\})}{\lambda}\frac{\phi(x^*)}{\pi(\mathbb{R}\setminus\{x^*\})}\right\}\end{aligned}$$

where

$$\pi_{x^*}(A)\overset{\text{def}}{=}\frac{\pi(A\setminus\{x^*\})}{\pi(\mathbb{R}\setminus\{x*\})}\qquad A\in\mathscr{B}(\mathbb{R})$$

Fix next $A\subset\mathbb{R}\setminus\{x^*\}$, and set $\phi(x)=\chi_A$. Then

$$\mathbb{E}\left[\exp\left[-\lambda\tau\right]\chi_A(X_\tau)\right]=\frac{\pi(\mathbb{R}\setminus\{x^*\})}{\lambda+\pi(\mathbb{R}\setminus\{x^*\})}\tilde\pi_{x^*}(A)$$

Taking $\lambda\searrow 0$, we see that X_τ has distribution $\tilde\pi_{x^*}$, and τ and X_τ are independent.

Exercises

(1) Show that (41) implies (42).

(2) Let's consider an alternate representation of (40). Fix $f \in C_b^2(\mathbb{R})$, $0 \le s_0 < s_1 \dots s_N \le s < t$, and $\{\varphi_n\}_{n=1}^N$. Consider the equality

$$\mathbb{E}\left[\left\{f(X_t) - f(X_s) - \int_{r=s}^t \frac{1}{2}\ddot{f}(X_r)dr\right\}\prod_{n=1}^N \varphi_n(X_{s_n})\right] = 0. \tag{48}$$

(a) Show that if (40) holds, then (48) holds.

(b) Show that if $\mathscr{F}_t \stackrel{\text{def}}{=} \sigma\{X_s;\ 0 \le s \le t\}$, then (48) implies (40).

(3) Prove (46).

CHAPTER 10

Martingales as time-changed Brownian motions

Let M be martingale with respect to a filtration $\{\mathscr{F}_t\}_{t\ge 0}$. Suppose that $\langle M\rangle$ is differentiable and that $\frac{d\langle M\rangle_t}{dt} \ge m_\circ$ for all $t>0$, for some $m_\circ > 0$. For all $t \ge 0$, define

$$\tau(t) \overset{\text{def}}{=} \inf\left\{s \ge 0 : \langle M\rangle_s \ge t\right\}.$$

Then

$$\langle M\rangle_{\tau(t)} = t, \tag{49}$$

so in a sense $\tau(t) = \langle M\rangle_t^{-1}$. Note that since $\langle M\rangle_{t/m_\circ} \ge t$, we must have that $\tau(t) \le t/m_\circ$, so $\tau(t)$ is bounded.

By Ito's formula, we have that

$$f(M_t) = f(M_0) + \frac{1}{2}\int_{s=0}^{t} \ddot{f}(M_s)d\left\langle M\right\rangle_s + \tilde{N}_t$$

where N is a martingale (with respect to $\{\mathscr{F}_t\}_{t\ge 0}$). We then have that

$$f(M_{\tau(t)}) = f(M_0) + \frac{1}{2}\int_{s=0}^{\tau(t)} \ddot{f}(M_s)d\left\langle M\right\rangle_s + \tilde{N}_{\tau(t)}. \tag{50}$$

Set

$$\tilde{\mathscr{F}}_t \overset{\text{def}}{=} \mathscr{F}_{\tau(t)}.$$

By optional sampling, $\tilde{N}$ is a martingale with respect to $\{\tilde{\mathscr{F}}_t\}_{t\ge 0}$. By differentiating (49), we get that

$$\dot{\tau}(t)\frac{d\left\langle M\right\rangle}{dt}(\tau(t)) = 1.$$

Making the transformation $u = \langle M\rangle_t$ in the integral term in (50), we get that

$$f(M_{\tau(t)}) = f(M_0) + \frac{1}{2}\int_{s=0}^{t} \ddot{f}(M_s)ds + \tilde{N}_{\tau(t)}.$$

Thus $\{M_{\tau(t)}\}_{t\ge 0}$ is a martingale with respect to $\{\tilde{\mathscr{F}}_t\}_{t\ge 0}$.

Exercises

(1) Suppose that f is a bounded predictable (and nonzero for simplicity) function. Show that

$$\lim_{\eta\searrow 0}\mathbb{P}\left\{\sup_{\substack{0\le s\le t\le T\\ |t-s|\le\eta}}\left|\int_{r=s}^{t} f_r dW_r\right| \ge \varepsilon\right\} = 0$$

for all $\varepsilon > 0$.

CHAPTER 11

Filtering

Suppose that we have a plant process X and an observation process Y which are given by the coupled SDE

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_\circ \\ dY_t &= h(X_t)dt + dV_t \\ Y_0 &= 0 \end{aligned}$$

Suppose that we want to compute

$$\pi_t(A) \stackrel{\text{def}}{=} \mathbb{P}\{X_t \in A|\mathscr{Y}_t\} \qquad A \in \mathscr{B}(\mathbb{R})$$

where

$$\mathscr{Y}_t \stackrel{\text{def}}{=} \sigma\{Y_s;\, s \le t\}.$$

Note that

$$Y_t = V_t + \int_{s=0}^t h(X_s)ds.$$

Let's use Girsanov's theorem to make Y into a Brownian motion. Define

$$L_t \stackrel{\text{def}}{=} \exp\left[-\int_{s=0}^t h(X_s)dV_s - \frac{1}{2}\int_{s=0}^t h^2(X_s)ds\right] = \exp\left[-\int_{s=0}^t h(X_s)dY_s + \frac{1}{2}\int_{s=0}^t h^2(X_s)ds\right]. \tag{51}$$

Define

$$\tilde{\mathbb{P}}_t(A) \stackrel{\text{def}}{=} \mathbb{E}\left[\chi_A L_t\right]. \tag{52}$$

Then $\tilde{\mathbb{P}}_t$ is a probability measure, and under $\tilde{\mathbb{P}}_t$, $\{(W_s, Y_s);\, 0 \le s \le t\}$ is a 2-dimensional standard Brownian motion (and W and Y are independent). We also have that

$$\mathbb{P}(A) = \tilde{\mathbb{E}}_t\left[\chi_A L_t^{-1}\right] \qquad A \in \mathscr{F}$$

and that

$$L_t^{-1} = \exp\left[\int_{s=0}^t h(X_s)dY_s - \frac{1}{2}\int_{s=0}^t h^2(X_s)ds\right] \tag{53}$$

For any $A \in \mathscr{B}(\mathbb{R})$ we have that

$$\tilde{\pi}_t(A) = \frac{\tilde{\mathbb{E}}_t\left[\chi_A(X_t)L_t^{-1}|\mathscr{Y}_t\right]}{\tilde{\mathbb{E}}_t[L_t^{-1}|\mathscr{Y}_t]}.$$

We want to find the evolution of the numerator and denominator. We will heavily use the statistical structure of things under $\tilde{\mathbb{P}}_t$. By Ito's formula (and (53)),

$$f(X_t)L_t = f(X_0)L_0^{-1} + \int_{s=0}^t (\mathscr{L}f)(X_s)L_s^{-1}ds + \int_{s=0}^t (hf)(X_s)L_s^{-1}dY_s + \int_{s=0}^t (\sigma f')(X_s)L_s^{-1}dW_s$$

for $0 \le t \le T$, where

$$(\mathscr{L}f)(x) = \frac{1}{2}\sigma^2(x)\ddot{f}(x) + b(x)\dot{f}(x)$$

Thus

$$\tilde{\mathbb{E}}_t\left[f(X_t)L_t\big|\mathscr{Y}_t\right] = \tilde{\mathbb{E}}_t\left[f(X_0)L_0^{-1}\big|\mathscr{Y}_t\right] + \tilde{\mathbb{E}}_t\left[\int_{s=0}^t (\mathscr{L}f)(X_s)L_s^{-1}ds\big|\mathscr{Y}_t\right]$$
$$+ \tilde{\mathbb{E}}_t\left[\int_{s=0}^t (hf)(X_s)L_s^{-1}dY_s\big|\mathscr{Y}_t\right] + \tilde{\mathbb{E}}_t\left[\int_{s=0}^t (\sigma f')(X_s)L_s^{-1}dW_s\big|\mathscr{Y}_t\right]$$

Next fix $s\in[0,t]$ and $\phi\in B(\mathbb{R})$. Since $\{Y_{t'}-Y_s\}$ is independent of $\mathscr{Y}_y$ under $\tilde{\mathbb{P}}_T$,

$$\tilde{\mathbb{E}}_t\left[\phi(X_s)L_s^{-1}\big|\mathscr{Y}_t\right] = \tilde{\mathbb{E}}_t\left[\phi(X_s)L_s^{-1}\big|\sigma\{Y_{t'}-Y_s;\, s\le t'\le t\}\vee\mathscr{Y}_s\right] = \tilde{\mathbb{E}}_t\left[\phi(X_s)L_s^{-1}\big|\mathscr{Y}_s\right].$$

Next, note that the statistics of $\{(W_r,Y_r);\, 0\le r\le s\}$ are the same under $\tilde{\mathbb{P}}_t$ and $\tilde{\mathbb{P}}_s$. Thus

$$\tilde{\mathbb{E}}_t\left[\phi(X_s)L_s^{-1}\big|\mathscr{Y}_t\right] = \tilde{\mathbb{E}}_s\left[\phi(X_s)L_s^{-1}\big|\mathscr{F}_s\right]. \tag{54}$$

We first have (from (54)) that

$$\tilde{\mathbb{E}}_t\left[f(X_0)L_0^{-1}\big|\mathscr{Y}_t\right] = \tilde{\mathbb{E}}_0\left[f(X_0)L_0^{-1}\big|\mathscr{Y}_0\right].$$

and

$$\tilde{\mathbb{E}}_t\left[\int_{s=0}^t (\mathscr{L}f)(X_s)L_s^{-1}ds\big|\mathscr{Y}_t\right] = \int_{s=0}^T \tilde{\mathbb{E}}_t\left[(\mathscr{L}f)(X_s)L_s^{-1}\big|\mathscr{Y}_t\right]ds = \int_{s=0}^T \tilde{\mathbb{E}}_s\left[(\mathscr{L}f)(X_s)L_s^{-1}\big|\mathscr{Y}_s\right]ds.$$

Next recall the $s_n^{(N)}$'s of (11). We then have that

$$\begin{aligned}
\tilde{\mathbb{E}}_t\left[\int_{s=0}^t (hf)(X_s)L_s^{-1}dY_s\bigg|\mathscr{Y}_t\right] &= \lim_{N\to\infty}\tilde{\mathbb{E}}_t\left[\sum_{n=1}^{N-1}(hf)(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\left\{Y_{s_{n+1}^{(N)}}-Y_{s_n^{(N)}}\right\}\bigg|\mathscr{Y}_t\right]\\
&= \lim_{N\to\infty}\sum_{n=1}^{N-1}\tilde{\mathbb{E}}_t\left[(hf)(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\left\{Y_{s_{n+1}^{(N)}}-Y_{s_n^{(N)}}\right\}\bigg|\mathscr{Y}_t\right]\\
&= \lim_{N\to\infty}\sum_{n=1}^{N-1}\tilde{\mathbb{E}}_t\left[(hf)(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\bigg|\mathscr{Y}_t\right]\left\{Y_{s_{n+1}^{(N)}}-Y_{s_n^{(N)}}\right\}\\
&= \lim_{N\to\infty}\sum_{n=1}^{N-1}\tilde{\mathbb{E}}_s\left[(hf)(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\bigg|\mathscr{Y}_s\right]\left\{Y_{s_{n+1}^{(N)}}-Y_{s_n^{(N)}}\right\}\\
&= \int_{s=0}^t \tilde{\mathbb{E}}_s\left[(hf)(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\bigg|\mathscr{Y}_s\right]dY_s
\end{aligned}$$

Finally note that under $\tilde{\mathbb{P}}_t$, $W_{s_{n+1}^{(N)}}-W_{s_n^{(N)}}$ is independent of $X_{s_n^{(N)}}$ and Y.We again use (11). We have that

$$\begin{aligned}
&\tilde{\mathbb{E}}_t\left[\int_{s=0}^t (\sigma f')(X_s)L_s^{-1}dW_s\bigg|\mathscr{Y}_t\right]\\
&= \lim_{N\to\infty}\tilde{\mathbb{E}}_t\left[\sum_{n=0}^{N-1}(\sigma f')(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\left\{W_{s_{n+1}^{(N)}}-W_{s_n^{(N)}}\right\}\bigg|\mathscr{Y}_t\right]\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}\tilde{\mathbb{E}}_t\left[(\sigma f')(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\left\{W_{s_{n+1}^{(N)}}-W_{s_n^{(N)}}\right\}\bigg|\mathscr{Y}_t\right]\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}\tilde{\mathbb{E}}_t\left[(\sigma f')(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\tilde{\mathbb{E}}\left[W_{s_{n+1}^{(N)}}-W_{s_n^{(N)}}\bigg|\mathscr{Y}_t\vee\sigma(X_{s_n^{(N)}})\right]\bigg|\mathscr{Y}_t\right]=0.
\end{aligned}$$

Collecting things together, we get that

$$\begin{aligned}\tilde{\mathbb{E}}_t\left[f(X_t)L_t\middle|\mathscr{Y}_t\right] &= \tilde{\mathbb{E}}_0\left[f(X_0)L_0^{-1}\middle|\mathscr{Y}_0\right] + \int_{s=0}^{T}\tilde{\mathbb{E}}_s\left[(\mathscr{L}f)(X_s)L_s^{-1}\middle|\mathscr{Y}_s\right]ds \\ &\quad + \int_{s=0}^{t}\tilde{\mathbb{E}}_s\left[(hf)(X_{s_n^{(N)}})L^{-1}_{s_n^{(N)}}\middle|\mathscr{Y}_s\right]dY_s\end{aligned} \tag{55}$$

Let's now assume that there is a (random) field $\{v(t,x);\ t\ge 0,\ x\in\mathbb{R}\}$ such that

$$\tilde{\mathbb{E}}_t\left[\chi_A(X_t)L_t^{-1}\middle|\mathscr{Y}_t\right] = \int_{x\in A} v(t,x)dx.$$

Then

$$\pi_t(A) = \frac{\int_{x\in A} v(t,x)dx}{\int_{x\in\mathbb{R}} v(t,x)dx}.$$

(note that in general $\int_{x\in\mathbb{R}} v(t,x)dx \neq 1$). From (54), we get the Zakai equation

$$dv(t,x) = \mathscr{L}^* v(t,x)dt + h(x)v(t,x)dY_t \tag{56}$$

Exercises

(1) Prove the second representation of (51)
(2) Show that $\tilde{\mathbb{P}}_t$ is a probability measure and that under $\tilde{\mathbb{P}}_t$, $\{(W_s,Y_s);\ 0\le s\le t\}$ is a 2-dimensional Brownian motion.
(3) If B is a Brownian motion and $0\le s\le t$, show that $\sigma\{B_r;\ r\le t\} = \sigma\{B_r;\ r\le s\}\vee\sigma\{B_r - B_s;\ s\le r\le t\}$.
(4) Show that (56) comes from (54).
(5) Suppose that $\lim_{n\to\infty} X_n = 0$ in L^1. Let $\mathscr{G}$ be a sub-sigma-algebra of $\mathscr{F}$. Show that $\lim_{n\to\infty}\mathbb{E}[X_n|\mathscr{G}] = 0$. Hint: write $X_n = X_n^+ - X_n^-$.

CHAPTER 12

Option Pricing

Suppose that we have an asset which evolves according to

$$dS_t = bS_t dt + \sigma S_t W_t$$
$$S_0 = S_\circ.$$

Suppose that we have a *European option*. Suppose that the payoff function is ϕ at expiry T; i.e., we get the payoff $\phi(S_T)$ at time T. We need to price the option at time 0. Let's suppose also that the current interest rate is r.

Let $\tilde{V}_t$ denote the value of the option at time t. We want to *replicate* $\tilde{V}$ by using the stock and a bond. Namely, we want to find processes w^a and w^b such that

$$\tilde{V}_t = w_t^a S_t + w_t^b B_t$$

where $\dot{B}_t = rB_t$. Moreover, we want w^a and w^b to be such that

$$d\tilde{V}_t = w_t^a dS_t + w^b \dot{B}_t dt; \tag{57}$$

technically this is *self-financing*.

Suppose that $\tilde{V}_t = V(t, S_t)$ where V is sufficiently regular. Then (57) is equivalent to

$$\left\{\frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial S}(t,S_t)bS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t,S_t)\right\} dt + \frac{\partial V}{\partial S}(t,S_t)\sigma S_t dW_t$$
$$= w^a bS_t dt + w^a \sigma S_t dW_t + w^b rB_t dt.$$

Let's equate the dW_t terms. We get that

$$w_t^a = \frac{\partial V}{\partial S}(t,S_t). \tag{58}$$

The right-hand side is called the *delta* of the option, and this is known as delta-hedging.

Let's now equate the dt terms and use (58). We get that

$$\begin{aligned}\frac{\partial V}{\partial t}(t,S_t) + \frac{\partial V}{\partial S}(t,S_t)bS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t,S_t) &= \frac{\partial V}{\partial S}(t,S_t)bS_t + w^b rB_t \\ &= \frac{\partial V}{\partial S}(t,S_t)bS_t + r\left(V(t,S_t) - w^a S_t\right) \\ &= \frac{\partial V}{\partial S}(t,S_t)bS_t + r\left(V(t,S_t) - \frac{\partial V}{\partial S_t}(t,S_t)S_t\right)\end{aligned}$$

In other words, V satisfies the PDE

$$\begin{aligned}\frac{\partial V}{\partial t}(t,S) + r\frac{\partial V}{\partial S}(t,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(t,S) &= rV(t,S) \qquad t < T \\ V(T,S) &= \phi(S).\end{aligned} \tag{59}$$

This is the famed *Black-Scholes PDE.*

Let's next see what happens if we represent V via Feynman-Kac. Suppose that

$$\begin{aligned}d\tilde{S}_t &= r\tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ \tilde{S}_0 &= S_\circ.\end{aligned} \tag{60}$$

Then by Ito's formula $e^{-rt}V(t,\tilde{S}_t)$ is a martingale so

$$V(t,S_\circ)=\mathbb{E}\left[e^{-rT}V(t,\tilde{S}_T)\right]=\mathbb{E}\left[e^{-rT}\phi(\tilde{S}_T)\right].$$

This is the *risk-neutral* measure.

1. Bonds

We can do something similar for bonds. A bond is a promise to pay a coupon at a certain time T, at least if the bond doesn't default. The yield of the bond compensates for the credit risk.

Let τ be the time that the bond defaults. Suppose that we have a bond B° which pays a coupon of \$1 at time T°. The dynamics of the bond are thus

$$B^\circ_t=\exp\left[-\int_{s=t}^{T^\circ}r(s)ds\right]\chi_{\{\tau>t\}}$$

Defining $J_t\overset{\text{def}}{=}\chi_{\{\tau\le t\}}$, we have that

$$\begin{aligned}dB^\circ_t&=r(t)B^\circ_t dt-B^\circ_{t-}dJ_t\\ B^\circ_T&=\chi_{\{\tau>T\}}.\end{aligned}$$

We want to understand the risk-neutral price of default. Fix $T\in(0,T^\circ)$ and consider a credit derivative which pays \$1 if at time T if $\tau>T$ (this will of course be another bond, but let's avoid that interpretation for the moment). The price of the credit derivative is $\tilde{V}(t)$. We want to replicate the credit derivative with a riskless asset Π (which grows at rate r^*) and with the bond B°; namely, we want that

$$\tilde{V}(t)=w^{(1)}_t\Pi_t+w^{(2)}_t B^\circ_t$$

for some weights $w^{(1)}$ and $w^{(2)}$. We want this porfrolio to be self-financing, i.e., that

$$\begin{aligned}d\tilde{V}(t)&=w^{(1)}_{t-}r\Pi_t dt+w^{(2)}_{t-}dB^\circ_t\\ &=w^{(1)}_{t-}r^*\Pi_{t-}dt+w^{(2)}_{t-}\left\{r(t)B^\circ_{t-}dt-B^\circ_{t-}dJ_t\right\}\end{aligned}$$

Let's now assume that

$$\tilde{V}(t)=V(t,B_t);$$

i.e., the price of the credit derivative is a function of time and the bond price. Then we have that

$$\begin{aligned}&\left\{\frac{\partial V}{\partial t}(t,B_{t-})+\frac{\partial V}{\partial B}(t,B_{t-})r(t)B_{t-}\right\}dt+\left\{V(t,0)-V(t,B_{t-})\right\}dJ_t\\ &\quad=w^{(1)}_{t-}r^*\Pi_{t-}dt+w^{(2)}_{t-}\left\{r(t)B_{t-}dt-B_{t-}dJ_t\right\}\end{aligned}$$

Equating the values of the jumps, let's set

$$w^{(2)}_t=-\frac{V(t,0)-V(t,B_t)}{B_t};\tag{61}$$

then

$$w^{(2)}_{t-}=-\frac{V(t,0)-V(t,B_{t-})}{B_{t-}}.$$

Using the fact that

$$w^{(1)}_t\Pi_t=V(t,B_t)-w^{(2)}_t B_t$$

we have that

$$\begin{aligned}\frac{\partial V}{\partial t}(t,B_{t-})+\frac{\partial V}{\partial B}(t,B_{t-})r(t)B_{t-}&=r^*V(t,B_{t-})+(r(t)-r^*)w^{(2)}_{t-}B_{t-}\\ &=r^*V(t,B_{t-})-(r(t)-r^*)\left\{V(t,0)-V(t,B_{t-})\right\}\\ &=r(t)V(t,B_{t-})-(r(t)-r^*)V(t,0)\end{aligned}$$

Thus the Black-Scholes PDE is

$$\frac{\partial V}{\partial t}(t,B)+\frac{\partial V}{\partial B}(t,B)r(t)B = r(t)V(t,B)-(r(t)-r^*)V(t,0) \tag{62}$$
$$= r^*V(t,B)-(r(t)-r^*)\left\{V(t,0)-V(t,B)\right\}$$

The boundary conditions on this PDE are

$$V(t,0)=0 \qquad \text{if } t<T$$
$$V(T,B)=1 \qquad \text{if } B>0$$

Using the first boundary condition in the PDE, we get that in fact

$$\frac{\partial V}{\partial t}(t,B)+\frac{\partial V}{\partial B}(t,B)r(t)B = r(t)V(t,B)$$

Turning things around, let's define $w^{(2)}$ as in (61) and suppose that

$$V(t,B_t)=w^{(1)}_t\Pi_t+w^{(2)}_tB_t.$$

We then have that

$$\begin{aligned}
dV(t,B_t) &= \left\{\frac{\partial V}{\partial t}(t,B_{t-})+\frac{\partial V}{\partial B}(t,B_{t-})r(t)B_{t-}\right\}dt+\left\{V(t,0)-V(t,B_{t-})\right\}dJ_t\\
&= r^*V(t,B_{t-})dt-(r(t)-r^*)\left\{V(t,0)-V(t,B_{t-})\right\}dt+\left\{V(t,0)-V(t,B_{t-})\right\}dJ_t\\
&= r^*V(t,B_{t-})dt+(r(t)-r^*)w^{(2)}_{t-}dt-w^{(2)}_{t-}dJ_t\\
&= r^*\left\{V(t,B_{t-})-w^{(2)}_{t-}B_{t-}\right\}+w^{(2)}_{t-}\left\{r(t)B_{t-}dt-B_{t-}dJ_t\right\}\\
&= r^*w^{(1)}_{t-}\Pi_{t-}dt+w^{(2)}_{t-}dB_t
\end{aligned}$$

so the portfolio is indeed self-financing.

Let's now do the Feynman-Kac formula for (62). Assume that τ^* has hazard function $r^*(t)-r$; i.e.,

$$\mathbb{P}\left\{\tau^*>t\right\}=\exp\left[-\int_{s=0}^t (r(s)-r^*)ds\right].$$

Defining

$$J_t \overset{\text{def}}{=} \chi_{\{\tau^*\le t\}},$$

we have that J jumps from 0 to 1 at time τ^* and thus that

$$dJ_t=(r(t)-r^*)(1-J_{t-})dt+dM_t$$

where M is a martingale. Let's finally set

$$B^*_t \overset{\text{def}}{=} \exp\left[-\int_{s=t}^T r(s)ds\right](1-J_t);$$

these are the dynamics of a bond with yield r^* and default rate $r(t)-r^*$. Note that thus

$$dB^*_t=r(t)B^*_{t-}dt-(r(t)-r^*)B_{t-}dt-\exp\left[\int_{s=t}^T r(s)ds\right]dM_t=r^*B^*_{t-}dt+\exp\left[\int_{s=t}^T r(s)ds\right]dM_t$$

Thus B^* has the required form of a martingale discounted by the risk-free rate.

Setting

$$Z_t \overset{\text{def}}{=} V(t,B^*_t)\exp\left[-r^*t\right],$$

we have that

$$\begin{aligned}
dZ_t &= \left\{\frac{\partial V}{\partial t}(t,B^*_{t-})+\frac{\partial V}{\partial B}(t,B^*_{t-})r(t)B^*_{t-}-r^*V(t,B^*_{t-})\right\}(1-J_{t-})dt+\{V(t,0)-V(t,B^*_{t-})\}dJ_t\\
&= (r(t)-r^*)V(t,B^*_{t-})dt-V(t,B^*_{t-})\left\{(r(t)-r)(1-J_{t-})dt+dM_t\right\}\\
&= -V(t,B^*_{t-})dM_t
\end{aligned}$$

Thus Z is a martingale, so

$$V(0, B_0^*) = e^{-r^*T}\mathbb{E}[V(T, B_T^*)] = e^{-r^*T}\mathbb{P}\{\tau^* > T\}.$$

This is what one would expect.

Exercises

(1) Show that if V satisfies (59), $w_t^a = \frac{\partial V}{\partial S}(t, S_t)$ and $w^b = B_t^{-1}\{V(t, S_t) - w_t^a S_t\}$, then (57) holds.
(2) Show that if $\tilde{S}$ satisfies (60) then $e^{-rt}V(t, \tilde{S}_t)$ is indeed a martingale.

CHAPTER 13

Feller Conditions

Suppose that we have an SDE

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_\circ \end{aligned} \tag{63}$$

Suppose that $a < x_\circ < b$. We want to understand how X exits the interval (a, b). We suppose that b and σ are continuous on (α, β). We suppose that $\sigma > 0$ on (a, b).

Define first $Y_t = p(X_t)$ where p is smooth. Then

$$dY_t = \left\{ bp' + \frac{1}{2}\sigma p'' \right\} (X_t)dt + (\sigma p')(X_t)dW_t$$

Let's assume that

$$bp' + \frac{1}{2}\sigma p'' \equiv 0$$

so that Y is a martingale. In other words, we want that

$$p''(x) = -\frac{2b}{\sigma^2}(x)p'(x).$$

Since $\sigma > 0$ on (α, β), a solution of this PDE exists on (a, b). Adding the initial condition $p(x_\circ) = 0$ and $p'(x_\circ) = 1$, we have that

$$p(x) = \int_{y=x_\circ}^{x} \exp\left[-\int_{z=x_\circ}^{y} \frac{2b}{\sigma^2}(z)dz \right] dy.$$

We note that $p' > 0$ so p is invertible. It is known as the *scale function*.

Defining

$$\tilde{\sigma}(y) \stackrel{\text{def}}{=} (p'\sigma)(p^{-1}(y)) \qquad y \in (\alpha, \beta)$$

we have that

$$\begin{aligned} dY_t &= \tilde{\sigma}(Y_t)dW_t \\ Y_\circ &= 0. \end{aligned}$$

Fix now $[\alpha, \beta] \subset\subset (a, b)$ containing $x_\circ$, define

$$\tau_{\alpha,\beta} \stackrel{\text{def}}{=} \inf \left\{ t > 0;\ X_t \notin [\alpha, \beta] \right\}.$$

Then

$$\kappa_{\alpha,\beta} \stackrel{\text{def}}{=} \inf_{y \in p[\alpha,\beta]} \tilde{\sigma}(y)$$

is positive. We first claim that X exits $[\alpha, \beta]$; i.e., Y exits $p[\alpha, \beta]$. Note that since $\alpha < x_\circ < \beta$, p is increasing, and $p(x_\circ) = 0$, $p(\alpha) < 0 < p(\beta)$. We have that

$$Y^2_{\tau_{\alpha,\beta}\wedge t} = \int_{s=0}^{\tau_{\alpha,\beta}\wedge t} \tilde{\sigma}^2(Y_s)ds + \int_{s=0}^{\tau_{\alpha,\beta}\wedge t} Y_s\tilde{\sigma}(Y_s)dW_s.$$

Thus

$$\mathbb{E}\left[Y^2_{\tau_{\alpha,\beta}\wedge t} \right] \ge \kappa^2 \mathbb{E}\left[\tau_{\alpha,\beta} \wedge t \right].$$

Thus

$$\mathbb{E}\left[\tau_{\alpha,\beta} \wedge t \right] \le \frac{1}{\kappa^2} \max\left\{ |p(\alpha)|^2, |p(\beta)|^2 \right\} < \infty$$

so in fact

$$\mathbb{E}\left[\tau_{\alpha,\beta}\right] \le \frac{1}{\kappa^2}\max\left\{|p(\alpha)|^2, |p(\beta)|^2\right\} \tag{64}$$

and hence $\mathbb{P}\{\tau_{\alpha,\beta}<\infty\}=1$.

Since $\mathbb{P}\{\tau_{\alpha,\beta}<\infty\}=1$,

$$\mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\alpha\right\}+\mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\beta\right\}=1.$$

Secondly, since Y is continuous and bounded on $[0,\tau_{\alpha,\beta}]$, we have that

$$\mathbb{E}\left[Y_{\tau_{\alpha,\beta}}\right]=\lim_{t\nearrow\infty}\mathbb{E}\left[Y_{\tau_{\alpha,\beta}\wedge t}\right]=\mathbb{E}[Y_0]=0.$$

Hence

$$p(\alpha)\mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\alpha\right\}+p(\beta)\mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\beta\right\}=0.$$

Thus

$$\mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\alpha\right\}=\frac{p(\beta)}{p(\beta)-p(\alpha)} \qquad \text{and} \qquad \mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\beta\right\}=\frac{-p(\alpha)}{p(\beta)-p(\alpha)}.$$

Define

$$\tau \overset{\text{def}}{=} \lim_{\substack{\alpha\searrow a\\ \beta\nearrow b}}\tau_{\alpha,\beta}$$

(by monotonicity, τ is well-defined). Note that by monotonicity,

$$p(a+)\overset{\text{def}}{=}\lim_{\alpha\searrow a}p(\alpha)$$

$$p(b-)\overset{\text{def}}{=}\lim_{\beta\nearrow b}p(\beta)$$

are well-defined (and possibly $p(a+)=-\infty$ or $p(b-)=\infty$).

First assume that $p(b-)=\infty$. For any $[\alpha,\beta]\subset(a,b)$ if $X_{\tau_{\alpha,\beta}}=\alpha$, then $\inf_{0\le t<\tau}X_t\le\alpha$. Thus

$$\mathbb{P}\left\{\inf_{0\le t<\tau}X_t\le\alpha\right\}\ge\mathbb{P}\left\{X_{\tau_{\alpha,\beta}}=\alpha\right\}=\frac{p(\beta)}{p(\beta)-p(\alpha)}.$$

Letting $\beta\nearrow 0$, we get that

$$\mathbb{P}\left\{\inf_{0\le t<\tau}X_t\le\alpha\right\}\ge 1.$$

Letting $\alpha\searrow a$, we get that

$$\mathbb{P}\left\{\inf_{0\le t<\tau}X_t=a\right\}=1.$$

Similarly, if $p(a+)=-\infty$, then

$$\mathbb{P}\left\{\sup_{0\le t<\tau}X_t=b\right\}=1. \tag{65}$$

Let's now see what happens if $p(a+)<\infty$ or $p(b-)<\infty$. Let's next use some results from martingale theory, letting $\mathscr{W}$ be the filtration of (6). Let's assume that $p(a+)>-\infty$. Let $\alpha_n\searrow a$ and $\beta_n\nearrow b$. For $n\in\mathbb{N}$ and $t\ge0$, define $\tilde{Y}^{(n)}_t\overset{\text{def}}{=}Y_{t\wedge\tau_{\alpha_n,\beta_n}}-p(a+)$. Then $\tilde{Y}^{(n)}_t\ge0$.

Remark 0.1. For a fixed $t\ge0$, $\{\tilde{Y}^{(n)}_t\}_{n\in\mathbb{N}}$ is a discrete-time martingale.

Remark 0.2. For a fixed $n\in\mathbb{N}$, $\{\tilde{Y}^{(n)}_t\}_{t\ge0}$ is a continuous-time martingale.

The submartingale convergence theorem and Remark 0.1 ensure that $\tilde{Y}_t\overset{\text{def}}{=}\lim_{n\to\infty}\tilde{Y}^{(n)}_t$ exists ($\mathbb{P}$-a.s.) for each $t\ge0$. Fix $0\le s\le t$ and $A\in\mathscr{W}_s$. Fix $n\in\mathbb{N}$; then

$$\mathbb{E}[\tilde{Y}_t\chi_A]=\mathbb{E}\left[\lim_{n'\to\infty}\tilde{Y}^{(n')}_t\chi_A\right]\le\varliminf_{n'\to\infty}\mathbb{E}[\tilde{Y}^{(n')}_t\chi_A]=\mathbb{E}[\tilde{Y}^{(n)}_t\chi_A]=\mathbb{E}[\tilde{Y}^{(n)}_s\chi_A]$$

where we have used Fatou's lemma. Hence Thus

$$\mathbb{E}[\tilde{Y}_t\big|\mathscr{W}_s]\le\tilde{Y}^{(n)}_s.$$

and upon letting $n \nearrow \infty$, we have that

$$\mathbb{E}[\tilde{Y}_t \big| \mathscr{W}_s] \le \tilde{Y}_s.$$

Thus $\tilde{Y}$ is a nonnegative supermartingale, and $\tilde{Y} \overset{\text{def}}{=} \lim_{t\to\infty} \tilde{Y}_t$ exists $\mathbb{P}$-a.s. If $t < \tau$, then $t < \tau_{\alpha_n,\beta_n} < \tau$ for large enough n, so in fact $\tilde{Y}^{(n)}_t = Y_t - p(a+)$ for n large enough. Thus $Y_t - p(a+) - \tilde{Y}_t$ if $t < \tau$, so in fact

$$\lim_{t\nearrow\tau} Y_t = \tilde{Y}. \tag{66}$$

A symmetric argument gives us (66) if $p(b-) < \infty$.

Suppose that $p(a+) > -\infty$ and $p(b-) = \infty$. Then p is a diffeomorphism from $[x_\circ, b)$ to $[0,\infty)$, so

$$\mathbb{P}\left\{\sup_{0\le t<\tau} X_t < b\right\} = \mathbb{P}\left\{\sup_{0\le t<\tau} Y_t < \infty\right\} = 1.$$

Similarly, suppose that $p(b-) < \infty$ and $p(a+) = -\infty$. Then p is a diffeomorphism from $(a, x_\circ]$ to $(-\infty, 0]$, so

$$\mathbb{P}\left\{\inf_{0\le t<\tau} X_t > a\right\} = \mathbb{P}\left\{\inf_{0\le t<\tau} Y_t > -\infty\right\} = 1.$$

Let's next assume that both $p(a+) = -\infty$ and $p(b-) = \infty$. Let U satisfy the integral equation

$$U(y) = 1 + \int_{z=0}^{y}\left\{\int_{w=0}^{z} \frac{2}{\tilde{\sigma}^2(w)} U(w)dw\right\} dz. \qquad y \in \mathbb{R} \tag{67}$$

To see that U exists and is well-defined, let

$$\begin{aligned} U_0(y) &= 1 \\ U_{n+1}(y) &= 1 + \int_{z=0}^{y}\left\{\int_{w=0}^{z} \frac{2}{\tilde{\sigma}^2(w)} U_n(w)dw\right\} dz \qquad n \in \mathbb{N}. \end{aligned} \tag{68}$$

The U_n's are well-defined on $\mathbb{R}$. For each $K > 0$, define

$$M_K \overset{\text{def}}{=} \sup_{|y|\le K} \frac{2}{\tilde{\sigma}^2(y)}.$$

For every $n \in \mathbb{N}$ with $n \ge 2$ and all $y \in [0, K]$,

$$\begin{aligned} |U_{n+1}(y) - U_n(y)| &\le \int_{z=0}^{y}\left\{\int_{w=0}^{z} \frac{2}{\tilde{\sigma}^2(w)} |U_n(w) - U_{n-1}(w)|dw\right\} dz \\ &\le \int_{w=0}^{y}\left\{\frac{2(y-w)}{\tilde{\sigma}^2(w)}\right\} |U_n(w) - U_{n-1}(w)|dw \\ &\le M_K \int_{w=0}^{y} |U_n(w) - U_{n-1}(w)|dw. \end{aligned}$$

Thus

$$|U_{n+1}(y) - U_n(y)| \le \frac{(M_K y)^n}{n!} \sup_{0\le y'\le K} |U_1(y') - U_0(y')|.$$

Reflecting things, we get that

$$|U_{n+1}(y) - U_n(y)| \le \frac{(M_K |y|)^n}{n!} \sup_{0\le y'\le K} |U_1(y') - U_0(y')|.$$

for all $y \in [-K, K]$ and $n \in \mathbb{N}$. Thus

$$\lim_{n,n'\to\infty} \sup_{|y|\le K} |U_n(y) - U_{n'}(y)|.$$

Thus there is a $U \in C(\mathbb{R})$ such that

$$\lim_{n\to\infty} \sup_{|y|\le K} |U_n(y) - U(y)| = 0.$$

We also note that if U and V satisfy (67), then

$$|U(y) - V(y)| \le M_K \int_{w=0}^{|y|} |U(w) - V(w)|dw$$

so $U = V$ by Gronwall's inequality; in other words, the solution of (67) is unique.

We note that the U_n's of (68) are nonnegative; thus $U \ge 0$ and thus in fact $U \ge 1$. Fix $K > 0$ and define

$$m_K \overset{\text{def}}{=} \inf_{|y| \le K} \frac{2}{\tilde{\sigma}^2(y)}.$$

Then for $y \ge K$, we have that

$$U(y) \ge \int_{z=K}^{y} \left\{ \int_{w=0}^{K} m_K dw \right\} dz = (y - K)K m_K.$$

In other words, $\overline{\lim}_{y \nearrow \infty} U(y) = \infty$. Similarly, $\lim_{y \searrow -\infty} U(y) = \infty$.

From a probabilistic standpoint, U is important since

$$Z_t \overset{\text{def}}{=} e^{-t} U(Y_t)$$

is a nonnegative martingale. As in our above martingale analysis, this means that

$$\tilde{Z}^{(t)}_n \overset{\text{def}}{=} Z_{t \wedge \tau_{\alpha_n, \beta_n}}$$

is a nonnegative martingale. Thus $\tilde{Z}_t \overset{\text{def}}{=} \lim_{n \to \infty} \tilde{Z}^{(t)}_n$ is well-defined and is a nonnegative supermartingale. Thus $Z \overset{\text{def}}{=} \lim_{t \nearrow \infty} \tilde{Z}_t$ is well-defined and

$$\mathbb{E}\left[\lim_{t \nearrow \tau} e^{-t} U(Y_t)\right] = \mathbb{E}[Z] \le \mathbb{E}[U(Y_0)] = 1.$$

If $\tau < \infty$, then $\lim_{t \nearrow \infty} U(Y_t) = \infty$. This leads to a contradiction unless $\mathbb{P}\{\tau < \infty\} = 0$. Thus we know that in fact $\mathbb{P}\{\tau = \infty\} = 1$.

Finally, assume that $p(a+) > -\infty$ and $p(b-) < \infty$. Then $\lim_{t \nearrow \tau} X_t = \lim_{n \to \infty} X_{\tau_{\alpha_n, \beta_n}}$ and

$$\chi_{\{\lim_{t \nearrow \tau} X_t = a\}} = \lim_{n \to \infty} \chi_{\{X_{\tau_{\alpha_n, \beta_n}} = \alpha_n\}} \qquad \text{and} \qquad \chi_{\{\lim_{t \nearrow \tau} X_t = b\}} = \lim_{n \to \infty} \chi_{\{X_{\tau_{\alpha_n, \beta_n}} = \beta_n\}}.$$

Thus by dominated convergence,

$$\begin{aligned} \mathbb{P}\left\{\lim_{t \nearrow \tau} X_t = a\right\} &= \lim_{n \to \infty} \mathbb{P}\left\{X_{\tau_{\alpha_n, \beta_n}} = \alpha_n\right\} = \lim_{n \to \infty} \frac{p(\beta_n)}{p(\beta_n) - p(\alpha_n)} = \frac{p(b-)}{p(b-) - p(a+)} \\ \mathbb{P}\left\{\lim_{t \nearrow \tau} X_t = b\right\} &= \lim_{n \to \infty} \mathbb{P}\left\{X_{\tau_{\alpha_n, \beta_n}} = \beta_n\right\} = \lim_{n \to \infty} \frac{-p(\alpha_n)}{p(\beta_n) - p(\alpha_n)} = \frac{-p(a+)}{p(b-) - p(a+)}. \end{aligned}$$

Let's combine things together.

If $p(a+) > -\infty$ and $p(b) = \infty$, then

$$\mathbb{P}\left\{\lim_{t \nearrow \tau} X_t = a\right\} = \mathbb{P}\left\{\sup_{0 \le t < \tau} X_r < b\right\} = 1.$$

If $p(a+) = -\infty$ and $p(b-) < \infty$, then

$$\mathbb{P}\left\{\lim_{t \nearrow \tau} X_t = b\right\} = \mathbb{P}\left\{\inf_{0 \le t < \tau} X_r > a\right\} = 1.$$

If $p(a+) > -\infty$ and $p(b-) < \infty$, then

$$\mathbb{P}\left\{\lim_{t \nearrow \tau} X_t = b\right\} = \frac{-p(a+)}{p(b-) - p(a+)} \qquad \text{and} \qquad \mathbb{P}\left\{\lim_{t \nearrow \tau} X_t = a\right\} = \frac{p(b-)}{p(b-) - p(a+)}.$$

Exercises

(1) Let W be a Brownian motion. Define $Y_t \stackrel{\text{def}}{=} W_t^3$.
   (a) Show that $dY_t = 3Y_t^{2/3} dW_t + 3Y_t^{1/3} dt$.
   (b) Find the scale function for Y.

(2) Show that if $p(a+) = -\infty$, then (65) holds.

(3) Show that if $p(b-) < \infty$, then $\lim_{\substack{t \nearrow \tau \\ t<\tau}} X_t$ exists.

(4) Consider again the CIR process (33). Show that λ is strictly positive if $2\bar{\lambda}\alpha \geq \sigma^2$.

CHAPTER 14

Wong-Zakai

Let W be a Brownian motion. For each $\delta \in (0,1)$, define

$$W^\delta_t \overset{\text{def}}{=} \left(\frac{t}{\delta} + 1 - \left\lfloor \frac{t}{\delta} \right\rfloor\right) W_{\lfloor t/\delta \rfloor \delta} + \left(\frac{t}{\delta} - \left\lfloor \frac{t}{\delta} \right\rfloor\right) W_{(\lfloor t/\delta \rfloor + 1)\delta}.$$

In other words, if $n\delta \le t < (n+1)\delta$,

$$\dot W^\delta_t = \frac{W_{(n+1)\delta} - W_{n\delta}}{\delta}.$$

Suppose now that b and σ are smooth drift and diffusion coefficients, and that they and their derivatives are bounded. Fix $x_\circ \in \mathbb{R}$. and consider the random ODE

$$\begin{aligned} \dot X^\delta_t &= b(X^\delta_t) + \sigma(X^\delta_t)\dot W^\delta_t \\ X^\delta_0 &= x_\circ. \end{aligned}$$

Where does X^δ go as $\delta \searrow 0$?

Let X^0 solve

$$\begin{aligned} dX_t &= b(X_t) + \sigma(X_t)dW_t + \frac{1}{2}(\sigma\sigma')(X_t) \\ X_0 &= x_\circ. \end{aligned}$$

We claim that for every $T > 0$,

$$\lim_{\delta \searrow 0} \sup_{0 \le t \le T} \mathbb{E}\left[\left|X^\delta_t - X_t\right|\right] = 0. \tag{69}$$

Throughout this chapter, we will use K to denote a constant (which may change from line to line) which depends only on b and σ and their derivatives.

Note that the small intervals compensate the δ^{-1} in the denominator. For $\delta > 0$, $n \in \mathbb{N}$, and $n\delta \le t < (n+1)\delta$,

$$|X^\delta_t - X^\delta_{n\delta}| \le K\left\{\delta + \left|W_{(n+1)\delta} - W_{n\delta}\right|\right\}. \tag{70}$$

For $\delta > 0$ and $t \ge 0$, define

$$\tau_\delta(t) \overset{\text{def}}{=} \lfloor to/\delta \rfloor \delta.$$

Then

$$X^\delta_t = x_\circ + \int_{s=0}^t b(X^\delta_s)ds + \int_{s=0}^t \sigma(X^\delta_s)\left\{\frac{W_{\tau_\delta(s)+\delta} - W_{\tau_\delta(s)}}{\delta}\right\}ds$$

Let's look at the second term. We have that

$$\int_{s=0}^t \sigma(X^\delta_s)\left(\frac{W_{\tau_\delta(s)+\delta} - W_{\tau_\delta(s)}}{\delta}\right)ds = \int_{s=0}^{\tau_\delta(t)} \sigma(X^\delta_s)\left(\frac{W_{\tau_\delta(s)+\delta} - W_{\tau_\delta(s)}}{\delta}\right)ds + \mathcal{E}^{A,\delta}_t$$

where

$$\mathcal{E}^{A,\delta}_t \overset{\text{def}}{=} \int_{s=\tau_\delta(t)}^t \sigma(X^\delta_s)\left(\frac{W_{\tau_\delta(s)+\delta} - W_{\tau_\delta(s)}}{\delta}\right)ds.$$

We calculate that

$$\left|\mathcal{E}^{A,\delta}_t\right| \le K\left|W_{\tau_\delta(t)+\delta} - W_{\tau_\delta(t)}\right|.$$

Fix now $n \in \mathbb{N}$; let's consider

$$\int_{n\delta}^{(n+1)\delta} \sigma(X^\delta_s)\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right) ds.$$

For $s \in [n\delta,(n+1)\delta)$,

$$\sigma(X^\delta_s) = \sigma(X^\delta_{n\delta}) + \int_{r=n\delta}^s (b\sigma')(X^\delta_r)dr + \int_{r=n\delta}^s (\sigma\sigma')(X^\delta_r)dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right)$$

and hence

$$\begin{aligned}\int_{n\delta}^{(n+1)\delta} \sigma(X^\delta_s)\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right) ds &= \sigma(X^\delta_{n\delta})\left\{W_{(n+1)\delta}-W_{n\delta}\right\} \\ &\quad + \int_{s=n\delta}^{(n+1)\delta}\int_{r=n\delta}^s (\sigma\sigma')(X^\delta_r)dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right)^2 ds + \mathcal{E}^{B,\delta}_n\end{aligned}$$

where

$$\mathcal{E}^{B,\delta}_n \overset{\text{def}}{=} \int_{s=n\delta}^{(n+1)\delta}\int_{r=n\delta}^s (b\sigma')(X^\delta_r)dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right) ds$$

We note that

$$\int_{s=n\delta}^{(n+1)\delta}\left\{\int_{r=n\delta}^s dr\right\} ds = \frac{\delta^2}{2}. \tag{71}$$

Thus

$$\left|\mathcal{E}^{B,\delta}_n\right| \le K\left|W_{(n+1)\delta}-W_{n\delta}\right|\delta$$

Next note that

$$\begin{aligned}&\int_{s=n\delta}^{(n+1)\delta}\int_{r=n\delta}^s (\sigma\sigma')(X^\delta_r)dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right)^2 ds \\ &\qquad = \int_{s=n\delta}^{(n+1)\delta}\int_{r=n\delta}^s (\sigma\sigma')(X^\delta_{n\delta})dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right)^2 ds + \mathcal{E}^{C,\delta}_n\end{aligned}$$

where

$$\mathcal{E}^{C,\delta}_n \overset{\text{def}}{=} \int_{s=n\delta}^{(n+1)\delta}\int_{r=n\delta}^s \left\{(\sigma\sigma')(X^\delta_r)-(\sigma\sigma')(X_{n\delta})\right\} dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right)^2 ds$$

By (70), we have that

$$|\mathcal{E}^{C,\delta}_n| \le K|W_{(n+1)\delta}-W_{n\delta}|^3.$$

Next note that

$$\left(W_{(n+1)\delta}-W_{n\delta}\right)^2 = 2\int_{s=n\delta}^{(n+1)\delta}(W_s-W_{n\delta})dW_s + \delta.$$

Thus

$$\int_{s=n\delta}^{(n+1)\delta}\int_{r=n\delta}^s (\sigma\sigma')(X^\delta_{n\delta})dr\left(\frac{W_{(n+1)\delta}-W_{n\delta}}{\delta}\right)^2 ds = \frac{1}{2}(\sigma\sigma')(X^\delta_{n\delta})\delta + \mathcal{E}^{D,\delta}_n$$

where

$$\mathcal{E}^{D,\delta}_n \overset{\text{def}}{=} \frac{1}{2}(\sigma\sigma')(X^\delta_{n\delta})\int_{s=n\delta}^{(n+1)\delta}(W_s-W_{n\delta})dW_s.$$

Combining things together, we have that

$$\begin{aligned}X^\delta_t &= x_\circ + \int_{s=0}^t b(X^\delta_s)ds + \sum_{0\le n\le\lfloor t/\delta\rfloor}\int_{s=n\delta}^{(n+1)\delta}\sigma(X^\delta_{n\delta})dW_s + \frac{1}{2}\sum_{0\le n\le\lfloor t/\delta\rfloor}\int_{s=n\delta}^{(n+1)\delta}(\sigma\sigma')(X^\delta_{n\delta})ds \\ &\quad + \mathcal{E}^{A,\delta}_t + \sum_{0\le n\le\lfloor t/\delta\rfloor}\left\{\mathcal{E}^{B,\delta}_n + \mathcal{E}^{C,\delta}_n + \mathcal{E}^{D,\delta}_n\right\}.\end{aligned} \tag{72}$$

We want this to look like

$$X_t = x_\circ + \int_{s=0}^t b(X_s)ds + \int_{s=0}^t \sigma(X_s)dW_s + \frac{1}{2}\int_{s=0}^t (\sigma\sigma')(X_s)ds. \tag{73}$$

Let's rewrite some of the terms in (72). We have that

$$\sum_{0\le n\le \lfloor t/\delta\rfloor} \int_{s=n\delta}^{(n+1)\delta} \sigma(X^\delta_{n\delta})dW_s = \int_{s=0}^t \sigma(X^\delta_{\tau_\delta(s)})dW_s + \mathcal{E}^{E,\delta}_t$$

where

$$\mathcal{E}^{E,\delta}_t = \sigma(X^\delta_{\tau_\delta(t)})\left\{W_t - W_{\tau_\delta(t)}\right\};$$

then

$$|\mathcal{E}^{E,\delta}_t| \le K|W_t - W_{\tau_\delta(t)}|.$$

We also have that

$$\sum_{0\le n\le \lfloor t/\delta\rfloor} \int_{s=n\delta}^{(n+1)\delta} \frac{1}{2}(\sigma\sigma')(X^\delta_{n\delta})ds = \int_{s=0}^t \frac{1}{2}(\sigma\sigma')(X^\delta_s)ds + \mathcal{E}^{F,\delta}_t + \mathcal{E}^{G,\delta}_t$$

where

$$\mathcal{E}^{F,\delta}_t \overset{\text{def}}{=} \int_{s=\tau_\delta(t)}^t \frac{1}{2}(\sigma\sigma')(X^\delta_{\tau_\delta(s)})ds$$

$$\mathcal{E}^{G,\delta}_t \overset{\text{def}}{=} \int_{s=0}^t \frac{1}{2}\left\{(\sigma\sigma')(X^\delta_{\tau_\delta(s)}) - (\sigma\sigma')(X^\delta_s)\right\}ds.$$

We note that

$$|\mathcal{E}^{F,\delta}| \le K\delta$$

$$|\mathcal{E}^{G,\delta}_t| \le K\int_{s=0}^t \left\{\delta + |W_s - W_{\tau_\delta(s)}|\right\}ds$$

We now have that

$$X^\delta_t = x_\circ + \int_{s=0}^t b(X^\delta_s)ds + \int_{s=0}^t \sigma(X^\delta_{\tau_\delta(s)})dW_s + \int_{s=0}^t \frac{1}{2}(\sigma\sigma')(X^\delta_s)ds + \tilde{\mathcal{E}}^\delta_t$$

where

$$\tilde{\mathcal{E}}^\delta_t \overset{\text{def}}{=} \mathcal{E}^{A,\delta}_t + \sum_{0\le n\le\lfloor t/\delta\rfloor}\left\{\mathcal{E}^{B,\delta}_n + \mathcal{E}^{C,\delta}_n + \mathcal{E}^{D,\delta}_n\right\} + \mathcal{E}^{E,\delta}_t + \mathcal{E}^{F,\delta}_t + \mathcal{E}^{G,\delta}_t$$

Let's compare the evolution of X with that of (73). We have that

$$X^\delta_t - X_t = \int_{s=0}^t \left\{B(X^\delta_s) - B(X_s)\right\}ds + \int_{s=0}^t \left\{\sigma(X^\delta_{\tau_\delta(s)}) - \sigma(X_s)\right\}dW_s + \tilde{\mathcal{E}}^\delta_t$$

where

$$B(x) \overset{\text{def}}{=} b(x) + \frac{1}{2}(\sigma\sigma')(x).$$

Motivated by our calculations on existence and uniqueness of SDE's, we now have that

$$\begin{aligned}\mathbb{E}[X^\delta_t - X_t|^2] &\le Kt\int_{s=0}^t \mathbb{E}[|X^\delta_s - X_s|^2]ds + K\int_{s=0}^t \mathbb{E}[|X^\delta_{\tau_\delta(s)} - X_s|^2]ds + K\mathbb{E}[|\tilde{\mathcal{E}}^\delta_t|^2] \\ &\le Kt\int_{s=0}^t \mathbb{E}[|X^\delta_s - X_s|^2]ds + K\int_{s=0}^t \mathbb{E}[|X^\delta_s - X_s|^2]ds + K\int_{s=0}^t \mathbb{E}[|X^\delta_s - X^\delta_{\tau_\delta(s)}|^2]ds \\ &\quad + K\mathbb{E}[|\tilde{\mathcal{E}}^\delta_t|^2]\end{aligned}$$

Let's finally bound the errors. We have that

$$\int_{s=0}^t \mathbb{E}[|X^\delta_s - X^\delta_{\tau_\delta(s)}|^2]ds \le K\int_{s=0}^t \left\{\delta^2 + \mathbb{E}[|W_s - W_{\tau_\delta(s)}|^2]\right\}ds \le K\int_{s=0}^t \left\{\delta^2 + \delta\right\}ds \le Kt\delta.$$

We also have that

$$\begin{aligned}
&\mathbb{E}[|\mathcal{E}_t^{A,\delta}|^2] \le K\mathbb{E}[|W_t - W_{\tau_\delta(t)}|^2] \le K\delta \\
&\mathbb{E}[|\mathcal{E}^{E,\delta}|^2] \le K\mathbb{E}[|W_t - W_{\tau_\delta(t)}|^2] \le K\delta \\
&\mathbb{E}[|\mathcal{E}^{F,\delta}|^2] \le K\delta^2 \\
&\mathbb{E}[|\mathcal{E}_t^{G,\delta}|^2] \le Kt\int_{s=0}^t \left\{\delta^2 + \mathbb{E}[|W_s - W_{\tau_\delta(s)}|^2]\right\} ds \le Kt^2\delta
\end{aligned}$$

We also have that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{n\le\lfloor t/\delta\rfloor}\mathcal{E}_n^{B,\delta}\right)^2\right] &\le K\lfloor t/\delta\rfloor \sum_{n\le\lfloor t/\delta\rfloor}\mathbb{E}[|W_{(n+1)\delta} - W_{n\delta}|^2]\delta^2 K\left(\lfloor t/\delta\rfloor\right)^2\delta^3 \le Kt^2\delta \\
\mathbb{E}\left[\left(\sum_{n\le\lfloor t/\delta\rfloor}\mathcal{E}_n^{C,\delta}\right)^2\right] &\le K\lfloor t/\delta\rfloor \sum_{n\le\lfloor t/\delta\rfloor}\mathbb{E}[|W_{(n+1)\delta} - W_{n\delta}|^6] K\left(\lfloor t/\delta\rfloor\right)^2\delta^3 \le Kt^2\delta
\end{aligned}$$

Finally,

$$\mathbb{E}\left[\left(\sum_{n\le\lfloor t/\delta\rfloor}\mathcal{E}_n^{D,\delta}\right)^2\right] \le K\mathbb{E}\left[\left(\int_{s=0}^{\tau_\delta(t)}(W_s - W_{\tau_\delta(s)})dW_s\right)^2\right] = K\int_{s=0}^{\tau_\delta(t)}\mathbb{E}[(W_s - W_{\tau_\delta(s)})^2]ds \le K\delta.$$

To summarize, we have

$$\mathbb{E}[X_t^\delta - X_t|^2] \le Kt\int_{s=0}^t \mathbb{E}[|X_s^\delta - X_s|^2]ds + K(1+t)\delta$$

Gronwall's inequality then implies that

$$\mathbb{E}[X_t^\delta - X_t|^2] \le K\exp\left[K(1+t)t\right](1+t)\delta.$$

Letting $\delta \searrow 0$, we get (69).

CHAPTER 15

Euler-Maruyama approximations

Let's now consider the stochastic differential equation

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \qquad 0 \le t \le T \\ X_0 &= x_\circ \end{aligned} \tag{74}$$

where b and σ are "nice" function and $x_\circ$ is some initial condition. Our choice of the time horizon T is arbitrary.

A natural way to approximate the solution X of (74) is via the *Euler-Maruyama scheme*. Fix $\delta > 0$ and define

$$\begin{aligned} X^\delta_{(n+1)\delta} &= X^\delta_{n\delta} + b(X_{n\delta})\delta + \sigma(X_{n\delta})\{W_{(n+1)\delta} - W_{n\delta}\} \\ X^\delta_0 &= x_\circ \end{aligned} \tag{75}$$

We want to compare $X^\delta_{n\delta}$ to $X_{n\delta}$. To do so, let's convert X^δ into an SDE. Define $\tau_\delta(t) \overset{\text{def}}{=} \lfloor t/\delta \rfloor \delta$. Let $\hat{X}^\delta$ be the solution of the ODE

$$\begin{aligned} d\hat{X}^\delta_t &= b(\hat{X}^\delta_{\tau_\delta(t)})dt + \sigma(\hat{X}^\delta_{\tau_\delta(t)})dW_t \qquad 0 \le t \le T \\ \hat{X}^\delta_0 &= x_\circ \end{aligned} \tag{76}$$

Then $X^\delta_{n\delta} = \hat{X}^\delta_{n\delta}$

To compare X and $\hat{X}^\delta$, we have

$$\begin{aligned} X_t - \hat{X}^\delta_t &= \int_{s=0}^t \{b(X_s) - b(\hat{X}^\delta_s)\}ds + \int_{s=0}^t \{\sigma(X_s) - f(\hat{X}^\delta_s)\}dW_s \\ &= \int_{s=0}^t \{b(X_s) - b(\hat{X}^\delta_s)\}ds + \int_{s=0}^t \{\sigma(X_s) - \sigma(\hat{X}^\delta_s\}dW_s \\ &\quad + \int_{s=0}^t \{b(\mathsf{X}^{(N)}_s) - b(\hat{X}^\delta_s)\}ds + \int_{s=0}^t \{\sigma(\hat{X}^\delta_s) - \sigma(\hat{X}^\delta_s)\}dW_s \end{aligned}$$

Thus

$$\mathbb{E}\left[\left|X_t - \hat{X}^\delta_t\right|^2\right] \le Kt\int_{s=0}^t \mathbb{E}\left[\left|X_s - \hat{X}^\delta_s\right|^2\right] ds + Kt\int_{s=0}^t \mathbb{E}\left[\left|\hat{X}^\delta_s - \hat{X}^\delta_{\tau_\delta(s)}\right|^2\right] ds$$

Since

$$\hat{X}^\delta_t = \hat{X}^\delta_{\tau_\delta(t)} + b(X^\delta_{\tau_\delta(t)}))(t - \tau_\delta(t)) + \sigma(X^\delta_{\tau_\delta(t)})\left(W_t - W_{\tau_\delta(t)}\right)$$

we have that

$$\mathbb{E}\left[\left|\hat{X}^\delta_t = \hat{X}^\delta_{\tau_\delta(t)}\right|^2\right] \le K\left\{\delta^2 + \delta\right\}.$$

Thus

$$\mathbb{E}\left[\left|X_t - \hat{X}^\delta_t\right|^2\right] \le K\exp\left[Kt^2\right]\delta.$$

We thus have that

$$\lim_{\delta \searrow 0} \sup_{0 \le t \le T} \mathbb{E}\left[\left|X_t - \hat{X}^\delta_t\right|^2\right] = 0.$$

CHAPTER 16

Large Deviations

Suppose that that $\{X_\varepsilon\}_{\varepsilon\in(0,1)}$ is a collection of $\mathbb{R}$-valued random variables such that, for some $f \in C^2[0,1]$,

$$\mathbb{P}\{X_\varepsilon \in A\} = c_\varepsilon \int_{x\in A} \exp\left[-\frac{1}{\varepsilon} f(x)\right] dx \tag{77}$$

for all $A \in \mathscr{B}[0,1]$, where

$$c_\varepsilon \stackrel{\text{def}}{=} \left\{\int_{x\in A} \exp\left[-\frac{1}{\varepsilon} f(x)\right] dx\right\}^{-1}$$

with $\lim_{N\to\infty} \varepsilon \ln c_\varepsilon = 0$.

For any $A \in \mathscr{B}[0,1]$,

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X_\varepsilon \in A\} \le - \inf_{x\in \bar{A}} f(x)$$
$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X_\varepsilon \in A\} \ge - \inf_{x\in A^\circ} f(x).$$

Let $\mathfrak{L}$ be Lebesgue measure on $([0,1], \mathscr{B}[0,1])$. The upper bound is simple;

$$\mathbb{P}\{X_\varepsilon \in A\} \le c_\varepsilon \mathfrak{L}(A) \exp\left[-\frac{1}{\varepsilon} \inf_{x\in A} f(x)\right] \le c_\varepsilon \exp\left[-\frac{1}{\varepsilon} \inf_{x\in \bar{A}} f(x)\right].$$

To get the lower bound, fix $x^* \in A^\circ$ and $\delta > 0$. Define $\mathcal{O} \stackrel{\text{def}}{=} \{x \in [0,1] : f(x) > f(x^*) - \delta\} \cap A^\circ$. Since f is continuous, $\mathcal{O}$ is open (in the topology $[0,1]$ inherits from $\mathbb{R}$) and thus $\mathfrak{L}(\mathcal{O}) > 0$. Hence

$$\mathbb{P}\{X_\varepsilon \in A\} \ge c_\varepsilon \mathfrak{L}(\mathcal{O}) \exp\left[-\frac{1}{\varepsilon}\{f(x^*) + \delta\}\right].$$

This gives the lower bound by taking $\varepsilon \to 0$ and then $\delta \to 0$, and they varying x^* over A°.

Our basic setup is as follows. We have a collection $\{X_\varepsilon\}_{\varepsilon\in(0,1)}$ of random variables which take values in some Polish space X. We will see that the following is the "correct" way to think about exponentially rare events.

Definition 0.3. We say that $\{X_\varepsilon\}_{\varepsilon\in(0,1)}$ has a large deviations principle with rate function $I : \mathsf{X} \to [0,\infty]$ if

(1) For each $s \ge 0$, $\Phi(s) \stackrel{\text{def}}{=} \{x \in \mathsf{X} : I(x) \le s\}$ is compact.
(2) For each closed subset F of X,

$$\varlimsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X_\varepsilon \in F\} \le - \inf_{x\in F} I(x)$$

(3) For each open subset G of X,

$$\varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{X_\varepsilon \in G\} \ge - \inf_{x\in G} I(x)$$

We will see why this is correct in a moment. It turns out, however, that there is an equivalent definition for a large deviations principle. We here let d be the metric on X.

Definition 0.4. We say that $\{X_\varepsilon\}_{\varepsilon\in(0,1)}$ has a large deviations principle with rate function $I : \mathsf{X} \to [0,\infty]$ if

(1) For each $s \ge 0$, $\Phi(s) \stackrel{\text{def}}{=} \{x \in \mathsf{X} : I(x) \le s\}$ is compact.

(2) For each $s \geq 0$ and each $\delta > 0$,

$$\varlimsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{d(X_\varepsilon, \Phi(s)) \geq \delta\} \leq -s$$

(3) For every $x^* \in \mathsf{X}$ and every $\delta > 0$,

$$\varlimsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{d(X_\varepsilon, x^*) < \delta\} \geq -I(x^*)$$

Let's now prove a large deviations result for Brownian motion. Fix a finite horizon $T > 0$ and define $C_0[0,T] = \{\varphi \in C[0,T] : \varphi(0) = 0\}$ for each $\varepsilon \in (0,1)$, define $X_\varepsilon(t) \stackrel{\text{def}}{=} \varepsilon W_t$ for all $t \in [0,T]$; then X_ε is a $C_0[0,T]$-valued random variable. For each $\varphi \in C_0[0,T]$, define

$$I(\varphi) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2}\int_0^T (\dot{\varphi}(s))^2\, ds & \text{if } \varphi \text{ is absolutely continuous and } \phi(0) = 0 \\ \infty\text{else}; \end{cases} \tag{78}$$

We claim that εW has a large deviations principle in $C_0[0,T]$ with rate functional given by (78).

Lemma 0.5. *For each $s \geq 0$, the level sets $\{\varphi \in C_0[0,T] : I(\varphi) \leq s\}$ of I are compact in $C_0[0,T]$.*

Proof. If $I(\varphi) \leq s$, then for any $0 \leq t_1 < t_2 \leq T$,

$$\begin{aligned}|\varphi(t_2) - \varphi(t_1)| = \left|\int_{s=t_1}^{t_2} \dot{\varphi}(s) ds\right| = t_2 - t_1 \left|\frac{1}{t_2 - t_1}\int_{s=t_1}^{t_2} \dot{\varphi}(s) ds\right| \\ \leq t_2 - t_1 \sqrt{\frac{1}{t_2 - t_1}\int_{s=t_1}^{t_2} |\dot{\varphi}(s)|\, ds} \leq \sqrt{t_2 - t_1}\sqrt{2s}.\end{aligned}$$

Thus $\overline{\Phi(s)}$ is compact. We thus need to show that $\Phi(s)$ is closed. This follows from the fact that

$$I(\varphi) = \sup\left\{\frac{1}{2}\sum_{j=0}^{n} \frac{|\varphi(t_{j+1}) - \varphi(t_j)|^2}{t_{j+1} - t_j} \mid 0 = t_0 < t_1 \cdots < t_n = 1\right\}. \tag{79}$$

□

Let's next prove the lower bound. We use a measure change.

Lemma 0.6. *For any $\varphi \in C_0[0,1]$ and $\delta > 0$,*

$$\varliminf_{\varepsilon \searrow 0} \varepsilon^2 \ln \mathbb{P}\left\{\|X^\varepsilon - \varphi\|_C < \delta\right\} \geq -I(\varphi)$$

Proof. The result is obvious if $I(\varphi) = \infty$. Assume that $I(\varphi) < \infty$; thus $\dot{\varphi}$ is well-defined, and we set

$$\tilde{W}^\varepsilon = W_t - \frac{1}{\varepsilon}\varphi(t) = W_t - \frac{1}{\varepsilon}\int_{s=0}^{t} \dot{\varphi}(s) ds;$$

then $X^\varepsilon = \varepsilon \tilde{W}$. Define

$$\tilde{\mathbb{P}}^\varepsilon(A) \stackrel{\text{def}}{=} \mathbb{E}\left[\chi_A \exp\left[\frac{1}{\varepsilon}\int_{s=0}^{T} \dot{\varphi}(s) ds - \frac{1}{2\varepsilon^2}\int_{s=0}^{T} (\dot{\varphi}(s))^2 ds\right]\right] \qquad A \in \mathscr{F}$$

By Girsanov's theorem, $\tilde{W}^\varepsilon$ is a Brownian motion under $\tilde{\mathbb{P}}^\varepsilon$. We thus have that

$$\begin{aligned}\mathbb{P}\left\{\|X^\varepsilon - \varphi\|_C < \delta\right\} = \tilde{\mathbb{E}}^\varepsilon\left[\chi_{\|X^\varepsilon\|_C < \delta} \exp\left[-\frac{1}{\varepsilon}\int_{s=0}^{1} \dot{\varphi}(s) dW_s - \frac{1}{\varepsilon^2} I(\varphi)\right]\right] \\ \geq \tilde{\mathbb{E}}^\varepsilon\left[\chi_{\|X^\varepsilon\|_C < \delta} \exp\left[-\frac{1}{\varepsilon}\left|\int_{s=0}^{T} \dot{\varphi}(s) dW_s\right|\right]\right] \exp\left[-\frac{1}{\varepsilon^2} I(\varphi)\right]\end{aligned}$$

Girsanov's theorem tells us that

$$\lim_{\varepsilon \to 0} \tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\} = 1,$$

so for $\varepsilon > 0$ sufficiently small, $\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}$ is positive. Thus we can use Jensen's inequality to calculate that

$$\begin{aligned}
&\tilde{\mathbb{E}}^\varepsilon\left[\chi_{\{\|X^\varepsilon\|_C<\delta\}}\exp\left[-\frac{1}{\varepsilon}\left|\int_{s=0}^1 \dot{\varphi}(s)dW_s\right|\right]\right] \\
&\qquad = \tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}\frac{\tilde{\mathbb{E}}^\varepsilon\left[\chi_{\{\|X^\varepsilon\|_C<\delta\}}\exp\left[-\frac{1}{\varepsilon}\left|\int_{s=0}^1 \dot{\varphi}(s)dW_s\right|\right]\right]}{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}} \\
&\qquad\qquad \geq \tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}\exp\left[-\frac{1}{\varepsilon}\frac{\tilde{\mathbb{E}}^\varepsilon\left[\chi_{\{\|X^\varepsilon\|_C<\delta\}}\left|\int_{s=0}^T \dot{\varphi}(s)dW_s\right|\right]}{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}}\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{\tilde{\mathbb{E}}^\varepsilon\chi_{\{\|X^\varepsilon\|_C<\delta\}}\left|\int_{s=0}^T \dot{\varphi}(s)dW_s\right|}{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}} &\leq \frac{\sqrt{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}}}{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}}\tilde{\mathbb{E}}^\varepsilon\left[\left|\int_{s=0}^1 \dot{\varphi}(s)dW_s\right|^2\right] \\
&= \frac{1}{\sqrt{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}}}\sqrt{2I(\varphi)}.
\end{aligned}$$

Combining things together, we get that

$$\mathbb{P}\left\{\|X^\varepsilon - \varphi\|_C < \delta\right\} \geq \tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}\exp\left[\frac{1}{\varepsilon}\sqrt{\frac{2I(\varphi)}{\tilde{\mathbb{P}}^\varepsilon\{\|X^\varepsilon\|_C < \delta\}}} - \frac{1}{\varepsilon^2}I(\varphi)\right]$$

This gives us the stated claim. □

To prove the upper bound, let's first approximate. For each $n \in \mathbb{N}$, define

$$\begin{aligned}
\tilde{X}^n_\varepsilon(t) &\overset{\text{def}}{=} (tn - \lfloor tn \rfloor)\, X_\varepsilon\left(\frac{\lfloor tn \rfloor + 1}{n}\right) + (1 - tn + \lfloor tn \rfloor)\, X_\varepsilon\left(\frac{\lfloor tn \rfloor}{n}\right) \\
&= (tn - \lfloor tn \rfloor)\, \varepsilon W_{(\lfloor tn \rfloor+1)/n} + (1 - tn + \lfloor tn \rfloor)\, \varepsilon W_{\lfloor tn \rfloor/n};
\end{aligned}$$

i.e., $\tilde{X}^n_\varepsilon$ is the linear interplant of X^ε_n with vertices a $n\mathbb{N}$. Let's next recall a result from the problem in Chapter 6;

$$\mathbb{P}\left\{\sup_{0\leq t\leq 1} W_t \geq L\right\} = 2\int_{r=L}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]dz.$$

Thus

$$\begin{aligned}
\mathbb{P}\left\{\sup_{0\leq t\leq 1}|W_t| \geq L\right\} &\leq \mathbb{P}\left\{\sup_{0\leq t\leq 1} W_t \geq L\right\} + \mathbb{P}\left\{\inf_{0\leq t\leq 1} W_t \leq -L\right\} = 4\int_{r=L}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]dz \\
&= 4\int_{r=0}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(L+z)^2}{2}\right]dz = 4e^{-L^2/2}\int_{r=0}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-Lz - \frac{z^2}{2}\right]dz \\
&\leq 4e^{-L^2/2}\int_{r=0}^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]dz = 2e^{-L^2/2}.
\end{aligned}$$

Fix now $n \in \mathbb{N}$, $\varepsilon > 0$, and $t \in [0,T]$. Then

$$\tilde{X}^{n,\varepsilon}(t) - X_\varepsilon(t) = (tn - \lfloor tn \rfloor)\left\{X_\varepsilon\left(\frac{\lfloor tn \rfloor + 1}{n}\right) - X_\varepsilon\left(\frac{\lfloor tn \rfloor}{n}\right)\right\} - \left\{X_\varepsilon(t) - X_\varepsilon\left(\frac{\lfloor tn \rfloor}{n}\right)\right\}$$

Thus

$$\begin{aligned}
|\tilde{X}^{n,\varepsilon}_t - X^\varepsilon_t| &\leq \varepsilon\left|W_{(\lfloor tn \rfloor+1)/n} - W_{\lfloor tn \rfloor/n}\right| + \varepsilon\left|W_t - W_{\lfloor tn \rfloor/n}\right| \\
&\leq 2\varepsilon \sup_{\lfloor tn \rfloor/n \leq t \leq (\lfloor tn \rfloor+1)/n}\left|W_t - W_{\lfloor tn \rfloor/n}\right|.
\end{aligned}$$

Thus

$$\mathbb{P}\left\{\|\tilde{X}^{n,\varepsilon}-X^\varepsilon\|_C\ge\delta\right\}\le\mathbb{P}\left\{\sup_{0\le k\le n-1}\sup_{\frac{k}{n}\le t\le\frac{k+1}{n}}\left|W_t-W_{\frac{k}{n}}\right|\ge\frac{\delta}{2\varepsilon}\right\}$$
$$\le\sum_{k=0}^{n-1}\mathbb{P}\left\{\sup_{\frac{k}{n}\le t\le\frac{k+1}{n}}\left|W_t-W_{\frac{k}{n}}\right|\ge\frac{\delta}{2\varepsilon}\right\}\le n\mathbb{P}\left\{\sup_{0\le t\le\frac{1}{n}}|W_t|\ge\frac{\delta}{2\varepsilon}\right\}$$
$$\le n\mathbb{P}\left\{\sup_{0\le t\le 1}|W_t|\ge\frac{\delta\sqrt{n}}{2\varepsilon}\right\}\le 2n\exp\left[-\frac{\delta^2 n}{8\varepsilon^2}\right].$$

This completes the proof. Thus for all $n\in\mathbb{N}$, $\varepsilon>0$, and $\delta>0$,

$$\mathbb{P}\left\{\|\tilde{X}^{n,\varepsilon}-X^\varepsilon\|_C\ge\delta\right\}\le\sqrt{\frac{2}{\pi}}n\exp\left[-\frac{\delta^2 n}{8\varepsilon^2}\right]$$

We can now prove the upper bound.

Lemma 0.7. *For $s\ge 0$ and $\delta>0$,*

$$\varlimsup_{\varepsilon\searrow 0}\varepsilon\ln\mathbb{P}\left\{\mathrm{dist}(X^\varepsilon,\Phi(s))\ge\delta\right\}\le -s.$$

Proof. We have that

$$\mathbb{P}\left\{\mathrm{dist}(X^\varepsilon,\Phi(s))\ge\delta\right\}\le\mathbb{P}\left\{\|X^\varepsilon-\tilde{X}^{n,\varepsilon}\|\ge\frac{\delta}{2}\right\}+\mathbb{P}\left\{\mathrm{dist}(\tilde{X}^{n,\varepsilon},\Phi(s))\ge\frac{\delta}{2}\right\}$$
$$\le 2n\exp\left[-\frac{\delta^2 n}{32\varepsilon^2}\right]+\mathbb{P}\left\{I\left(\tilde{X}^{n,\varepsilon}\right)\ge s\right\}.$$

We note that

$$I\left(\tilde{X}^{n,\varepsilon}\right)=\frac{n\varepsilon^2}{2}\sum_{k=1}^{n-1}\left|W_{(k+1)/n}-W_{k/n}\right|^2$$

We thus have that for $\alpha>0$

$$\mathbb{P}\left\{I\left(\tilde{X}^{n,\varepsilon}\right)\ge s\right\}\le\mathbb{P}\left\{\frac{1-\alpha}{\varepsilon^2}I\left(\tilde{X}^{n,\varepsilon}\right)\ge\frac{s(1-\alpha)}{\varepsilon^2}\right\}\le\exp\left[-\frac{s(1-\alpha)}{\varepsilon^2}\right]\mathbb{E}\left[\exp\left[\frac{(1-\alpha)}{\varepsilon^2}I\left(\tilde{X}^{n,\varepsilon}\right)\right]\right]$$
$$=\exp\left[-\frac{s(1-\alpha)}{\varepsilon^2}\right]\mathbb{E}\left[\prod_{k=0}^{n-1}\exp\left[\frac{n(1-\alpha)}{2}\left(W_{(k+1)/n}-W_{k/n}\right)^2\right]\right]$$
$$=\exp\left[-\frac{s(1-\alpha)}{\varepsilon^2}\right]\left(\sqrt{\frac{n}{2\pi}}\int_{z\in\mathbb{R}}\exp\left[-\frac{\alpha n z^2}{2}\right]dz\right)^n=\exp\left[-\frac{s(1-\alpha)}{\varepsilon^2}\right]\alpha^{-n/2}.$$

Collect things together. ☐

Theorem 0.8. *We have that $\{X_\varepsilon\}_{\varepsilon>0}$ has an LDP in $C_0[0,1]$ with rate functional I as in (78), which is equivalently written as*

$$I(\varphi)=\begin{cases}\frac{1}{2}\int_{s=0}^1(\dot\varphi(s))^2ds & \text{if } \dot\varphi\in L^2\\ \infty & \text{else.}\end{cases}\tag{80}$$

Proof. Collect the above calculations together ☐

Exercises

(1) Show that Definitions 0.3 and 0.4 are equivalent.
(2) We here study (79).

(a) Suppose that $0 = t_0 < t_1 \cdots < t_n = T$. Show that

$$\sum_{j=0}^{n} \frac{|\varphi(t_{j+1}) - \varphi(t_j)|^2}{t_{j+1} - t_j} \le \int_{t=0}^{1} (\dot{\varphi}(t))^2 dt.$$

(b) Fix an open subset $\mathcal{O}$ of $[0,1]$. Show that there is a $\{\varphi_n\}_{n\in\mathbb{N}} \subset C([0,1];[0,1])$ such that $\varphi_n \nearrow \chi_{\mathcal{O}}$. Hint: construct the φ_n using the distance function to $[0,1] \setminus \mathcal{O}$.

(c) Since Lebesgue measure on $([0,T], \mathscr{B}[0,T])$ is regular, show that for any $A \in \mathscr{B}[0,T]$ and any $\delta > 0$, there is a $\varphi \in C([0,1];[0,1])$ such that

$$\int_{t=0}^{T} |\chi_A(t) - \varphi(t)|^2 dt < \delta$$

(d) Assume that $\psi \in L^2[0,T]$. Show that for $\delta > 0$, there is a $\psi^* \in C[0,T]$ such that

$$\int_{t=0}^{T} |\psi(t) - \psi^*(t)|^2 \, dt < \delta$$

(e) Show that if $\varphi \in C^1[0,T]$, then

$$I(\varphi) \le \sup \left\{ \frac{1}{2} \sum_{j=0}^{n} \frac{|\varphi(t_{j+1}) - \varphi(t_j)|^2}{t_{j+1} - t_j} \mid 0 = t_0 \le t_1 \cdots \le t_n = T \right\}.$$

(f) Show that if $I(\varphi) < \infty$, then there is a $\varphi^* \in C^1[0,T]$ such that

$$\int_{t=0}^{T} |\dot{\varphi}(t) - \dot{\varphi}^*(t)|^2 \, dt < \delta.$$

(g) Show that if $I(\varphi) < \infty$, then

$$I(\varphi) \le \sup \left\{ \frac{1}{2} \sum_{j=0}^{n} \frac{|\varphi(t_{j+1}) - \varphi(t_j)|^2}{t_{j+1} - t_j} \mid 0 = t_0 \le t_1 \cdots \le t_n = T \right\}.$$

(3) Show that since (79) holds, the level sets of I are closed in $C[0,T]$.

CHAPTER 17

Malliavin Calculus

Consider the SDE

$$dX_t = \sigma(X_t)dW_t$$
$$X_0 = x_\circ$$

Assume that $\sigma \in C_b^\infty(\mathbb{R})$ (i.e., σ and all of its derivatives are uniformly bounded) and that there is a $\sigma_\circ > 0$ such that $\sigma(x) \geq \sigma_\circ$ for all $x \in \mathbb{R}$. We want to show that for each $t > 0$, X has a *density*; i.e, that there is a measurable map $p_t : \mathbb{R} \to [0, \infty)$ such that

$$\mathbb{P}\{X_t \in A\} = \int_{x \in A} p_t(x)dx. \qquad A \in \mathscr{B}(\mathbb{R})$$

This is highly *nontrivial.* Let's first of all cast our interest in a different way.

PROPOSITION 0.9. *Assume that there is a $K > 0$ such that*

$$|\mathbb{E}[f''(X_t)]| \leq K \sup_{x \in \mathbb{R}} |f(x)| \tag{81}$$

for every $f \in C_b^1(\mathbb{R})$. Then p_t exists.

PROOF. Define

$$\varphi(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[f_\theta(X_t)\right] \qquad \theta \in \mathbb{R}$$

where

$$f_\theta(x) \stackrel{\text{def}}{=} \exp\left[\sqrt{-1}\theta x\right] \qquad x \in \mathbb{R}$$

for all $\theta \in \mathbb{R}$ (i.e., φ is the characteristic function of X_t). if (81) holds, we have that

$$\theta^2 |\varphi(\theta)| = \mathbb{E}\left[\left|f_\theta^{\prime,\prime}(X_t)\right|\right] \leq K \sup_{x \in \mathbb{R}} |f_\theta(x)| = K.$$

Thus

$$|\varphi(\theta)| \leq \frac{K}{1+\theta^2}, \qquad \theta \in \mathbb{R}$$

so φ is integrable and

$$p_t(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\theta \in \mathbb{R}} e^{-\sqrt{-1}\theta x} \varphi(\theta) d\theta$$

is well-defined. For any bounded, integrable, and continuous function g, we then have that

$$\begin{aligned}\mathbb{E}[g(X)] &= \lim_{\varepsilon \to 0} \int_{x \in \mathbb{R}} g(x) \left\{ \frac{1}{2\pi} \int_{\theta \in \mathbb{R}} \exp\left[-\frac{\varepsilon}{2}\|\theta\|^2 - \sqrt{-1}\theta x\right] \varphi(\theta) d\theta \right\} dx \\ &= \int_{x \in \mathbb{R}} g(x) p_t(x) dx\end{aligned}$$

The result follows. □

Let's see how to get this sort of inequality. Fix a bounded predictable function ξ and define

$$dX_t^\varepsilon = \sigma(X_t^\varepsilon)\{dW_t - \varepsilon \xi_t dt\}$$
$$X_0^\varepsilon = x_\circ$$

In other words,

$$X_t^\varepsilon = x_\circ + \int_{s=0}^t \sigma(X_s^\varepsilon)dW_s - \varepsilon\int_{s=0}^t \sigma(X_s^\varepsilon)\xi_s ds. \tag{82}$$

Let's also define

$$G(\varepsilon) \stackrel{\text{def}}{=} \exp\left[\varepsilon\int_{s=0}^t \xi_s dW_s - \frac{\varepsilon^2}{2}\int_{s=0}^t \xi_s^2 ds\right].$$

By Girsanov's theorem, we then have that

$$\mathbb{E}\left[f(X_t^\varepsilon)G(\varepsilon)\right] = \mathbb{E}\left[f(X_t)\right]. \tag{83}$$

Let's differentiate this. Differentiating (82) at $\varepsilon = 0$, we get

$$\Xi_t = \int_{s=0}^t \sigma'(X_s)\Xi_s dW_s - \int_{s=0}^t \sigma(X_s)\xi_s ds \tag{84}$$

We will explicitly solve this in a moment. Returning to (83), we have that

$$\mathbb{E}[f'(X_t)\Xi_t] = \mathbb{E}\left[-f(X_t)\left\{\int_{s=0}^t \xi_s dW_s\right\}\right].$$

This looks promising if we can do something like bound Ξ from below (this is not really right, but it points our thoughts in useful directions).

Let's now solve for Ξ. Define

$$M_t \stackrel{\text{def}}{=} \exp\left[\int_{s=0}^t \sigma'(X_s)dW_s - \frac{1}{2}\int_{s=0}^t (\sigma'(X_s))^2 ds\right].$$

Then

$$\Xi_t = \int_{s=0}^t M_t M_s^{-1}\sigma(X_s)\xi_s ds. \tag{85}$$

Note that M satisfies

$$\begin{aligned} dM_t &= \sigma'(X_t)M_t dW_t \\ M_0 &= 1. \end{aligned}$$

We haven't yet chosen ξ. We want to choose ξ to get something like a lower bound on Ξ. This makes sense if we choose

$$\xi_s = M_s^{-1}\sigma(X_s).$$

Then

$$\Xi_t = M_t\int_{s=0}^t \left(M_s^{-2}\sigma^2(X_s)\right)^2 ds.$$

Let's define

$$N_t \stackrel{\text{def}}{=} M_t^{-1} = \exp\left[-\int_{s=0}^t \sigma'(X_s)dW_s + \frac{1}{2}\int_{s=0}^t (\sigma'(X_s))^2 ds\right];$$

then X and N satisfy the joint SDE

$$\begin{aligned} dX_t &= \sigma(X_t)dW_t \\ X_0 &= x_\circ \\ dN_t &= -\sigma'(X_t)N_t dW_t + (\sigma'(X_t))^2 N_t dt \\ N_0 &= 1 \end{aligned}$$

Let's now put ε's in a number of places. Consider the joint SDE

$$\begin{aligned} dX_t^\varepsilon &= \sigma(X_t^\varepsilon)\left\{dW_t - \varepsilon N_t^\varepsilon \sigma(X_t^\varepsilon)dt\right\} \\ X_0^\varepsilon &= x_\circ \\ dN_t^\varepsilon &= -\sigma'(X_t^\varepsilon)N_t^\varepsilon dW_t \\ N_0^\varepsilon &= 1 \end{aligned}$$

For $\varepsilon > 0$, define

$$W_t^\varepsilon \stackrel{\text{def}}{=} W_t - \varepsilon \int_{s=0}^t N_s^\varepsilon \sigma(X_s^\varepsilon) ds \qquad 0 \le t \le T$$

If F is any sufficiently nice function of the Brownian motion W, define

$$DF(W) \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \varepsilon^{-1} \left\{ F(W^\varepsilon) - F(W) \right\}.$$

We can but won't be precise about this. In any case, we have that

$$DX_t = \Xi_t = N_t^{-1} \int_{s=0}^t (N_s \sigma(X_s))^2 ds.$$

Thus if f is differentiable,

$$D(f \circ X_t) = f'(X_t)\Xi_t.$$

Also note that D satisfies Liebniz's rule.

Let's also define

$$\delta_t \stackrel{\text{def}}{=} \int_{s=0}^t N_s \sigma(X_s) dW_s.$$

For any sufficiently nice random variable F, we then have that

$$\mathbb{E}[DF] = -\mathbb{E}\left[F\delta_t\right].$$

Thus

$$\begin{aligned}
\mathbb{E}[f'(X_t)F] &= \mathbb{E}\left[D(f \circ X_t)\frac{F}{\Xi_t}\right] \\
&= \mathbb{E}\left[D\left((f \circ X_t)\frac{F}{\Xi_t}\right)\right] - \mathbb{E}\left[f(X_t)D\frac{F}{\Xi_t}\right] \\
&= -\mathbb{E}\left[f(X_t)\frac{F\delta_t}{\Xi_t}\right] - \mathbb{E}\left[f(X_t)D\frac{F}{\Xi_t}\right] \\
&= -\mathbb{E}\left[f(X_t)D^*F\right]
\end{aligned}$$

where

$$D^*F \stackrel{\text{def}}{=} D\frac{F}{\Xi_t} + \frac{F\delta_t}{\Xi_t}.$$

Thus, defining $\mathbf{1}$ to be the identically one function, we have that

$$\mathbb{E}[f''(X_t)] = -\mathbb{E}\left[f'(X_t)D^*\mathbf{1}\right] = \mathbb{E}\left[f(X_t)D^*D^*\mathbf{1}\right]$$

We have (81) if

$$\mathbb{E}\left[|D^*D^*\mathbf{1}|\right] < \infty. \tag{86}$$

In fact, it is fairly easy to see that

$$\mathbb{E}\left[\left|\frac{1}{\Xi_t}\right|^p\right] < \infty$$

for every $p > 0$, and this gives (86) and thus (81).

Exercises

(1) Show that (85) indeed satisfies (84).

CHAPTER 18

Stochastic Averaging

Consider the 2-dimensional SDE

$$
\begin{aligned}
d\theta_t^\varepsilon &= \frac{1}{\varepsilon}dt \\
dZ_t^\varepsilon &= \sigma(\theta_t^\varepsilon)dW_t \\
\theta_0^\varepsilon &= 0 \\
Z_0^\varepsilon &= 0.
\end{aligned}
$$

We require here that σ be 1-periodic and smooth. We want to think of this as a diffusion on a cylinder, with θ^ε being the angular variable and Z^ε being the axial variable. We want to understand the behavior of this SDE as $\varepsilon \searrow 0$ This is a 2-dimensional Markov process, where θ^ε is the fast variable and Z^ε is the slow variable. We want to show that the slow variable Z^ε has a Markovian limit.

We have arranged our problem to be as simple as possible; we have that

$$Z_t^\varepsilon = \int_{s=0}^t \sigma(s/\varepsilon)dW_s.$$

We want to use machinery which is fairly general.

First, we claim that the laws of the Z^ε's are *tight*. Define $\bar{\sigma} \overset{\text{def}}{=} \|\sigma\|_C$. By making a time change, we have that

$$Z_t^\varepsilon = V^\varepsilon_{\int_{r=0}^t \sigma^2(r/\varepsilon)dr}$$

for some ε-dependent Brownian motion V^ε. Thus for any $\delta > 0$ and $T > 0$,

$$\lim_{\eta\searrow 0} \mathbb{P}\left\{ \sup_{\substack{0\le s\le t\le T \\ |t-s|\le\eta}} |Z_t^\varepsilon - Z_s^\varepsilon| \ge \eta \right\} \le \mathbb{P}\left\{ \sup_{\substack{0\le s'\le t'\le T\bar{\sigma} \\ |t-s|\le\bar{\sigma}\eta}} |V_t^\varepsilon - Z_s^\varepsilon| \ge \eta \right\} = 0.$$

Thus the laws of the Z^ε's have at least one convergent subsequence.

Let's next identify the limit. We claim that $\lim_{\varepsilon\searrow 0} Z^\varepsilon = Z$, where $Z_t = \bar{\sigma}W_t$ with

$$\bar{\sigma} \overset{\text{def}}{=} \left\{ \int_{\theta=0}^1 \sigma^2(\theta)d\theta \right\}^{1/2}.$$

Namely, we want to show that for any $f \in C_b^2(\mathbb{R})$ and any $0 \le s_0 \le s_1 \le s_n \le s \le t$ and any $\{\phi_n\}_{n=1}^N \subset C(\mathbb{R})$,

$$\lim_{\varepsilon\searrow 0} \mathbb{E}\left[\left\{ f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_{r=s}^t \frac{\bar{\sigma}^2}{2}\ddot{f}(Z_r^\varepsilon)dr \right\} \prod_{n=1}^N \phi_n(Z_{s_n}^\varepsilon) \right] = 0.$$

Ito's formula tells us that

$$\mathbb{E}\left[\left\{ f(Z_t^\varepsilon) - f(Z_s^\varepsilon) - \int_{r=s}^t \frac{\sigma^2(r/\varepsilon)}{2}\ddot{f}(Z_r^\varepsilon)dr \right\} \prod_{n=1}^N \phi_n(Z_{s_n}^\varepsilon) \right] = 0.$$

Thus we really want to show that

$$\lim_{\varepsilon\searrow 0} \mathbb{E}\left[\int_{r=s}^t \left\{ \sigma^2(r/\varepsilon) - \bar{\sigma}^2 \right\} \ddot{f}(Z_r^\varepsilon)dr \prod_{n=1}^N \phi_n(Z_{s_n}^\varepsilon) \right] = 0. \tag{87}$$

To show (87), define

$$\Phi(t) \overset{\text{def}}{=} \int_{r=0}^{t} \left\{\sigma^2(r) - \bar{\sigma}^2\right\} dr.$$

We note that Φ is bounded and differentiable. We have that

$$\begin{aligned}\varepsilon\Phi(t/\varepsilon) f''(Z_t^\varepsilon) &= \int_{r=0}^{t} \left\{\sigma(r/\varepsilon) - \bar{\sigma}^2\right\} f''(Z_r^\varepsilon) dr \\ &\quad + \frac{\varepsilon}{2}\int_{r=0}^{t} \Phi(r/\varepsilon)\sigma^2(r/\varepsilon) f^{(4)}(Z_r^\varepsilon) dr + \varepsilon \int_{r=0}^{t} \Phi(r/\varepsilon)\sigma(r/\varepsilon) f^{(3)}(Z_r^\varepsilon) dW_r.\end{aligned}$$

In other words,

$$\begin{aligned}&\mathbb{E}\left[\int_{r=s}^{t} \left\{\sigma^2(r/\varepsilon) - \bar{\sigma}^2\right\} \ddot{f}(Z_r^\varepsilon) dr \prod_{n=1}^{N} \phi_n(Z_{s_n}^\varepsilon)\right] \\ &\qquad = \varepsilon\mathbb{E}\left[\left\{\Phi(t/\varepsilon) f''(Z_t^\varepsilon) - \Phi(s/\varepsilon) f''(Z_s^\varepsilon)\right\} \prod_{n=1}^{N} \phi_n(Z_{s_n}^\varepsilon)\right] \\ &\qquad\quad - \frac{\varepsilon}{2}\mathbb{E}\left[\int_{r=s}^{t} \Phi(r/\varepsilon)\sigma^2(r/\varepsilon) f^{(4)}(Z_r^\varepsilon) dr \prod_{n=1}^{N} \phi_n(Z_{s_n}^\varepsilon)\right].\end{aligned}$$

This gives us the claim.