1

INTRODUCTION

1.1 BASIC TERMINOLOGY

In a course in real analysis, the normal procedure is to begin with a definition of the real numbers, either by means of a set of axioms or by a constructive procedure which starts with the "God--given" set of positive integers. The set of all integers is constructed, and from this the rational numbers are obtained, and finally the reals. A discussion of this type is part of the area of logic and foundations rather than real analysis, and we will postpone it until much later. For now we'll take the point of view that we know what the real numbers are: A real number is an integer plus an infinite decimal, for example, 65.7204 . . .

If the decimal ends in all nines, we have two representations of the same real number, for example,

\[ 5.237999\ldots = 5.238000\ldots \]

We will often talk about sets of real numbers, and therefore a modest amount of set-theoretic terminology is necessary before we can get anywhere. You have probably seen most of this in another course, so we will proceed rather quickly.
1.1.1 Definitions and Comments

The union of two sets $A$ and $B$, denoted by $A \cup B$, is the set of points belonging to either $A$ or $B$ (or both; from now on, the word "or" always has the so-called inclusive connotation "or both" unless otherwise specified).

The intersection of two sets $A$ and $B$, denoted by $A \cap B$, is the set of points belonging to both $A$ and $B$.

The complement of a set $A$, denoted by $A^c$, is the set of points not belonging to $A$. (Generally, we will be working in a fixed space $\Omega$ (sometimes called the universe), for example, the set of real numbers or perhaps the set of pairs of real numbers, that is, the Euclidean plane. All sets under discussion will consist of various points of $\Omega$, and thus $A^c$ is the set of points of $\Omega$ that do not belong to $A$).

Unions, intersections, and complements may be represented by pictures called Venn diagrams that are probably familiar to many readers; see Fig. 1.1.1.

Unions and intersections may be defined for more than two sets, in fact for an arbitrary collection of sets.

The union of sets $A_1, A_2, \ldots$, denoted by $A_1 \cup A_2 \cup \ldots$ or by $\bigcup_i A_i$, is the set of points belonging to at least one of the $A_i$; the intersection of $A_1, A_2, \ldots$, denoted by $A_1 \cap A_2 \cap \ldots$ or by $\bigcap_i A_i$, is the set of points belonging to all the $A_i$. The union of sets $A_1, \ldots, A_n$ is often written as $\bigcup_{i=1}^n A_i$, and the union of an infinite sequence $A_1, A_2, \ldots$ is denoted by $\bigcup_{i=1}^{\infty} A_i$, with similar notation for intersection.

There are a few identities involving unions, intersections, and complements that come up occasionally. For example, the distributive law holds: for arbitrary sets $A, B, C$,

$$(1) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(the word "distributive" is used because in this formula, at least, intersection behaves like ordinary multiplication and union like addition).
Figure 1.1.1  Union, Intersection, and Complement
The formula may be verified by drawing a Venn diagram (Fig. 1.1.2) or by showing that the sets on the left and right sides of the equality have the same members, as follows. (The symbol $\in$ means "belongs to"). If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$, so that $x \in B$ or $x \in C$.

**Case 1.** $x \in A$ and $x \in B$; then $x \in A \cap B$; hence $x \in (A \cap B) \cup (A \cap C)$.

**Case 2.** $x \in A$ and $x \in C$; then $x \in A \cap C$; hence $x \in (A \cap B) \cup (A \cap C)$.

**Caution.** So far we have shown only that the set on the left is a subset of the set on the right. In general, the set $D$ is said to be a subset of the set $E$ if every point of $D$ also belongs to $E$ (see Fig. 1.1.3). We use the notation $D \subseteq E$; sometimes we say that $E$ contains $D$ or the $D$ is contained in $E$. In order to show that $D = E$ we must also show that every point of $E$ belongs to $D$. If $D \subseteq E$ but $D \neq E$, we say that $D$ is a proper subset of $E$, sometimes written $D \subset E$. Note that according to the definition, a set is a subset of itself.

To return to the proof of the distributive law, suppose $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$; thus, we know definitely that $x \in A$, and also $x \in B$ or $x \in C$, as desired.

![Venn Diagram]

_A $\cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Figure 1.1.2** Distributive Law
The following identities, called the De Morgan laws, are often useful:

\[(A \cap B)^c = A^c \cup B^c,\]
\[(A \cup B)^c = A^c \cap B^c.\]

As above, the identities can be verified by a Venn diagram or a formal argument using the definitions of union, intersection, and complement. In fact the De Morgan laws can be extended to an arbitrary collection of sets, as follows:

\[(\bigcap_{i} A_i)^c = \bigcup_{i} A_i^c,\]
\[(\bigcup_{i} A_i)^c = \bigcap_{i} A_i^c.\]

The Venn diagram approach is not useful with such a large collection of sets, and the formal method must be used (Problem 1).

To complete the necessary set-theoretic terminology, the empty set (set with no members) will be denoted by \(\emptyset\). If \(A\) is any set, then \(\emptyset \subseteq A\). (If you produce an element \(x \in \emptyset\), I will be delighted to show that \(x\) belongs to \(A\).) We will look at this idea in more detail in Section 1.4. The sets \(A_i\) are said to be disjoint or mutually exclusive if there is no overlap between any pair of sets; that is,

\[A_i \cap A_j = \emptyset \quad \text{whenever} \quad i \neq j.\]
As an example, let $A = \{1, 2, 3\}, B = \{1, 2\}, C = \{3, 5\}$ (the notation is standard: a set is described by listing its members, so that, for example, $A$ is the set consisting of the numbers 1, 2, and 3). In this case, $A$, $B$, $C$ are not disjoint, although $A \cap B \cap C = \emptyset$. For disjointness we must have $A \cap B = A \cap C = B \cap C = \emptyset$, and here we have $A \cap B = \{1, 2\} \neq \emptyset$, $A \cap C = \{3\} \neq \emptyset$. Since $B \cap C = \emptyset$, we may say that $B$ and $C$ are disjoint.

The *set-theoretic difference* between $A$ and $B$ is defined by

$$A \setminus B = \{x \in \Omega: x \in A, x \notin B\} = A \cap B^c.$$ 

**Problems for Section 1.1**

1. Prove the De Morgan laws for an arbitrary collection of sets (Eq. (3)).

2. Prove that union distributes over intersection; i.e.,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

3. Show that the union of three arbitrary sets can be written as a disjoint union in the following way:

$$A \cup B \cup C = A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C).$$

4. Continuing Problem 3, if $A_1, A_2, \ldots$ are arbitrary sets, show that

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup (A_1^c \cap A_2) \cup \cdots \cup (A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n) \cup \cdots$$

5. If $A$, $B$, $C$, $D$ are arbitrary sets, express the set of elements belonging to at least two of the sets $A$, $B$, $C$, $D$, using unions and intersections of $A$, $B$, $C$, $D$.

6. Repeat Problem 5 with at least two replaced by exactly two and use complements as well as unions and intersections.
1.2 FINITE AND INFINITE SETS; COUNTABLY INFINITE AND UNCOUNTABLY INFINITE SETS

Sometimes we need to know something about the size of a set, in particular, whether it is finite. If a set is infinite, it may be useful to know if it can somehow be counted or if it is uncountable. The definitions are as follows.

A **finite set** is one that can be put in one-to-one correspondence with \(\{1, 2, \ldots, n\}\) for some positive integer \(n\) (by convention, the empty set is regarded as finite). An **infinite set** is a set that is not finite. A **countably infinite set** is one that can be put in one-to-one correspondence with the entire set of positive integers. This means simply that the points can be labeled 1, 2, 3, \ldots A set is **uncountably infinite** if it is infinite but not countably infinite. It is convenient to call a set **countable** if it is either finite or countably infinite; thus, **uncountable** is synonymous with uncountably infinite.

Possibly you have seen the classic arguments that the set of rational numbers is countably infinite, but the set of all real numbers is uncountably infinite. To count the positive rationals, we devise an explicit scheme (see Fig. 1.2.1). The procedure amounts to counting the points of an infinite rectangular array, and is slightly inefficient because each rational number appears infinitely often: for example,

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & \cdots & r_1 = 1 & r_6 = \frac{1}{4} \\
2 & 2 & 2 & 2 & 2 & \cdots & r_2 = \frac{1}{2} & r_7 = \frac{2}{3} \\
3 & 3 & 3 & 3 & 3 & \cdots & r_3 = 2 & r_8 = \frac{3}{2} \\
4 & 4 & 4 & 4 & 4 & \cdots & r_4 = 3 & r_9 = 4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & r_5 = \frac{1}{3} & r_{10} = 5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{array}
\]

*Figure 1.2.1* Counting the Rational Numbers
1/2 = 2/4 = 3/6, and so on. After the first appearance of a number (say $r_2 = 1/2$ in Fig. 1.2.1), all other appearances are skipped in making the count. To show that the reals are uncountable, we use the Cantor diagonal process. (This idea will occur several times later on.) Suppose we were able to count the real numbers between 0 and 1; list the numbers, in decimal form, as follows:

\begin{equation}
\begin{align*}
 r_1 &= .a_{11}a_{12}a_{13} \ldots , \\
 r_2 &= .a_{21}a_{22}a_{23} \ldots , \\
 r_3 &= .a_{31}a_{32}a_{33} \ldots .
\end{align*}
\end{equation}

Then form the real number $r = .b_1b_2b_3 \ldots$, where $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}, \ldots$. To avoid the ambiguity caused by expansions ending in all nines or all zeros, we can, if we like, take $1 \leq b_n \leq 8$ for all $n$. Then $r$ is a real number between 0 and 1, but cannot appear on the list.

There is an extensive theory of infinite sets, but we will be content with a few basic results.

1.2.1 THEOREM. There are $2^n$ subsets of $\{1, 2, \ldots, n\}$.

Proof. If $S \subseteq \{1, 2, \ldots, n\}$, then $1 \in S$ or $1 \notin S$ (two choices), $2 \in S$ or $2 \notin S$ (two choices), $\ldots$, $n \in S$ or $n \notin S$ (two choices). The total number of subsets is the same as the total number of choices, namely, $2 \times 2 \times \cdots \times 2 = 2^n$. ■

1.2.2 THEOREM. There are uncountably many subsets of the positive integers.

Proof. Make a correspondence between subsets of the positive integers and binary representations of real numbers between 0 and 1, as follows:

\[.10010110 \ldots \] means $1 \in S, 2 \notin S, 3 \notin S, 4 \in S, 5 \notin S, 6 \in S,$ \\
\[7 \in S, 8 \notin S, \quad \text{etc.}\]

Since there are uncountably many reals between 0 and 1, there are uncountably many subsets of $\{1, 2, \ldots\}$.1 ■

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1Note that the real number $0.01000\ldots = 0.001111\ldots$ actually yields two subsets, $\{2\}$ and $\{3, 4, 5, 6, 7, \ldots\}$. Since this increases the number of subsets relative to the reals, the conclusion of 1.2.2 is undisturbed.
1.2.3 THEOREM. A countable union of countable sets is countable. In other words, if for each \( n = 1, 2, \ldots \), \( A_n \) is countable, then \( \bigcup_{n=1}^{\infty} A_n \) is countable.

Proof. List the members of the \( A_i \) as follows:

\[
\begin{align*}
A_1 & : \quad a_{11} \quad a_{12} \quad \ldots \\
A_2 & : \quad a_{21} \quad a_{22} \quad \ldots \\
A_3 & : \quad a_{31} \quad a_{32} \quad \ldots \\
& \vdots
\end{align*}
\]

Then count \( \bigcup_{n=1}^{\infty} A_n \) by the same procedure we used to count the rationals. □

Problems for Section 1.2

1. Give an alternative proof of Theorem 1.2.2, as follows. If \( S_1, S_2, \ldots \) is a list of subsets of \( \{1, 2, \ldots \} \), construct a subset \( S \) that cannot possibly be on the list.

2. Show that there are only countably many finite subsets of the positive integers.

3. Verify informally that the mapping

\[
(x, y) \rightarrow \frac{1}{2}[(x + y)(x + y + 1) + 2x)]
\]

is a one-to-one correspondence between ordered pairs \((x, y)\) of nonnegative integers and nonnegative integers (see diagram).

\[
\begin{array}{cccc}
(0, 0) & (1, 0) & (2, 0) & (3, 0) \\
(0, 1) & (1, 1) & (2, 1) & \cdots \\
(0, 2) & (1, 2) & \cdots \\
(0, 3) & & &
\end{array}
\]

4. The method of Fig. 1.2.1 shows that the positive rationals are countably infinite. How would you modify the procedure so as to count all the rationals?
5. Suppose that the rational numbers between 0 and 1 are listed as in (4). We then pick a rational \( r = .r_1r_2r_3 \ldots \) with \( r_n \neq a_{nn}, n = 1, 2, \ldots \). Why doesn’t this show that the rationals are uncountable?

6. Show that it is impossible to list the rational numbers in increasing order.

7. Show that for any positive integer \( n \) the set of all \((x_1, \ldots, x_n)\), where the \( x_i \) are rational, is countable.

1.3 DISTANCE AND CONVERGENCE

One of the basic ideas of analysis is that of convergence; a sequence of numbers \( x_n \) converges to a number \( x \) if, as \( n \) gets very large, \( x_n \) gets very close to \( x \); in other words, the distance between \( x_n \) and \( x \) gets very small. The key concept is that of distance; as long as we have a distance function, we can talk about convergence. You are familiar with several distance functions. If \( x \) and \( y \) are points on the real line, the distance between them is \( |x - y| \); if \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) are points in the plane, the Euclidean distance between them is \( [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} \). What properties must a distance function satisfy in order that we can talk about convergence sensibly? It turns out that only a few are needed, as follows.

1.3.1 Definitions and Comments

A metric or distance function on a set \( \Omega \) is an assignment, to each pair of points \((x, y)\), \( x, y \in \Omega \), of a nonnegative real number \( d(x, y) \), such that for all \( x, y, z \in \Omega \) we have

\[
\begin{align*}
(a) & \quad d(x, x) = 0; \quad d(x, y) > 0 \quad \text{if} \quad x \neq y, \\
(b) & \quad d(x, y) = d(y, x), \\
(c) & \quad d(x, z) \leq d(x, y) + d(y, z).
\end{align*}
\]

Statement (c) is called the triangle inequality; if \( x, y, \) and \( z \) are vertices of a triangle in the plane, (c) says that the length of one side of the triangle cannot exceed the sum of the lengths of the other two sides.
A set \( \Omega \) on which a distance function is defined is called a metric space. Our basic metric spaces will be the set of real numbers, to be denoted from now on by \( R \), and Euclidean \( p \)-space \( R^p \), the set of all \( p \)-tuples \((x_1, \ldots, x_p)\) of real numbers. The metric on \( R^p \) is given by

\[
d[(x_1, \ldots, x_p), (y_1, \ldots, y_p)] = \left[ \sum_{j=1}^{p} (x_j - y_j)^2 \right]^{1/2}.
\]

When \( p = 1 \), we have \( R^p = R \), \( d(x, y) = |x - y| \). We know because of our familiarity with elementary geometry that \( d \) is a metric when \( p \leq 3 \), but this must be proved when \( p > 3 \). However, it's probably best to wait until we have more experience before doing this, so let's accept the result for now. [See Section 3.2, Problem 5 for a proof.]

On any metric space there is a natural notion of convergence: if \( x_1, x_2, \ldots \) is a sequence of points in \( \Omega \), we say that the sequence converges to \( x \) (notation \( x_n \to x \)) if \( d(x_n, x) \to 0 \) as \( n \to \infty \); in other words,

\[
\lim_{n \to \infty} d(x_n, x) = 0.
\]

The definition of limit might have been baffling when you first encountered it in calculus, but let's look at it again. The statement \( \lim_{n \to \infty} d(x_n, x) = 0 \) means that given \( \varepsilon > 0 \) there is a positive integer \( N \) such that if \( n \geq N \) then \( d(x_n, x) < \varepsilon \). For example, suppose \( \varepsilon = 10^{-6} \). Then if we go far enough out in the sequence (it might turn out, say, that we have to go beyond \( N = 10^8 \)), then all \( x_n \) from that point on (that is, \( n \geq 10^8 \)) are within distance \( 10^{-6} \) of the point \( x \). Thus after a certain point, all the \( x_n \)'s are very close to \( x \).

The distance function and the convergence concept allow us to study the structure of various sets in a metric space. For example, in \( R^2 \) the set of points whose distance from the origin is less than \( r \) is the interior of the circle of radius \( r \) and center at \((0, 0)\). In general, an open ball in a metric space \( \Omega \) is a set of the form

\[
B_r(x) = \{y \in \Omega : d(x, y) < r\} \quad \text{where} \quad x \in \Omega \quad \text{and} \quad r > 0.
\]

(Again, the set-theoretic notation is standard; \( B_r(x) \) is the set of points \( y \) in \( \Omega \) whose distance from \( x \) is less than \( r \). When it is clear what space
we are working in, we may omit " ∈ Ω." A closed ball is a set of the form
\[ C_r(x) = \{ y ∈ Ω : d(x, y) ≤ r \} . \]

Let \( E \) be a subset of the metric space \( Ω \); \( E \) is said to be an open set if for every \( x ∈ E \) there is an open ball \( B_r(x) \) that is entirely contained in \( E \); that is, \( B_r(x) ⊆ E \).

For example, in \( R^2 \), \( E = \{(x, y) : 0 < x < 1, y > 2\} \) is the interior of an infinite rectangular strip and is therefore open. Intuitively, an open set is one that does not contain any of its boundary points.

Now consider \( E_1 = \{(x, y) : 0 < x < 1, y = 0\} \); \( E_1 \) is not open because an open ball in \( R^2 \) with center at \((x, 0)\) cannot lie entirely within \( E_1 \) (Fig. 1.3.1). Note, however, that \( A = \{x ∈ R : 0 < x < 1\} \) is open, because in \( R \) an open ball is just an open interval: \( B_r(x) = \{y ∈ R : x - r < y < x + r\} \) (see Fig. 1.3.2). Thus, in talking about open sets, be sure you know which space you are working in. This applies also to closed sets, to be discussed next. Before going on, let's introduce the standard notation for open, closed, and semiclosed intervals of \( R \):

\[
\begin{align*}
(a, b) &= \{x ∈ R : a < x < b\}, \\
(a, b] &= \{x ∈ R : a < x ≤ b\}, \\
[a, b) &= \{x ∈ R : a ≤ x < b\}, \\
[a, b] &= \{x ∈ R : a ≤ x ≤ b\}, \\
(a, ∞) &= \{x ∈ R : x > a\},
\end{align*}
\]

Figure 1.3.1 Example of a Nonopen Subset of \( R^2 \)
1.3 DISTANCE AND CONVERGENCE

Figure 1.3.2 An Open Subset of \( \mathbb{R} \)

\[
(a, \infty) = \{ x \in \mathbb{R} : x \geq a \}, \\
(-\infty, b) = \{ x \in \mathbb{R} : x < b \}, \\
(-\infty, b] = \{ x \in \mathbb{R} : x \leq b \}.
\]

The adjective "closed" is appropriate for \([a, b]\) because in the following sense it is not possible to get out of the set. If \(x_1, x_2, \ldots\) is a sequence of points in \([a, b]\) and \(x_n \to x\), then \(x\) must also be in \([a, b]\). In fact, this is the general definition of a closed set in a metric space:

If \(E\) is a subset of the metric space \(\Omega\), \(E\) is said to be closed if whenever \(x_1, x_2, \ldots\) is a sequence of points in \(E\) converging to the point \(x \in \Omega\), we must have \(x \in E\).

Some familiar examples of closed sets in \(\mathbb{R}^2\) are the closed balls \(C_r(x, y) = \{ (x', y') : (x' - x)^2 + (y' - y)^2 \leq r^2 \}\) and the closed rectangular boxes \(\{ (x, y) : a \leq x \leq b, c \leq y \leq d \}\). In \(\mathbb{R}\), the interval \((0, 1]\) is not closed because the sequence of numbers \(1/n \) converges to 0, which is outside the interval. (There are sequences (such as \(x_n = 1/2 + 1/n, n = 2, 3, \ldots\) in \((0, 1]\) that converge to a limit in \((0, 1]\), but not every sequence in \((0, 1]\) converging to a limit has its limit in \((0, 1]\).) Note also that \((0, 1]\) is not open because an open interval centered at 1 cannot be a subset of \((0, 1]\). Intuitively, a closed set is one that contains every one of its boundary points; in this case, the boundary point 0 is outside the set.

The following relation between open and closed sets is basic.

1.3.2 THEOREM. A set \(E\) is open if and only if its complement \(E^c\) is closed.

Proof. Suppose \(E\) is open. If \(x_1, x_2, \ldots\) is a sequence of points in \(E^c\), and \(x_n \to x\), we must show \(x \in E^c\). Assume, on the contrary, that \(x \in E\). Since \(E\) is open, there is an open ball \(B_r(x)\) entirely contained in \(E\), and since \(x_n \to x\), we have \(x_n \in B_r(x)\) for all sufficiently large \(n\) (see Fig. 1.3.3(a)). But this contradicts the assumption that all \(x_n\) belong to \(E^c\).
Now assume \( E^c \) closed but \( E \) not open. To say that \( E \) is open means that every point \( x \in E \) has an open ball \( B_r(x) \subseteq E \), so \( E \) not open means that not every point in \( E \) has this property. If \( x \) is a "bad point" of \( E \), then no open ball \( B_r(x) \) is a subset of \( E \); thus, for every \( r > 0 \), \( B_r(x) \not\subseteq E \). (The symbol \( \not\subseteq \) means "is not a subset of"; thus, we can find a point \( y \in B_r(x) \) with \( y \notin E \) (see Fig. 1.3.3(b)).) Now if we take \( r = 1/n \) and select \( x_n \in B_r(x) \) with \( x_n \notin E \), we have \( d(x_n, x) < 1/n \rightarrow 0 \), so \( x_n \rightarrow x \). But \( x_n \in E^c \), a closed set, so \( x \in E^c \), contradicting the assumption that \( x \in E \).

**Problems for Section 1.3**

1. Determine whether the following sets are open in \( R \), closed in \( R \), or neither.
   (a) \( \{ x \in R : x \leq 0 \} \)
   (b) \( \{ x \in R : x < 0 \text{ or } x > 1 \} \)
   (c) \( \{ x \in R : x < 0 \text{ or } x \geq 1 \} \)
   (d) \( \{ x \in R : x^2 - 3x + 2 < 0 \} \)
   (e) \( [4, \infty) \)

2. Repeat Problem 1 for the following subsets of \( R^2 \).
   (a) \( \{ (x,y) : |x| + |y| < 1 \} \)
   (b) \( \{ (x,y) : 0 \leq y \leq \varepsilon^2 \} \)
   (c) \( \{ (x,y) : \max(x,y) = 3 \} \)
   (d) \( \{ (x,y) : 0 < y \leq \varepsilon^2 \} \)
3. You may have the impression that a set cannot be both open and closed, but this is not the case. Show that \( R \) is both open and closed in \( R \) (similarly, \( R^2 \) is both open and closed in \( R^2 \)).

4. Give an example of another set that is both open and closed in \( R \).

5. Let \( \Omega = \{x \in R : x \leq 0 \text{ or } x \geq 1\}, E = \{x \in R : x \geq 1\} \). Although \( E \) is not an open subset of \( R \), show that \( E \) is open (as well as closed) in \( \Omega \) (we will not worry about this situation until later, when we talk about connected sets).

6. Show that any finite set is closed.

7. Show that limits are unique; i.e., if \( x_n \to a \) and \( x_n \to b \) then \( a = b \).

8. Prove the “squeeze theorem”; i.e., if \( \{a_n\} \) and \( \{b_n\} \) are sequences of real numbers converging to the same limit \( L \), and \( a_n \leq x_n \leq b_n \) for all \( n \), then \( x_n \to L \) also.

### 1.4 MINICOURSE IN BASIC LOGIC

Mathematical reasoning sometimes depends on the logical structure of the particular statements and not on their mathematical content. The statements might come from analysis, algebra, geometry, or other areas, but the underlying reasoning process is still the same. In this section we look at this reasoning process.

#### 1.4.1 Truth Tables

One very common type of mathematical statement is the proposition, which has a definite truth value (true (T) or false (F)). New propositions can be created from old ones by using the connectives “and,” “or,” “not,” and “implies.” The truth or falsity of the new proposition can be determined by a (probably familiar) device called a truth table:

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<th>or</th>
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<tr>
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<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Thus,  
"$A$ and $B$" is T iff $A$ and $B$ are both T  
"$A$ or $B$" is T iff $A$ or $B$ (or both) are T  
"not $A$" is T iff $A$ is F

The "implies" connective is more complicated:\[2\]:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$A \rightarrow B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Thus, "$A$ implies $B$" is F only when $A$ is T and $B$ is F

It may seem strange to regard $A \rightarrow B$ as true ("by default") when $A$ is F, but here is a supporting argument. Consider the following statement about positive integers:

If $n$ is divisible by 4 then $n$ is divisible by 2. If you agree that this is a true statement, then the truth table for $A \rightarrow B$ is forced. Take $n = 5$ to get the FF case, and $n = 6$ to get the FT case.

Another view: the only way to disprove $A \rightarrow B$ is to produce a situation in which $A$ is true but $B$ is false.

We now have a formal justification that $\emptyset \subseteq A$ for any set $A$; i.e., if $x \in \emptyset$, then $x \in A$. The condition "$x \in \emptyset$" is always false, so "if $x \in \emptyset$ then $x \in A$" is vacuously true or true by default.

1.4.2 Types of Proof

Two very common ideas that occur in the construction of proofs are proof by contradiction and contrapositive.

\[2\] If there is any chance of confusing the assertion "$A$ implies $B$" with a statement about limits, we will use $A \Rightarrow B$ instead of $A \rightarrow B$.\]
Proof by contradiction. If we wish to prove $A$, we assume (on the contrary) that “not $A$” holds; i.e., $A$ is false. If we reach a contradiction, then the assumption “not $A$” is untenable, so $A$ must hold.

Example. See both parts of the proof of Theorem 1.3.2.

Contrapositive. To prove $A \rightarrow B$, we may instead prove the equivalent proposition not $B \rightarrow$ not $A$, the so-called “contrapositive” of $A \rightarrow B$. The two propositions are equivalent because they have the same truth table:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>not $B$</th>
<th>not $A$</th>
<th>not $B \rightarrow$ not $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
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<td>F</td>
<td>F</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Example. Suppose $A_1$, $A_2$, and $S$ are sets and $A_1 \subseteq A_2$. If $S$ is not a subset of $A_2$, then $S$ is not a subset of $A_1$. The contrapositive is easier to handle: it says that if $S$ is a subset of $A_1$, then $S$ is a subset of $A_2$.

1.4.3 Quantifiers

There are mathematical statements or formulas that do not have a definite truth value. For example, suppose we are working in $R$, and consider the statement

$$x + 3 > 10.$$  

This is true for certain values of $x$, namely, those numbers $> 7$, but false for all other $x$.

Similarly, in $R^2$, the statement $x^2 + y^2 < 1$ holds for certain values of $(x, y)$ and not for others. Thus the “property” $x^2 + y^2 < 1$ defines a subset of $R^2$, sometimes called a predicate, where the property holds.
One way to convert a statement containing variables (such as $x$ and $y$) into a sentence with a definite truth value is to use quantifiers, which are of two types:

the universal quantifier $\forall$ (read "for all"),
and the existential quantifier $\exists$ (read "there exists").

For example, if $x$ stands for a real number, then $\exists x (x + 3 > 10)$ is true because there is an $x$ such that $x + 3 > 10$ (any $x > 7$ will do). But $\forall x (x + 3 > 10)$ is false, since it is definitely not the case that for every $x$, $x + 3 > 10$.

Both types of quantifiers can appear in the same sentence. For example, $\forall x \exists y (x + y = 10)$ is true since for every $x$ there is a $y$ such that $x + y = 10$, namely, $y = 10 - x$. But $\exists x \forall y (x + y = 10)$ is false; it says that there is an $x$ such that for every $y$, $x + y = 10$. This certainly cannot hold for a fixed $x$ and all possible $y$.

1.4.4 Mathematical Induction

Let's look at another proof of Theorem 1.2.1: There are $2^n$ subsets of \{1,2, \ldots ,n\}. When $n = 1$, we are saying that there are two subsets of \{1\}, and this is correct; we have \{1\} itself and the empty set. Now if we assume that \{1,2, \ldots ,n\} has $2^n$ subsets, we can prove that \{1,2, \ldots ,n+1\} has $2^{n+1}$ subsets, as follows.

Let $A$ be any subset of \{1,2, \ldots ,n+1\}. There are two mutually exclusive possibilities:

Case 1. $n+1$ does not belong to $A$. Then $A$ is a subset of \{1,2, \ldots ,n\}, so there are $2^n$ such $A$'s.

Case 2. $n+1$ belongs to $A$. Then $A$ consists of $n+1$ plus a subset of \{1,2, \ldots ,n\}, so again there are $2^n$ such $A$'s.

The total number of subsets of \{1,2, \ldots ,n+1\} is therefore $2^n + 2^n = 2^{n+1}$, as asserted.

How do we know that our result is valid for all $n$? We have verified the case $n = 1$, and we know that if it holds for $n = 1$, then it holds for
$n = 2$; thus, we have the case $n = 2$. If the theorem is true for $n = 2$, then it is true for $n = 3$, so we have the case $n = 3$. This “domino effect” will eventually reach any positive integer $n$.

Our proof uses the technique of mathematical induction, which may be expressed formally as follows.

For each positive integer $n$, let $P(n)$ be a statement. A proof by induction that $P(n)$ holds for all $n$ consists of the following steps:

**Basis.** $P(1)$.

**Inductive Step.** If $P(n)$, then $P(n + 1)$.

Conclude that $P(n)$ is true for all $n$.

In a proof by strong induction, the inductive step is replaced by the statement: If $P(1), \ldots, P(n)$ all hold, then so does $P(n + 1)$ (the conclusion is the same).

**Example (Well-Ordering Principle).** If $A$ is a nonempty set of positive integers, then $A$ has a smallest element.

**Proof.** Suppose that $A$ has no smallest element; show $A = \emptyset$. Here, $P(n)$ is “$n \notin A$”. Now, $1 \notin A$, for if $1 \in A$ then 1 is the smallest element of $A$. If $1 \notin A$, $2 \notin A$, \ldots, $n \notin A$, then $n + 1 \notin A$, for if $n + 1 \in A$, then $n + 1$ is the smallest element of $A$.

We will encounter only a few proofs by induction (see Section 2.4, Problem 6, and Review Problem 4 in Chapter 2), but there will be many examples of inductive procedures. Most commonly, we will generate a sequence $x_1, x_2, \ldots$ where $x_n$ depends on $x_1, \ldots, x_{n-1}$.

**1.4.5 Negations**

The second part of the proof of Theorem 1.3.2 involves a basic reasoning process in abstract mathematics: going from a statement to its negative. Perhaps the intuition may be aided by the following mechanical procedure, using quantifiers. The statement that $E$ is open means for all $x \in E$ there exists a real number $r > 0$ such that $B_r(x) \subseteq E$; that is,

$$(\forall x \in E)(\exists r > 0)(B_r(x) \subseteq E).$$
Now if we have a mathematical statement of the form \((\forall x)P(x)\) (for every \(x\), \(P(x)\) is true), the negation is \((\exists x)(\neg P(x))\); in other words, for some \(x\), \(P(x)\) is false. Similarly, the negation of \((\exists x)P(x)\) is \((\forall x)(\neg P(x))\). Thus, to obtain the negation, we simply reverse all the quantifiers and change the statement on the extreme right of the expression to its negative. Therefore, "\(E\) is not open" means

\[(\exists x \in E)(\forall r > 0)(B_r(x) \nsubseteq E);\]

there is an \(x \in E\) such that for every \(r > 0\), the open ball \(B_r(x)\) is not a subset of \(E\), just as we found in the proof of Theorem 1.3.2.

**Note.** In a phrase such as \(\forall x \in E\), the fragment "\(\in E\)" refers to the range of the variable \(x\) and is *not changed* when taking the negative.

**Problems for Section 1.4**

1. The *converse* of \(A \rightarrow B\) is \(B \rightarrow A\). Give an example in which \(A \rightarrow B\) is true but \(B \rightarrow A\) is false.

2. Verify the De Morgan laws for propositions; i.e.,

   not \((A \text{ and } B)\) is equivalent to \((\neg A) \text{ or } (\neg B)\);

   not \((A \text{ or } B)\) is equivalent to \((\neg A) \text{ and } (\neg B)\).

3. When we say that \(P\) and \(Q\) are equivalent, we mean that \(P\) and \(Q\) have the same truth value regardless of the truth or falsity of the component propositions that make up \(P\) and \(Q\). (See the discussion of contrapositives in the text.) Show that this amounts to the assertion that the proposition "\(P\) if and only if \(Q\)," i.e., \((P \rightarrow Q)\) and \((Q \rightarrow P)\), is true.

   notation \(P \iff Q\)

4. Suppose \(x\) and \(y\) range over the positive real numbers (i.e., \(x, y > 0\)). Determine whether the following statements are true or false:
   
   (a) \(\forall x \exists y (x < y)\)

   (b) \(\exists x \forall y (x \leq y)\)

5. Write the statement \(\lim_{n \to \infty} d(x_n, x) = 0\) using universal and existential quantifiers.
6. Use the technique given in the text to take the negation of the statement of Problem 5. Then express your result in ordinary language.

7. Suppose we have proved "if P then not P." Show that we do not necessarily have a contradiction, but we can conclude "not P."

1.5 LIMIT POINTS AND CLOSURE

The following problem frequently arises in analysis: we have a point \( x \) and a set \( E \), and for each open ball \( B_r(x) \) we wish to select, if possible, a point \( y \in B_r(x) \cap E \). If we can do this for every \( r > 0 \), and furthermore, the point \( y \) can be chosen to be unequal to \( x \), then \( x \) is called a limit point or cluster point of \( E \). Formally, \( x \) is a limit point of \( E \) if every open ball \( B_r(x) \) contains a point \( y \in E \) with \( y \neq x \). Thus there are points in \( E \) arbitrarily close to \( x \) (not counting \( x \) itself); i.e., every deleted open ball about \( x \) \( (B_r(x) \text{ with } x \text{ removed}) \) contains a point of \( E \). For example, let \( E = (0, 1] \cup \{2\} = \{x \in \mathbb{R} : 0 < x \leq 1 \text{ or } x = 2\} \) (see Fig. 1.5.1). If \( 0 \leq x \leq 1 \), every open interval \( (x - r, x + r) \) has a point of \( E \) other than \( x \), so that every point in \( [0, 1] \) is a limit point of \( E \). (Note that a limit point of \( E \) need not belong to \( E \), for example, the point 0.) An isolated point of \( E \) is a point belonging to \( E \) that is not a limit point. In Fig. 1.5.1, \( E \) has one isolated point, namely 2.

If \( x \) is a limit point of \( E \), we can find points \( x_n \in B_{1/n}(x) \) with \( x_n \neq x \). Thus, \( x \) is the limit of a sequence of points \( x_n \in E \) with \( x_n \) never equal to \( x \). We can now describe precisely those points that can be expressed as limits of sequences from \( E \).

1.5.1 THEOREM. Suppose \( E \) is a subset of the metric space \( \Omega \). Let \( E' \) be the set of limit points of \( E \), and define \( \bar{E} = E \cup E' \); \( E \) is called the closure of \( E \). Then

(a) The point \( x \) belongs to \( \bar{E} \) if and only if there is a sequence of points \( x_n \in E \) with \( x_n \to x \).
(b) \( \bar{E} \) is a closed set.

Figure 1.5.1 Limit Points and Isolated Points
(c) If $C$ is a closed set and $E \subseteq C$, then $\bar{E} \subseteq C$; thus, $\bar{E}$ is the smallest closed set containing $E$.

(d) $E$ is closed if and only if $E = \bar{E}$.

**Proof**

(a) Let $x \in \bar{E}$. If $x \in E$, take $x_n = x$ for all $n$; if $x \in E'$, then there are points $x_n \in E$ with $x_n \rightarrow x$ (and $x_n$ never $= x$).

Conversely, if $x_n \in E$ and $x_n \rightarrow x$, we have two possibilities (a proof by cases). If $x \in E$, then $x \in \bar{E}$ by definition of $\bar{E}$; if $x \notin E$, then any open ball $B_r(x)$ will contain the $x_n$ for all sufficiently large $n$ (because $x_n \rightarrow x$). Since $x_n \in E$, we have $x_n \neq x$, so $x \in E' \subseteq \bar{E}$.

(b) Let $x_n \in \bar{E}$, $x_n \rightarrow x$; we must show $x \in \bar{E}$. Assume the contrary, that is, $x \notin E$ and $x \notin E'$. Then for some $r > 0$, $B_r(x)$ contains no points of $E$. But $x_n \in B_r(x)$ for all sufficiently large $n$; consider any such $x_n$ (Fig. 1.5.2). Since $x_n \in B_r(x)$, we have $x_n \notin E$, and since $x_n \in \bar{E}$ we must have $x_n \in E'$. Choose $s > 0$ so small that $B_s(x_n) \subseteq B_r(x)$, and select $z \in B_s(x_n)$ such that $z \in E$. This contradicts the fact that $B_r(x)$ has no points of $E$.

(c) If $x \in \bar{E}$, then by (a) we have a sequence of points $x_n \in E$ with $x_n \rightarrow x$. By hypothesis, $E \subseteq C$; therefore $x_n \in C$. Also by hypothesis, $C$ is closed; therefore $x \in C$.

(d) If $E = \bar{E}$, then $E$ is closed by (b). Conversely, if $E$ is closed, then by (c) (with $C = E$) we have $\bar{E} \subseteq E$. But $E$ is always a subset of $\bar{E}$, so $E = \bar{E}$.

Intuitively, $\bar{E}$ is $E$ together with all of its boundary points.

![Figure 1.5.2](image-url) Proof of Theorem 1.5.1
1.5 LIMIT POINTS AND CLOSURE

Problems for Section 1.5

1. In the text, we showed that if $x$ is a limit point of $E$ then there is a sequence of points $x_n \in E$ with $x_n$ never equal to $x$. Show that, conversely, if $x_n \in E$, $x_n$ never equal to $x$, and $x_n \to x$, then $x$ is a limit point of $E$.

2. Let $E = \{(x,y) : 0 \leq x < 1, 0 < y \leq 1\}$. Find $E'$ and $\overline{E}$.

3. In $\mathbb{R}$, let $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Find $E'$ and $\overline{E}$.

4. If $E'$ is the set of limit points of $E$, show that $E'$ is closed.

5. Show that $x$ is a limit point of $E$ if and only if $x$ is a limit point of $\overline{E}$.

6. Let $E'$ be the set of limit points of $E$. If $x$ is a limit point of $E'$, show that $x$ is a limit point of $E$.

7. Let $E = (0, 1) \cup \{2\frac{1}{2}, 2\frac{1}{4}, 2\frac{1}{8}, \ldots, 2 + \frac{1}{n}, \ldots\}$.
   \begin{enumerate}
   \item Find $E'$.
   \item Exhibit a limit point of $E$ that is not a limit point of $E'$.
   \end{enumerate}

8. We may give a formal definition of boundary, as follows: $x$ is a boundary point of the set $E$ if every open ball $B_r(x)$ contains both a point of $E$ and a point of $E^c$; $x$ is an interior point of $E$ if there is an $r > 0$ such that $B_r(x) \subseteq E$. Show that $x$ is a boundary point of $E$ iff $x \in \overline{E}$ and $x$ is not an interior point of $E$.

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REVIEW PROBLEMS FOR CHAPTER 1

1. Call a set of positive integers co-finite if its complement is finite. For example, \{1, 3, 4, 8, 9, 10, 11, \ldots\} is co-finite because its complement \{2, 5, 6, 7\} is finite. Are there countably or uncountably many co-finite subsets of the positive integers? Justify your answer.

2. Consider the following statement: If $x$ is a limit point of $\overline{E}$, then $x$ is a limit point of $E$. If the statement is true, explain why. If it is false, give an explicit counterexample.

3. We defined an open ball as a set of the form

$$B_r(x) = \{y \in \Omega : d(x, y) < r\}, \quad r > 0.$$
INTRODUCTION

Prove formally (without appealing to a picture) that $B_r(x)$ is actually an open set.

4. Consider the following mathematical statement involving quantifiers:

$$(\forall \epsilon > 0)(\exists N)(\forall n, m \geq N)(d(x_n, x_m) < \epsilon).$$

(This says that $\{x_n\}$ is a Cauchy sequence; we study such sequences in Section 2.4).

(a) Write the negation of the statement (again using quantifiers).

(b) If the negation holds, show that for some $\epsilon > 0$, there are distinct integers $n_1, m_1, n_2, m_2, \ldots$ such that $d(x_{n_k}, x_{m_k}) \geq \epsilon$ for all $k$.

5. Give an example of a nonempty set of real numbers with no limit point.