Solutions to Problems

Chapter 1

1. The primary ideals are (0) and (p^n), p prime.

2. $R/Q \cong k[y]/(y^2)$, and zero-divisors in this ring are of the form $cy + (y^2)$, $c \in k$, so they are nilpotent. Thus Q is primary. Since $\sqrt{Q} = P = (x, y)$, Q is P-primary.

3. If $Q = P_0^n$ with P_0 prime, then $\sqrt{Q} = P_0$, so by Problem 2, $P_0 = (x, y)$. But $x \in Q$ and $x \notin P_0^n$ for $n \geq 2$, so $Q \neq P_0^n$ for $n \geq 2$. Since $y \in P_0$ but $y \notin Q$, we have $Q \neq P_0$ and we reach a contradiction.

4. $\overline{P}$ is prime since $R/\overline{P} \cong k[y]$, an integral domain. Thus $\overline{P}^2$ is a prime power and its radical is the prime ideal $\overline{P}$. But it is not primary, because $\overline{x}\,\overline{y} = \overline{z}^2 \in \overline{P}^2$, $\overline{x} \notin \overline{P}^2$, $\overline{y} \notin \overline{P}$.

5. We have $I \subseteq P_1 \cap P_2^2$ and $I \subseteq P_1 \cap Q$ by definition of the ideals involved. For the reverse inclusions, note that if $f(x,y)x = g(x,y)y^2$ (or $f(x,y)x = g(x,y)y$), then $g(x,y)$ must involve x and $f(x,y)$ must involve y, so $f(x,y)$ is a polynomial multiple of xy.

   Now P_1 is prime (because $R/P_1 \cong k[y]$, a domain), hence P_1 is P_1-primary. P_2 is maximal and $\sqrt{P_2^2} = \sqrt{Q} = P_2$. Thus P_2^2 and Q are P_2-primary. [See (1.1.1) and (1.1.2). Note also that the results are consistent with the first uniqueness theorem.]

6. Let $\mathcal{M}$ be the maximal ideal of R, and $k = R/\mathcal{M}$ the residue field. Let $M_k = k \otimes_R M = (R/\mathcal{M}) \otimes_R M \cong M/\mathcal{M}M$. Assume $M \otimes_R N = 0$. Then $M_k \otimes_k N_k = (k \otimes_R M) \otimes_k (k \otimes_R N) = [(k \otimes_R M) \otimes_k k] \otimes_R N = (k \otimes_R M) \otimes_R N = k \otimes_R (M \otimes_R N) = 0$. Since M_k and N_k are finite-dimensional vector spaces over a field, one of them must be 0. [$k^r \otimes_k k^s = (k \otimes_k k^s)^r$ because tensor product commutes with direct sum, and this equals $(k^s)^r = k^{rs}$.] If $M_k = 0$, then $M = \mathcal{M}M$, so by NAK, $M = 0$. Similarly, $N_k = 0$ implies $N = 0$.

7. We have $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(n,m)\mathbb{Z}$, which is 0 if n and m are relatively prime.

8. $(M \otimes_R N)_S \cong R_S \otimes_R (M \otimes_R N) \cong (R_S \otimes_R M) \otimes_R N \cong M_S \otimes_R N \cong (M_S \otimes_{R_S} R_S) \otimes_R N \cong M_S \otimes_{R_S} (R_S \otimes_R N) \cong M_S \otimes_{R_S} N_S$.

9. By Problem 8, $(M \otimes_R N)_P \cong M_P \otimes_{R_P} N_P$ as R_P-modules. Thus $P \notin \operatorname{Supp}(M \otimes_R N)$ iff $M_P \otimes_{R_P} N_P = 0$. By Problem 6, this happens iff $M_P = 0$ or $N_P = 0$, that is, $P \notin \operatorname{Supp} M$ or $P \notin \operatorname{Supp} N$.

10. The first assertion follows from (1.6.4) and (1.6.6). Since the preimage of a prime ideal under a ring homomorphism is prime, the second assertion follows from (1.6.4).
11. Say $P_i^n = 0$. Then $x \in \mathcal{M}_i$ iff $\pi_i(x) \in P_i$ iff $\pi_i(x^n) = 0$ iff $x^n \in I_i$, and the result follows.
12. Since I_i consists of those elements that are 0 in the i^{th} coordinate, the zero ideal is the intersection of the I_i, and $I_i \not\supseteq \cap_{j\neq i} I_j$. By Problem 11, the decomposition is primary. Now $I_i \subseteq \sqrt{I_i} = \mathcal{M}_i$, and $I_i + I_j = R$ for $i \neq j$. Thus $\mathcal{M}_i + \mathcal{M}_j = R$, so the $\mathcal{M}_i$ are distinct and the decomposition is reduced.
13. By Problem 12, the $\mathcal{M}_i$ are distinct and hence minimal. By the second uniqueness theorem (1.4.5), the I_i are unique (for a given R). Since $R_i \cong R/I_i$, the R_i are unique up to isomorphism.
14. By (1.6.9), the length $l_{R_P}(M_P)$ will be finite iff every element of $\mathrm{AP}_{R_P}(M_P)$ is maximal. Now R_P is a local ring with maximal ideal PR_P. By the bijection of (1.4.2), $l_{R_P}(M_P) < \infty$ iff there is no $Q \in \mathrm{AP}(M)$ such that $Q \subset P$. By hypothesis, $P \in \operatorname{Supp} M$, so by (1.5.8), P contains some $P' \in \mathrm{AP}(M)$, and under the assumption that $l_{R_P}(M_P)$ is finite, P must coincide with P'. The result follows.

Chapter 2

1. Let $Q_1 = (2+i),\ \ Q_2 = (2-i)$. An integer divisible by $2+i$ must also be divisible by the complex conjugate $2-i$, hence divisible by $(2+i)(2-i) = 5$. Thus $Q_1 \cap \mathbb{Z} = (5)$, and similarly $Q_2 \cap \mathbb{Z} = (5)$.
2. We have $x^2 = y^3$, hence $(x/y)^2 = y$. Thus $\alpha^2 - y = 0$, so α is integral over R. If $\alpha \in R$, then $\alpha = x/y = f(x,y)$ for some polynomial f in two variables with coefficients in k. Thus $x = yf(x,y)$. Written out longhand, this is $X + I = Yf(X,Y) + I$, and consequently $X - Yf(X,Y) \in I = (X^2, Y^3)$. This is impossible because there is no way that a linear combination $g(X,Y)X^2 + h(X,Y)Y^3$ can produce X.
3. Since the localization functor is exact, we have (a) implies (b), and (b) implies (c) is immediate. To prove that (c) implies (a), consider the exact sequence

$$0 \longrightarrow \operatorname{im} f \xrightarrow{\ i\ } \ker g \xrightarrow{\ \pi\ } \ker g/\operatorname{im} f \longrightarrow 0$$

   Applying the localization functor, we get the exact sequence

$$0 \longrightarrow (\operatorname{im} f)_P \xrightarrow{\ i_P\ } (\ker g)_P \xrightarrow{\ \pi_P\ } (\ker g/\operatorname{im} f)_P \longrightarrow 0$$

   for every maximal (indeed for every prime) ideal P. But by basic properties of localization,

$$(\ker g/\operatorname{im} f)_P = (\ker g)_P/(\operatorname{im} f)_P = \ker g_P/\operatorname{im} f_P,$$

   which is 0 for every maximal ideal P, by (c). By (1.5.1), $\ker g/\operatorname{im} f = 0$, in other words, $\ker g = \operatorname{im} f$, proving (a).
4. In the injective case, apply Problem 3 to the sequence

$$0 \longrightarrow M \xrightarrow{\ f\ } N,$$

and in the surjective case, apply Problem 3 to the sequence

$$M \xrightarrow{f} N \longrightarrow 0.$$

5. This follows because $S^{-1}(\cap A_i) \subseteq \cap_i S^{-1}(A_i)$ for arbitrary rings (or modules) A_i.
6. Taking $S = R \setminus Q$ and applying Problem 5, we have the following chain of inclusions, where P ranges over all maximal ideals of R:

$$M_Q = (\cap_P R_P)_Q \subseteq \cap_P (R_P)_Q \subseteq (R_Q)_Q = R_Q.$$

7. Since R is contained in every R_P, we have $R \subseteq M$, hence $R_Q \subseteq M_Q$ for every maximal ideal Q. Let $i : R \to M$ and $i_Q : R_Q \to M_Q$ be inclusion maps. By Problem 6, $R_Q = M_Q$, in particular, i_Q is surjective. Since Q is an arbitrary maximal ideal, i is surjective by Problem 4, so $R = M$. But $R \subseteq \cap_{P \text{prime}} R_P \subseteq M$, and the result follows.
8. The implication (a) implies (b) follows from (2.2.6), and (b) immediately implies (c). To prove that (c) implies (a), note that if for every i, K is the fraction field of A_i, where the A_i are domains that are integrally closed in K, then $\cap_i A_i$ is integrally closed. It follows from Problem 7 that R is the intersection of the R_Q, each of which is integrally closed (in the same fraction field K). Thus R is integrally closed.
9. The elements of the first field are $a/f + PR_P$ and the elements of the second field are $(a + P)/(f + P)$, where in both cases, $a, f \in R, f \notin P$. This tells you exactly how to construct the desired isomorphism.

Chapter 3

1. Assume that $(V, \mathcal{M}_V) \leq (R, \mathcal{M}_R)$, and let α be a nonzero element of R. Then either α or α^{-1} belongs to V. If $\alpha \in V$ we are finished, so assume $\alpha \notin V$, hence $\alpha^{-1} \in V \subseteq R$. Just as in the proof of Property 9 of Section 3.2, α^{-1} is not a unit of V. (If $b \in V$ and $b\alpha^{-1} = 1$, then $\alpha = \alpha\alpha^{-1}b = b \in V$.) Thus $\alpha^{-1} \in \mathcal{M}_V = \mathcal{M}_R \cap V$, so α^{-1} is not a unit of R. This is a contradiction, as α and its inverse both belong to R.
2. 2. By definition of h, $\ker h = \mathcal{M}_V$. Since h_1 extends h, $\ker h = (\ker h_1) \cap V$, that is, $\mathcal{M}_V = \mathcal{M}_{R_1} \cap V$. Since $R_1 \supseteq V$, the result follows.
3. By hypothesis, $(V, \mathcal{M}_V)$ is maximal with respect to domination, so $(V, \mathcal{M}_V) = (R_1, \mathcal{M}_{R_1})$. Therefore $V = R_1$, and the proof is complete.
4. If $(R, \mathcal{M}_R)$ is not dominated in this way, then it is a maximal element in the domination ordering, hence R itself is a valuation ring.

Chapter 4

1. We have $f \in I^d$ iff all terms of f have degree at least d, so if we identify terms of degree at least $d + 1$ with 0, we get an isomorphism between I^d/I^{d+1} and the homogeneous polynomials of degree d. Take the direct sum over all $d \geq 0$ to get the desired result.
2. If $x \in M_n$ and $f(x) \in N_{n+1}$, then $f(x) + N_{n+1} = 0$, so $x \in M_{n+1}$.

3. The result holds for $n = 0$ because $M_0 = M$ and $N_0 = N$. If it is true for n, let $x \in f^{-1}(N_{n+1})$. Since $N_{n+1} \subseteq N_n$, it follows that x belongs to $f^{-1}(N_n)$, which is contained in M_n by the induction hypothesis. By Problem 2, the result is true for $n + 1$.

4. Using the additional hypothesis and Problem 3, we have $f^{-1}(0) \subseteq f^{-1}(\cap N_n) = \cap f^{-1}(N_n) \subseteq \cap M_n = 0$.

5. By (4.1.8) we have

$$(I^{m+k}M) \cap N = I^k((I^m M) \cap N) \subseteq I^k N \subseteq (I^k M) \cap N.$$

6. Since $g_n \circ f_n = 0$ for all n, we have $g \circ f = 0$. If $g(y) = 0$, then y is represented by a sequence $\{y_n\}$ with $y_n \in M_n$ and $g_n(y_n) = 0$ for sufficiently large n. Thus for some $x_n \in M'_n$ we have $y_n = f_n(x_n)$. The elements x_n determine $x \in M'$ such that $y = f(x)$, proving exactness.

7. Since $\hat{R} \otimes_R R \cong \hat{R}$ and tensor product commutes with direct sum, h_M is an isomorphism when M is free of finite rank. In general, we have an exact sequence $0 \longrightarrow N \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$ with F free of finite rank. Thus the following diagram is commutative, with exact rows.

$$\begin{array}{ccccccccc}
 & & \hat{R} \otimes_R N & \longrightarrow & \hat{R} \otimes_R F & \longrightarrow & \hat{R} \otimes_R M & \longrightarrow & 0 \\
 & & \downarrow h_N & & \downarrow h_F & & \downarrow h_M & & \\
0 & \longrightarrow & \hat{N} & \xrightarrow[\hat{f}]{} & \hat{F} & \xrightarrow[\hat{g}]{} & \hat{M} & \longrightarrow & 0
\end{array}$$

See (4.2.7) for the last row. Since $\hat{g}$ is surjective and h_F is an isomorphism, it follows that h_M is surjective.

8. By hypothesis, N is finitely generated, so by Problem 7, h_N is surjective. Since h_F is an isomorphism, h_M is injective by the four lemma. (See TBGY, 4.7.2, part (ii).)

9. Take inverse limits in (4.2.9).

10. Consider the diagram for Problem 7, with M finitely generated. No generality is lost; see TBGY, (10.8.1). Then all vertical maps are isomorphisms, so if we augment the first row by attaching $0 \rightarrow$ on the left, the first row remains exact. Thus the functor $\hat{R} \otimes_R$ — is exact, proving that $\hat{R}$ is flat.

11. Since M is isomorphic to its completion, we may regard $\hat{M}$ as the set of constant sequences in M. If x belongs to M_n for every n, then x converges to 0, hence x and 0 are identified in $\hat{M}$. By (4.2.4), the topology is Hausdorff.

12. I is finitely generated, so by Problem 8, $h_I : \hat{R} \otimes_R I \rightarrow \hat{I}$ is an isomorphism. Since $\hat{R}$ is flat over R by Problem 10, $\hat{R} \otimes_R I \rightarrow \hat{R} \otimes_R R \cong \hat{R}$ is injective, and the image of this map is $\hat{R}I$.

13. By Problem 12, $(I^n)\hat{} \cong \hat{R}I^n = (\hat{R}I)^n \cong (\hat{I})^n$.

14. The following diagram is commutative, with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^n/I^{n+1} & \longrightarrow & R/I^{n+1} & \longrightarrow & R/I^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\hat{I})^n/(\hat{I})^{n+1} & \longrightarrow & \hat{R}/(\hat{I})^{n+1} & \longrightarrow & \hat{R}/(\hat{I})^n & \longrightarrow & 0
\end{array}$$

The second and third vertical maps are isomorphisms by (4.2.9), so the first vertical map is an isomorphism by the short five lemma.

15. By (4.2.9) and Problem 9, $\hat{R}$ is complete with respect to the $\hat{I}$-adic topology. Suppose that $a \in \hat{I}$. Since $a^n + a^{n+1} + \cdots + a^m \in (\hat{I})^n$ for all n, the series $1 + a + a^2 + \cdots + a^n$ converges to some $b \in \hat{R}$. Now $(1-a)(1+a+a^2+\cdots+a^n) = 1 - a^{n+1}$, and we can let n approach infinity to get $(1-a)b = 1$. Thus $a \in \hat{I} \Rightarrow 1-a$ is a unit in $\hat{R}$. Since ax belongs to $\hat{I}$ for every $x \in \hat{R}$, $1+ax$ is also a unit. By (0.2.1), $a \in J(\hat{R})$.

16. By (4.2.9), $R/\mathcal{M} \cong \hat{R}/\hat{\mathcal{M}}$, so $\hat{R}/\hat{\mathcal{M}}$ is a field, hence $\hat{\mathcal{M}}$ is a maximal ideal. By Problem 15, $\hat{\mathcal{M}}$ is contained in every maximal ideal, and it follows that $\hat{\mathcal{M}}$ is the unique maximal ideal of $\hat{R}$.

Chapter 5

1. The function 2^n is its own difference.

2. If P is a prime ideal containing $\operatorname{ann}(M/\mathcal{M}M)$, then $P \supseteq \mathcal{M}$, hence $P = \mathcal{M}$ by maximality of $\mathcal{M}$. Conversely, we must show that $\mathcal{M} \supseteq \operatorname{ann}(M/\mathcal{M}M)$. This will be true unless $\operatorname{ann}(M/\mathcal{M}M) = R$. In this case, 1 annihilates $M/\mathcal{M}M$, so $\mathcal{M}M = M$. By NAK, $M = 0$, contradicting the hypothesis.

3. Let $S = R \setminus P$. Then $(R/I)_P = 0$ iff $S^{-1}(R/I) = 0$ iff $S^{-1}R = S^{-1}I$ iff $1 \in S^{-1}I$ iff $1 = a/s$ for some $a \in I$ and $s \in S$ iff $I \cap S \neq \emptyset$ iff I is not a subset of P.

4. By Going Up [see (2.2.3)], any chain of distinct prime ideals of R can be lifted to a chain of distinct prime ideals of S, so $\dim S \geq \dim R$. A chain of distinct prime ideals of S contracts to a chain of prime ideals of R, distinct by (2.2.1). Thus $\dim R \geq \dim S$.

5. Since S/J is integral over the subring R/I, it follows from (5.3.1) and Problem 4 that $\text{coht } I = \dim R/I = \dim S/J = \text{coht } J$.

6. If J is a prime ideal of S, then $I = J \cap R$ is a prime ideal of R. The contraction of a chain of prime ideals of S contained in J is a chain of prime ideals of R contained in I, and distinctness is preserved by (2.2.1). Thus $\text{ht } J \leq \text{ht } I$. Now let J be any ideal of S, and let P be a prime ideal of R such that $P \supseteq I$ and $\text{ht } P = \text{ht } I$. (If the height of I is infinite, there is nothing to prove.) As in the previous problem, S/J is integral over R/I, so by Lying Over [see (2.2.2)] there is a prime ideal Q containing J that lies over P. Thus with the aid of the above proof for J prime, we have $\text{ht } J \leq \text{ht } Q \leq \text{ht } P = \text{ht } I$.

7. First assume J is a prime ideal of S, hence I is a prime ideal of R. A descending chain of distinct prime ideals of R starting from I can be lifted to a descending chain of distinct prime ideals of S starting from J, by Going Down [see (2.3.4)]. Thus $\text{ht } J \geq \text{ht } I$. For any ideal J, let Q be a prime ideal of S with $Q \supseteq J$. Then $P = Q \cap R \supseteq I$.

By what we have just proved, ht $Q \geq$ ht P, and ht $P \geq$ ht I by definition of height. Taking the infimum over Q, we have ht $J \geq$ ht I. By Problem 6, ht $J =$ ht I.

8. The chain of prime ideals $(\overline{X}) \subset (\overline{X}, \overline{Y}) \subset (\overline{X}, \overline{Y}, \overline{Z})$ gives $\dim R \geq 2$. Since XY (or equally well XZ), belongs to the maximal ideal (X, Y, Z) and is not a zero-divisor, we have $\dim R \leq \dim S/(XY) = \dim S - 1 = 2$ by (5.4.7) and (5.4.9).
9. The height of P is 0 because the ideals $(\overline{Y})$ and $(\overline{Z})$ are not prime. For example, $\overline{X} \notin (\overline{Y})$ and $\overline{Z} \notin (\overline{Y})$, but $\overline{X}\ \overline{Z} = \overline{0} \in (\overline{Y})$. Since $R/P \cong k[[X]]$ has dimension 1, P has coheight 1 by (5.3.1).

Chapter 6

1. By (6.1.3), $\dim R/P = \dim R - t = \dim R - \operatorname{ht} P$. By (5.3.1), $\dim R/P = \operatorname{coht} P$, and the result follows.
2. Let J be the ideal $(\overline{Z},\ \overline{X} + \overline{Y})$. If $\mathcal{M} = (X, Y, Z)$ is the unique maximal ideal of S, then $\overline{\mathcal{M}}^2 = (\overline{X}^2, \overline{Y}^2, \overline{Z}^2, \overline{Y}\ \overline{Z}) \subseteq J \subseteq \mathcal{M}$, so J is an ideal of definition. (Note that $\overline{X}\ \overline{Y} = \overline{X}\ \overline{Z} = 0$, $\overline{X}(\overline{X} + \overline{Y}) = \overline{X}^2$, and $\overline{Y}(\overline{X} + \overline{Y}) = \overline{Y}^2$.) By (6.1.2), $\{\overline{Z},\ \overline{X} + \overline{Y}\}$ is a system of parameters. Since $\overline{Z}\ \overline{X} = 0$, $\overline{Z}$ is a zero-divisor.

Chapter 7

1. Note that $\ker f, \operatorname{im} f$, and $\ker g$ are all equal to $\{0, 2\}$.
2. We have $\operatorname{im} \partial_n = \ker f_{n-1} = 0$ and $\ker g_n = \operatorname{im} f_n = B_n$. Thus g_n is the zero map, so $\ker \partial_n = \operatorname{im} g_n = 0$. Therefore ∂_n is an injective zero map, which forces $C_n = 0$.
3. This follows from the base change formula $R/I \otimes_R M \cong M/IM$ with $I = \mathcal{M}$ (see TBGY, S7.1).
4. We have $g^* : 1 \otimes e_i \to 1 \otimes x_i$, which is an isomorphism. (The inverse is $1 \otimes x_i \to 1 \otimes e_i$.) Thus $\operatorname{im} f^* = \ker g^* = 0$. Since f^* is the zero map, δ is surjective. But $\ker u_M$ is 0 by hypothesis, so $\delta = 0$. This forces coker $u_K = 0$.
5. By Problem 4, $K = \mathcal{M}K$. Since M is a Noetherian R-module, K is finitely generated, so by NAK we have $K = 0$. Thus $0 = \operatorname{im} f = \ker g$, so g is injective.
6. Since free implies projective implies flat always, it suffices to show that flat implies free. If M is flat, then the functor $N \to N \otimes_R M$ is exact. If $\mathcal{M}$ is the maximal ideal of R, then the map $\mathcal{M} \otimes_R M \to R \otimes_R M \cong M$ via $a \otimes x \to ax$ is injective. But this map is just u_M, and the result follows from Problems 3-5.
7. We have the short exact sequence $0 \to I \to R \to R/I \to 0$, which induces, for any R-module N, the exact sequence

$$\operatorname{Hom}_R(R/I, N) \to \operatorname{Hom}_R(R, N) \to \operatorname{Hom}_R(I, N) \to \operatorname{Ext}^1_R(R/I, N).$$

The last term is 0 by hypothesis, hence the map $i^* : \operatorname{Hom}_R(R, N) \to \operatorname{Hom}_R(I, N)$ is surjective. This says, by Baer's criterion (TBGY 10.6.4), that N is injective.

8. The left side is at least equal to the right side by (7.2.5), so assuming that the right side is at most n, it suffices to show that $\mathrm{id}_R N \le n$ for all N. Given an exact sequence as in (7.2.4) part 4, dimension shifting yields $\mathrm{Ext}_R^{n+1}(R/I, N) \cong \mathrm{Ext}_R^1(R/I, C_{n-1})$. By (7.2.4), $\mathrm{Ext}_R^1(R/I, C_{n-1}) = 0$, so by (7.2.1) and Problem 7, C_{n-1} is injective. By (7.2.4), $\mathrm{id}_R N \le n$.

9. If (a) holds, only the second assertion of (b) requires proof. Apply Tor to the exact sequence $0 \to N' \to N \to N'' \to 0$ to get the exact sequence

$$0 = \mathrm{Tor}_1^R(M, N'') \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0.$$

We may replace $M \otimes_R N$ by $(M \otimes_R S) \otimes_S N$, and similarly for the other two tensor products. By exactness, $M \otimes_R S$ is flat. Now assuming (b), we have $\mathrm{Tor}_1^R(M, F) = 0$ for every free S-module F, because Tor commutes with direct sums. If N is an arbitrary S-module, we have a short exact sequence $0 \to K \to F \to N \to 0$ with F free. The corresponding (truncated) long exact sequence is

$$0 = \mathrm{Tor}_1^R(M, F) \to \mathrm{Tor}_1^R(M, N) \to M \otimes_R K \to M \otimes_R F \to M \otimes_R N \to 0.$$

As before, we replace $M \otimes_R K$ by $(M \otimes_R S) \otimes_S K$, and similarly for the other two tensor products. The map whose domain is $(M \otimes_R S) \otimes_S K$ is induced by the inclusion of K into F, and is therefore injective, because $M \otimes_R S$ is a flat S-module by hypothesis. Thus the kernel of the map, namely $\mathrm{Tor}_1^R(M, N)$, is zero.

Chapter 8

1. To ease the notation we will omit all the overbars and adopt the convention that all calculations are mod $(X^3 - Y^2)$. We have $(X^2 + X + 1)(X - 1) = X^3 - 1 = Y^2 - 1 = (Y - 1)(Y + 1)$. Now $X^2 + X + 1$ and $Y + 1$ are units in R because they do not vanish when $X = Y = 1$, assuming that the characteristic of K is not 2 or 3. Thus $X - 1$ and $Y - 1$ are associates.

2. The maximal ideal is not principal because $\overline{X}$ and $\overline{Y}$ cannot both be multiples of a single polynomial. To show that $\dim R = 1$, we use (5.6.7). Since $K(Y)$ has transcendence degree 1 over K and $K(X, Y)/(X^3 - Y^2)$ is algebraic over $K(Y)$, (we are adjoining a root of $X^3 - Y^2$), it follows that the dimension of $K[X, Y]/(X^3 - Y^2)$ is 1. By (5.3.1), the coheight of $(X^3 - Y^2)$ is 1, and the corresponding sequence of prime ideals is $(X^3 - Y^2), (X, Y)$. Thus localization at $(\overline{X}, \overline{Y})$ has no effect on dimension, so $\dim R = 1$. (In general, prime ideals of a localized ring A_P correspond to prime ideals of A that are contained in P, so localization may reduce the dimension.)

3. By definition, the Hilbert polynomial is the composition length $l_k(I^n/I^{n+1})$. Since monomials of degree n in r variables form a basis for the polynomials of degree n, we must count the number of such monomials, which is

$$\binom{n+r-1}{r-1} = \frac{(n+r-1)(n+r-2)\cdots(n+2)(n+1)}{(r-1)!}$$

This is a polynomial of degree $r - 1$ in the variable n.

4. This follows from Problem 3 and additivity of length (5.2.3).
5. Fix a nonzero element $b \in B_d$. (Frequently, b is referred to as a homogeneous element of degree d.) By definition of a graded ring, we have $bA_n \subseteq B_{n+d}$ for $n \geq 0$. Then

$$l_k(B_{n+d}) \geq l_k(bA_n) = l_k(A_n) \geq l_k(B_n).$$

Since $l_k(A_n) = \binom{n+r-1}{r-1}$, the result follows.

6. If R is regular, we may define the graded k-algebra homomorphism φ of Problems 3-5 with $r = d$. Since the Hilbert polynomial has degree $d - 1$, φ is an isomorphism. Conversely, an isomorphism of graded k-algebras induces an isomorphism of first components, in other words,

$$(k[X_1, \dots, X_d])_1 \cong \mathcal{M}/\mathcal{M}^2.$$

But the k-vector space on the left has a basis consisting of all monomials of degree 1. Since there are exactly d of these, we have $\dim_k \mathcal{M}/\mathcal{M}^2 = d$. By (8.1.3), R is regular.