Solutions to Problems

Chapter 1

1. The primary ideals are \((0)\) and \((p^n)\), \(p\) prime.
2. \(R/Q \cong k[y]/(y^2)\), and zero-divisors in this ring are of the form \(cy + (y^2)\), \(c \in k\), so they are nilpotent. Thus \(Q\) is primary. Since \(\sqrt{Q} = P = (x, y)\), \(Q\) is \(P\)-primary.
3. If \(Q = P_0^n\) with \(P_0\) prime, then \(\sqrt{Q} = P_0\), so by Problem 2, \(P_0 = (x, y)\). But \(x \in Q\) and \(x \notin P_0^n\) for \(n \geq 2\), so \(Q \neq P_0^n\) for \(n \geq 2\). Since \(y \in P_0\) but \(y \notin Q\), we have \(Q \neq P_0\) and we reach a contradiction.
4. \(\overline{P}\) is prime since \(R/\overline{P} \cong k[y]\), an integral domain. Thus \(\overline{P}^2\) is a prime power and its radical is the prime ideal \(\overline{P}\). But it is not primary, because \(\overline{x} \overline{y} = \overline{z}^2 \in \overline{P}^2\), \(\overline{x} \notin \overline{P}^2\), \(\overline{y} \notin \overline{P}\).
5. We have \(I \subseteq P_1 \cap P_2^2\) and \(I \subseteq P_1 \cap Q\) by definition of the ideals involved. For the reverse inclusions, note that if \(f(x, y)x = g(x, y)y^2\) (or \(f(x, y)x = g(x, y)y)\), then \(g(x, y)\) must involve \(x\) and \(f(x, y)\) must involve \(y\), so \(f(x, y)\) is a polynomial multiple of \(xy\).

Now \(P_1\) is prime (because \(R/P_1 \cong k[y]\), a domain), hence \(P_1\) is \(P_1\)-primary. \(P_2\) is maximal and \(\sqrt{P_2^2} = \sqrt{Q} = P_2\). Thus \(P_2^2\) and \(Q\) are \(P_2\)-primary. [See (1.1.1) and (1.1.2). Note also that the results are consistent with the first uniqueness theorem.]
6. Let \(M\) be the maximal ideal of \(R\), and \(k = R/M\) the residue field. Let \(M_k = k \otimes_R M = (R/M) \otimes_R M \cong M/MM\). Assume \(M \otimes_R N = 0\). Then \(M_k \otimes_k N_k = (k \otimes_R M) \otimes_k (k \otimes_R N) = [(k \otimes_R M) \otimes_k k] \otimes_R N = (k \otimes_R M) \otimes_R N = k \otimes_R (M \otimes_R N) = 0\).

Since \(M_k\) and \(N_k\) are finite-dimensional vector spaces over a field, one of them must be 0. \([k^r \otimes_k k^s = (k \otimes_k k^s)^r\] because tensor product commutes with direct sum, and this equals \((k^s)^r = k^{rs}\).] If \(M_k = 0\), then \(M = MM\), so by NAK, \(M = 0\). Similarly, \(N_k = 0\) implies \(N = 0\).
7. We have \(\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(n, m)\mathbb{Z}\), which is 0 if \(n\) and \(m\) are relatively prime.
8. \((M \otimes_R N)_S \cong R_S \otimes_R (M \otimes_R N) \cong (R_S \otimes_R M) \otimes_R N \cong M_S \otimes_R N \cong (M_S \otimes_{R_S} R_S) \otimes_R N \cong M_S \otimes_{R_S} (R_S \otimes_R N) \cong M_S \otimes_{R_S} N_S\).
9. By Problem 8, \((M \otimes_R N)_P \cong M_P \otimes_{R_P} N_P\) as \(R_P\)-modules. Thus \(P \notin \text{Supp}(M \otimes_R N)\) iff \(M_P \otimes_{R_P} N_P = 0\). By Problem 6, this happens iff \(M_P = 0\) or \(N_P = 0\), that is, \(P \notin \text{Supp} M\) or \(P \notin \text{Supp} N\).
10. The first assertion follows from (1.6.4) and (1.6.6). Since the preimage of a prime ideal under a ring homomorphism is prime, the second assertion follows from (1.6.4).

11. Say $P^n_i = 0$. Then $x \in M_i$ iff $\pi_i(x) \in P_i$ iff $\pi_i(x^n) = 0$ iff $x^n \in I_i$, and the result follows.

12. Since $I_i$ consists of those elements that are 0 in the $i^{th}$ coordinate, the zero ideal is the intersection of the $I_i$, and $I_i \not\supseteq \bigcap_{j \neq i} I_j$. By Problem 11, the decomposition is primary. Now $I_i \subseteq \sqrt{I_i} = M_i$, and $I_i + I_j = R$ for $i \neq j$. Thus $M_i + M_j = R$, so the $M_i$ are distinct and the decomposition is reduced.

13. By Problem 12, the $M_i$ are distinct and hence minimal. By the second uniqueness theorem (1.4.5), the $I_i$ are unique (for a given $R$). Since $R_i \cong R/I_i$, the $R_i$ are unique up to isomorphism.

14. By (1.6.9), the length $l_{R_P}(M_P)$ will be finite iff every element of $AP_{R_P}(M_P)$ is maximal. Now $R_P$ is a local ring with maximal ideal $PR_P$. By the bijection of (1.4.2), $l_{R_P}(M_P) < \infty$ iff there is no $Q \in AP(M)$ such that $Q \subset P$. By hypothesis, $P \in \text{Supp} M$, so by (1.5.8), $P$ contains some $P' \in AP(M)$, and under the assumption that $l_{R_P}(M_P)$ is finite, $P$ must coincide with $P'$. The result follows.

Chapter 2

1. Let $Q_1 = (2 + i)$, $Q_2 = (2 - i)$. An integer divisible by $2 + i$ must also be divisible by the complex conjugate $2 - i$, hence divisible by $(2 + i)(2 - i) = 5$. Thus $Q_1 \cap \mathbb{Z} = (5)$, and similarly $Q_2 \cap \mathbb{Z} = (5)$.

2. We have $x^2 = y^3$, hence $(x/y)^2 = y$. Thus $\alpha^2 - y = 0$, so $\alpha$ is integral over $R$. If $\alpha \in R$, then $\alpha = x/y = f(x, y)$ for some polynomial $f$ in two variables with coefficients in $k$. Thus $x = yf(x, y)$. Written out longhand, this is $X + I = Yf(X, Y) + I$, and consequently $X - Yf(X, Y) \in I = (X^2, Y^3)$. This is impossible because there is no way that a linear combination $g(X, Y)X^2 + h(X, Y)Y^3$ can produce $X$.

3. Since the localization functor is exact, we have (a) implies (b), and (b) implies (c) is immediate. To prove that (c) implies (a), consider the exact sequence

$$0 \longrightarrow \text{im } f \overset{i}{\longrightarrow} \ker g \overset{\pi}{\longrightarrow} \ker g/\text{im } f \longrightarrow 0$$

Applying the localization functor, we get the exact sequence

$$0 \longrightarrow (\text{im } f)_P \overset{i_P}{\longrightarrow} (\ker g)_P \overset{\pi_P}{\longrightarrow} (\ker g/\text{im } f)_P \longrightarrow 0$$

for every maximal (indeed for every prime) ideal $P$. But by basic properties of localization,

$$(\ker g/\text{im } f)_P = (\ker g)_P/(\text{im } f)_P = \ker g_P/\text{im } f_P,$$

which is 0 for every maximal ideal $P$, by (c). By (1.5.1), $\ker g/\text{im } f = 0$, in other words, $\ker g = \text{im } f$, proving (a).

4. In the injective case, apply Problem 3 to the sequence

$$0 \longrightarrow M \overset{f}{\longrightarrow} N,$$
and in the surjective case, apply Problem 3 to the sequence

\[ M \xrightarrow{f} N \xrightarrow{} 0. \]

5. This follows because \( S^{-1}(\cap A_i) \subseteq \cap_i S^{-1}(A_i) \) for arbitrary rings (or modules) \( A_i \).

6. Taking \( S = R \setminus Q \) and applying Problem 5, we have the following chain of inclusions, where \( P \) ranges over all maximal ideals of \( R \):

\[ M_Q = (\cap_P R_P)Q \subseteq \cap_P (R_P)Q \subseteq (R_Q)Q = R_Q. \]

7. Since \( R \) is contained in every \( R_P \), we have \( R \subseteq M \), hence \( R_Q \subseteq M_Q \) for every maximal ideal \( Q \). Let \( i : R \to M \) and \( i_Q : R_Q \to M_Q \) be inclusion maps. By Problem 6, \( R_Q = M_Q \), in particular, \( i_Q \) is surjective. Since \( Q \) is an arbitrary maximal ideal, \( i \) is surjective by Problem 4, so \( R = M \). But \( R \subseteq \cap_{P\text{ prime}} R_P \subseteq M \), and the result follows.

8. The implication (a) implies (b) follows from (2.2.6), and (b) immediately implies (c). To prove that (c) implies (a), note that if for every \( i \), \( K \) is the fraction field of \( A_i \), where the \( A_i \) are domains that are integrally closed in \( K \), then \( \cap_i A_i \) is integrally closed. It follows from Problem 7 that \( R \) is the intersection of the \( R_Q \), each of which is integrally closed (in the same fraction field \( K \)). Thus \( R \) is integrally closed.

9. The elements of the first field are \( a/f + PR_P \) and the elements of the second field are \( (a + P)/(f + P) \), where in both cases, \( a, f \in R, f \notin P \). This tells you exactly how to construct the desired isomorphism.

Chapter 3

1. Assume that \((V,M_V) \leq (R,M_R)\), and let \( \alpha \) be a nonzero element of \( R \). Then either \( \alpha \) or \( \alpha^{-1} \) belongs to \( V \). If \( \alpha \in V \) we are finished, so assume \( \alpha \notin V \), hence \( \alpha^{-1} \in V \subseteq R \).

    Just as in the proof of Property 9 of Section 3.2, \( \alpha^{-1} \) is not a unit of \( V \). (If \( b \in V \) and \( b\alpha^{-1} = 1 \), then \( \alpha = \alpha \alpha^{-1} b = b \in V \).) Thus \( \alpha^{-1} \in M_V = M_R \cap V \), so \( \alpha^{-1} \) is not a unit of \( R \). This is a contradiction, as \( \alpha \) and its inverse both belong to \( R \).

2. By definition of \( h \), \( \ker h = M_V \). Since \( h_1 \) extends \( h \), \( \ker h = (\ker h_1) \cap V \), that is, \( M_V = M_{R_1} \cap V \). Therefore \( V = R_1 \), and the proof is complete.

3. By hypothesis, \((V,M_V)\) is maximal with respect to domination, so \((V,M_V) = (R_1,M_{R_1})\). Therefore \( V = R_1 \), and the proof is complete.

4. If \((R,M_R)\) is not dominated in this way, then it is a maximal element in the domination ordering, hence \( R \) itself is a valuation ring.

Chapter 4

1. We have \( f \in I^d \) iff all terms of \( f \) have degree at least \( d \), so if we identify terms of degree at least \( d + 1 \) with 0, we get an isomorphism between \( I^d/I^{d+1} \) and the homogeneous polynomials of degree \( d \). Take the direct sum over all \( d \geq 0 \) to get the desired result.

2. If \( x \in M_n \) and \( f(x) \in N_{n+1} \), then \( f(x) + N_{n+1} = 0 \), so \( x \in M_{n+1} \).
3. The result holds for \( n = 0 \) because \( M_0 = M \) and \( N_0 = N \). If it is true for \( n \), let \( x \in f^{-1}(N_{n+1}) \). Since \( N_{n+1} \subseteq N_n \), it follows that \( x \) belongs to \( f^{-1}(N_n) \), which is contained in \( M_n \) by the induction hypothesis. By Problem 2, the result is true for \( n + 1 \).

4. Using the additional hypothesis and Problem 3, we have \( f^{-1}(0) \subseteq f^{-1}(\cap N_n) = \cap f^{-1}(N_n) \subseteq \cap M_n = 0 \).

5. By (4.1.8) we have

\[(I^{n+k}M) \cap N = I^k((I^n M) \cap N) \subseteq I^k N \subseteq (I^k M) \cap N.\]

6. Since \( g_n \circ f_n = 0 \) for all \( n \), we have \( g \circ f = 0 \). If \( g(y) = 0 \), then \( y \) is represented by a sequence \( \{y_n\} \) with \( y_n \in M_n \) and \( g_n(y_n) = 0 \) for sufficiently large \( n \). Thus for some \( x_n \in M'_n \) we have \( y_n = f_n(x_n) \). The elements \( x_n \) determine \( x \in M' \) such that \( y = f(x) \), proving exactness.

7. Since \( \hat{R} \otimes_R R \cong \hat{R} \) and tensor product commutes with direct sum, \( h_M \) is an isomorphism when \( M \) is free of finite rank. In general, we have an exact sequence

\[
0 \rightarrow N \xrightarrow{f} F \xrightarrow{g} M \rightarrow 0
\]

with \( F \) free of finite rank. Thus the following diagram is commutative, with exact rows.

\[
\begin{array}{c}
\hat{R} \otimes_R N \xrightarrow{h_N} \hat{R} \otimes_R F \xrightarrow{h_F} \hat{R} \otimes_R M \rightarrow 0
\end{array}
\]

See (4.2.7) for the last row. Since \( \hat{g} \) is surjective and \( h_F \) is an isomorphism, it follows that \( h_M \) is surjective.

8. By hypothesis, \( N \) is finitely generated, so by Problem 7, \( h_N \) is surjective. Since \( h_F \) is an isomorphism, \( h_M \) is injective by the four lemma. (See TBGY, 4.7.2, part (ii).)

9. Take inverse limits in (4.2.9).

10. Consider the diagram for Problem 7, with \( M \) finitely generated. No generality is lost; see TBGY, (10.8.1). Then all vertical maps are isomorphisms, so if we augment the first row by attaching \( 0 \rightarrow \) on the left, the first row remains exact. Thus the functor \( \hat{R} \otimes_R \) — is exact, proving that \( \hat{R} \) is flat.

11. Since \( M \) is isomorphic to its completion, we may regard \( \hat{M} \) as the set of constant sequences in \( M \). If \( x \) belongs to \( M_n \) for every \( n \), then \( x \) converges to 0, hence \( x \) and 0 are identified in \( M \). By (4.2.4), the topology is Hausdorff.

12. \( I \) is finitely generated, so by Problem 8, \( h_I : \hat{R} \otimes_R I \rightarrow \hat{I} \) is an isomorphism. Since \( \hat{R} \) is flat over \( \hat{R} \) by Problem 10, \( \hat{R} \otimes_R I \rightarrow \hat{R} \otimes_R R \cong \hat{R} \) is injective, and the image of this map is \( \hat{R} \).

13. By Problem 12, \( (I^n) \cong (\hat{R}I)^n \cong (\hat{I})^n \).

14. The following diagram is commutative, with exact rows.
The function $2^n$ is its own difference.

2. If $P$ is a prime ideal containing $\text{ann}(M/\mathcal{M}M)$, then $P \supseteq \mathcal{M}$, hence $P = \mathcal{M}$ by maximality of $\mathcal{M}$. Conversely, we must show that $\mathcal{M} \supseteq \text{ann}(M/\mathcal{M}M)$. This will be true unless $\text{ann}(M/\mathcal{M}M) = R$. In this case, 1 annihilates $M/\mathcal{M}M$, so $\mathcal{M}M = M$. By NAK, $M = 0$, contradicting the hypothesis.

3. Let $S = R \setminus P$. Then $(R/I)_P = 0$ iff $S^{-1}(R/I) = 0$ iff $S^{-1}R = S^{-1}I$ iff 1 is a unit in $S$ (if $I \cap S \neq \emptyset$). Thus $\dim R \geq \dim S$.

4. By Going Up [see (2.2.3)], any chain of distinct prime ideals of $R$ can be lifted to a chain of distinct prime ideals of $S$, so $\dim S \geq \dim R$. A chain of distinct prime ideals of $S$ contracts to a chain of prime ideals of $R$, distinct by (2.2.1). Thus $\dim R \geq \dim S$.

5. Since $S/J$ is integral over the subring $R/I$, it follows from (5.3.1) and Problem 4 that $\text{coht} I = \dim R/I = \dim S/J = \text{coht} J$.

6. If $J$ is a prime ideal of $S$, then $I = J \cap R$ is a prime ideal of $R$. The contraction of a chain of prime ideals of $S$ contained in $J$ is a chain of prime ideals of $R$ contained in $I$, and distinctness is preserved by (2.2.1). Thus $\text{ht} J \leq \text{ht} I$. Now let $J$ be any ideal of $S$, and let $P$ be a prime ideal of $R$ such that $P \supseteq I$ and $\text{ht} P = \text{ht} I$. (If the height of $I$ is infinite, there is nothing to prove.) As in the previous problem, $S/J$ is integral over $R/I$, so by Lying Over [see (2.2.2)] there is a prime ideal $Q$ containing $J$ that lies over $P$. Thus with the aid of the above proof for $J$ prime, we have $\text{ht} J \leq \text{ht} Q \leq \text{ht} P = \text{ht} I$.

7. First assume $J$ is a prime ideal of $S$, hence $I$ is a prime ideal of $R$. A descending chain of distinct prime ideals of $R$ starting from $I$ can be lifted to a descending chain of distinct prime ideals of $S$ starting from $J$, by Going Down [see (2.3.4)]. Thus $\text{ht} J \geq \text{ht} I$. For any ideal $J$, let $Q$ be a prime ideal of $S$ with $Q \supseteq J$. Then $P = Q \cap R \supseteq I$.

Chapter 5

1. The function $2^n$ is its own difference.

2. If $P$ is a prime ideal containing $\text{ann}(M/\mathcal{M}M)$, then $P \supseteq \mathcal{M}$, hence $P = \mathcal{M}$ by maximality of $\mathcal{M}$. Conversely, we must show that $\mathcal{M} \supseteq \text{ann}(M/\mathcal{M}M)$. This will be true unless $\text{ann}(M/\mathcal{M}M) = R$. In this case, 1 annihilates $M/\mathcal{M}M$, so $\mathcal{M}M = M$. By NAK, $M = 0$, contradicting the hypothesis.

3. Let $S = R \setminus P$. Then $(R/I)_P = 0$ iff $S^{-1}(R/I) = 0$ iff $S^{-1}R = S^{-1}I$ iff 1 is a unit in $S$ (if $I \cap S \neq \emptyset$). Thus $\dim R \geq \dim S$.

4. By Going Up [see (2.2.3)], any chain of distinct prime ideals of $R$ can be lifted to a chain of distinct prime ideals of $S$, so $\dim S \geq \dim R$. A chain of distinct prime ideals of $S$ contracts to a chain of prime ideals of $R$, distinct by (2.2.1). Thus $\dim R \geq \dim S$.

5. Since $S/J$ is integral over the subring $R/I$, it follows from (5.3.1) and Problem 4 that $\text{coht} I = \dim R/I = \dim S/J = \text{coht} J$.

6. If $J$ is a prime ideal of $S$, then $I = J \cap R$ is a prime ideal of $R$. The contraction of a chain of prime ideals of $S$ contained in $J$ is a chain of prime ideals of $R$ contained in $I$, and distinctness is preserved by (2.2.1). Thus $\text{ht} J \leq \text{ht} I$. Now let $J$ be any ideal of $S$, and let $P$ be a prime ideal of $R$ such that $P \supseteq I$ and $\text{ht} P = \text{ht} I$. (If the height of $I$ is infinite, there is nothing to prove.) As in the previous problem, $S/J$ is integral over $R/I$, so by Lying Over [see (2.2.2)] there is a prime ideal $Q$ containing $J$ that lies over $P$. Thus with the aid of the above proof for $J$ prime, we have $\text{ht} J \leq \text{ht} Q \leq \text{ht} P = \text{ht} I$.

7. First assume $J$ is a prime ideal of $S$, hence $I$ is a prime ideal of $R$. A descending chain of distinct prime ideals of $R$ starting from $I$ can be lifted to a descending chain of distinct prime ideals of $S$ starting from $J$, by Going Down [see (2.3.4)]. Thus $\text{ht} J \geq \text{ht} I$. For any ideal $J$, let $Q$ be a prime ideal of $S$ with $Q \supseteq J$. Then $P = Q \cap R \supseteq I$.
By what we have just proved, \( \text{ht } Q \geq \text{ht } P \), and \( \text{ht } P \geq \text{ht } I \) by definition of height. Taking the infimum over \( Q \), we have \( \text{ht } J \geq \text{ht } I \). By Problem 6, \( \text{ht } J = \text{ht } I \).

8. The chain of prime ideals \((X) \subset (X,Y) \subset (X,Y,Z)\) gives \( \dim R \geq 2 \). Since \( XY \) (or equally well \( XZ \)), belongs to the maximal ideal \((X,Y,Z)\) and is not a zero-divisor, we have \( \dim R \leq \dim S/(XY) = \dim S - 1 = 2 \) by (5.4.7) and (5.4.9).

9. The height of \( P \) is 0 because the ideals \((Y)\) and \((Z)\) are not prime. For example, \( X/\in \in (Y) \) and \( Z/\in \in (Y) \), but \( XZ = 0 \in (Y) \). Since \( R/P \sim k[[X]] \) has dimension 1, \( P \) has coheight 1 by (5.3.1).

Chapter 6

1. By (6.1.3), \( \dim R/P = \dim R - t = \dim R - \text{ht } P \). By (5.3.1), \( \dim R/P = \text{coht } P \), and the result follows.

2. Let \( J \) be the ideal \( (Z, X + Y) \). If \( M = (X,Y,Z) \) is the unique maximal ideal of \( S \), then \( \overline{M}^2 = (X^2, Y^2, Z^2, YZ) \subseteq J \subseteq M \), so \( J \) is an ideal of definition. (Note that \( X \neq Y \) and \( Z \neq Y \), but \( XZ = 0 \).) By (6.1.2), \( \{Z, X + Y\} \) is a system of parameters. Since \( X \neq Y \), \( Z \) is a zero-divisor.

Chapter 7

1. Note that \( \ker f, \text{im } f, \) and \( \ker g \) are all equal to \( \{0, 2\} \).

2. We have \( \text{im } \partial_n = \ker f_{n-1} = 0 \) and \( \ker g_n = \text{im } f_n = B_n \). Thus \( g_n \) is the zero map, so \( \ker \partial_n = \text{im } g_n = 0 \). Therefore \( \partial_n \) is an injective zero map, which forces \( C_n = 0 \).

3. This follows from the base change formula \( R/I \otimes_R M \cong M/IM \) with \( I = M \) (see TBGY, S7.1).

4. We have \( g^*: 1 \otimes e_i \to 1 \otimes x_i \), which is an isomorphism. (The inverse is \( 1 \otimes x_i \to 1 \otimes e_i \).)

Thus \( \text{im } f^* = \ker g^* = 0 \). Since \( f^* \) is the zero map, \( \delta \) is surjective. But \( \ker u_M = 0 \) by hypothesis, so \( \delta = 0 \). This forces \( \text{coker } u_K = 0 \).

5. By Problem 4, \( K = MK \). Since \( M \) is a Noetherian \( R \)-module, \( K \) is finitely generated, so by NAK we have \( K = 0 \). Thus \( 0 = \text{im } f = \ker g \), so \( g \) is injective.

6. Since free implies projective implies flat always, it suffices to show that flat implies free. If \( M \) is flat, then the functor \( N \to N \otimes_R M \) is exact. If \( M \) is the maximal ideal of \( R \), then the map \( \mathcal{M} \otimes_R M \to R \otimes_R M \cong M \) via \( a \otimes x \to ax \) is injective. But this map is just \( u_M \), and the result follows from Problems 3-5.

7. We have the short exact sequence \( 0 \to I \to R \to R/I \to 0 \), which induces, for any \( R \)-module \( N \), the exact sequence

\[
\text{Hom}_R(R/I, N) \to \text{Hom}_R(R, N) \to \text{Hom}_R(I, N) \to \text{Ext}_R^1(R/I, N).
\]

The last term is 0 by hypothesis, hence the map \( i^*: \text{Hom}_R(R, N) \to \text{Hom}_R(I, N) \) is surjective. This says, by Baer’s criterion (TBGY 10.6.4), that \( N \) is injective.
8. The left side is at least equal to the right side by (7.2.5), so assuming that the right side is at most \( n \), it suffices to show that \( \text{id}_R N \leq n \) for all \( N \). Given an exact sequence as in (7.2.4) part 4, dimension shifting yields \( \text{Ext}^{n+1}_R(R/I, N) \cong \text{Ext}^1_R(R/I, C_{n-1}) \). By (7.2.4), \( \text{Ext}^1_R(R/I, C_{n-1}) = 0 \), so by (7.2.1) and Problem 7, \( C_{n-1} \) is injective. By (7.2.4), \( \text{id}_R N \leq n \).

9. If (a) holds, only the second assertion of (b) requires proof. Apply Tor to the exact sequence \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) to get the exact sequence

\[
0 = \text{Tor}^R_1(M, N') \rightarrow M \otimes_R N' \rightarrow M \otimes_R N \rightarrow M \otimes_R N'' \rightarrow 0.
\]

We may replace \( M \otimes_R N \) by \( (M \otimes_R S) \otimes_S N \), and similarly for the other two tensor products. By exactness, \( M \otimes_R S \) is flat. Now assuming (b), we have \( \text{Tor}_1^R(M, F) = 0 \) for every free \( S \)-module \( F \), because Tor commutes with direct sums. If \( N \) is an arbitrary \( S \)-module, we have a short exact sequence \( 0 \to K \to F \to N \to 0 \) with \( F \) free. The corresponding (truncated) long exact sequence is

\[
0 = \text{Tor}_1^R(M, F) \rightarrow \text{Tor}_1^R(M, N) \rightarrow M \otimes_R K \rightarrow M \otimes_R F \rightarrow M \otimes_R N \rightarrow 0.
\]

As before, we replace \( M \otimes_R K \) by \( (M \otimes_R S) \otimes_S K \), and similarly for the other two tensor products. The map whose domain is \( (M \otimes_R S) \otimes_S K \) is induced by the inclusion of \( K \) into \( F \), and is therefore injective, because \( M \otimes_R S \) is a flat \( S \)-module by hypothesis. Thus the kernel of the map, namely \( \text{Tor}_1^R(M, N) \), is zero.

Chapter 8

1. To ease the notation we will omit all the overbars and adopt the convention that all calculations are mod \( (X^3 - Y^2) \). We have \( (X^2 + X + 1)(X - 1) = X^3 - 1 = Y^2 - 1 = (Y - 1)(Y + 1) \). Now \( X^2 + X + 1 \) and \( Y + 1 \) are units in \( R \) because they do not vanish when \( X = Y = 1 \), assuming that the characteristic of \( K \) is not 2 or 3. Thus \( X - 1 \) and \( Y - 1 \) are associates.

2. The maximal ideal is not principal because \( \mathbf{X} \) and \( \mathbf{Y} \) cannot both be multiples of a single polynomial. To show that \( \dim R = 1 \), we use (5.6.7). Since \( K(Y) \) has transcendence degree 1 over \( K \) and \( K(X, Y)/(X^3 - Y^2) \) is algebraic over \( K(Y) \), (we are adjoining a root of \( X^3 - Y^2 \)), it follows that the dimension of \( K[X, Y]/(X^3 - Y^2) \) is 1. By (5.3.1), the colheight of \( (X^3 - Y^2) \) is 1, and the corresponding sequence of prime ideals is \( (X^3 - Y^2), (X, Y) \). Thus localization at \( (\mathbf{X}, \mathbf{Y}) \) has no effect on dimension, so \( \dim R = 1 \). (In general, prime ideals of a localized ring \( A_P \) correspond to prime ideals of \( A \) that are contained in \( P \), so localization may reduce the dimension.)

3. By definition, the Hilbert polynomial is the composition length \( l_k(I^n/I^{n+1}) \). Since monomials of degree \( n \) in \( r \) variables form a basis for the polynomials of degree \( n \), we must count the number of such monomials, which is

\[
\binom{n + r - 1}{r - 1} = \frac{(n + r - 1)(n + r - 2) \cdots (n + 2)(n + 1)}{(r - 1)!}
\]

This is a polynomial of degree \( r - 1 \) in the variable \( n \).
4. This follows from Problem 3 and additivity of length (5.2.3).

5. Fix a nonzero element $b \in B_d$. (Frequently, $b$ is referred to as a homogeneous element of degree $d$.) By definition of a graded ring, we have $bA_n \subseteq B_{n+d}$ for $n \geq 0$. Then

$$l_k(B_{n+d}) \geq l_k(bA_n) = l_k(A_n) \geq l_k(B_n).$$

Since $l_k(A_n) = \binom{n+r-1}{r-1}$, the result follows.

6. If $R$ is regular, we may define the graded $k$-algebra homomorphism $\varphi$ of Problems 3-5 with $r = d$. Since the Hilbert polynomial has degree $d - 1$, $\varphi$ is an isomorphism. Conversely, an isomorphism of graded $k$-algebras induces an isomorphism of first components, in other words,

$$(k[X_1, \ldots, X_d])_1 \cong \mathcal{M}/\mathcal{M}^2.$$

But the $k$-vector space on the left has a basis consisting of all monomials of degree 1. Since there are exactly $d$ of these, we have $\dim_k \mathcal{M}/\mathcal{M}^2 = d$. By (8.1.3), $R$ is regular.