Exercises

Chapter 1

1. What are the primary ideals of $\mathbb{Z}$?
2. Let $R = k[x, y]$ where k is a field. Show that $Q = (x, y^2)$ is P-primary, and identify P.
3. Continuing Problem 2, show that Q is not a power of a prime ideal.
4. Let $R = k[x, y, z]/I$ where $I = (xy - z^2)$. Let $\overline{x} = x + I,\ \overline{y} = y + I,\ \overline{z} = z + I$. If $\overline{P} = (\overline{x}, \overline{z})$, show that $\overline{P}^2$ is a power of a prime ideal and its radical is prime, but it is not primary.
5. Let $R = k[x, y]$ where k is a field, and let $P_1 = (x),\ P_2 = (x, y),\ Q = (x^2, y),\ I = (x^2, xy)$. Show that $I = P_1 \cap P_2^2$ and $I = P_1 \cap Q$ are both primary decompositions of I.
6. Let M and N be finitely generated modules over a local ring R. Show that $M \otimes_R N = 0$ if and only if either M or N is 0.
7. Continuing Problem 6, show that the result fails to hold if R is not local.
8. Let S be a multiplicative subset of R, and $M_S = S^{-1}M$. Use base change formulas in the tensor product to show that $(M \otimes_R N)_S \cong M_S \otimes_{R_S} N_S$ as R_S-modules.
9. If M and N are finitely generated R-modules, show that $\operatorname{Supp}(M \otimes_R N) = \operatorname{Supp} M \cap \operatorname{Supp} N$.

   In Problems 10-13, we consider uniqueness in the structure theorem (1.6.7) for Artinian rings.

10. Let $R = \prod_1^r R_i$, where the R_i are Artinian local rings, and let π_i be the projection of R on R_i. Show that each R_i has a unique prime ideal P_i, which is nilpotent. Then show that $\mathcal{M}_i = \pi_i^{-1}(P_i)$ is a maximal ideal of R.
11. Let $I_i = \ker \pi_i, i = 1, \dots, r$. Show that $\sqrt{I_i} = \mathcal{M}_i$, so by (1.1.2), I_i is $\mathcal{M}_i$-primary.
12. Show that $\cap_1^r I_i$ is a reduced primary decomposition of the zero ideal.
13. Show that in (1.6.7), the R_i are unique up to isomorphism.
14. Let M be finitely generated over the Noetherian ring R, and let P be a prime ideal in the support of M. Show that $l_{R_P}(M_P) < \infty$ if and only if P is a minimal element of $\mathrm{AP}(M)$.

Chapter 2

1. Let $R = \mathbb{Z}$ and $S = \mathbb{Z}[i]$, the Gaussian integers. Give an example of two prime ideals of S lying above the same prime ideal of R. (By (2.2.1), there cannot be an inclusion relation between the prime ideals of S.)
2. Let $R = k[X,Y]/I$, where k is a field and I is the prime ideal $(X^2 - Y^3)$. Write the coset $X + I$ simply as x, and $Y + I$ as y. Show that $\alpha = x/y$ is integral over R, but $\alpha \notin R$. Thus R is not integrally closed.
3. Suppose we have a diagram of R-modules

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

   with $\operatorname{im} f \subseteq \ker g$. Show that the following conditions are equivalent.

   (a) The given sequence is exact.

   (b) The sequence

$$M'_P \xrightarrow{f_P} M_P \xrightarrow{g_P} M''_P$$

   is exact for every prime ideal P.

   (c) The localized sequence of (b) is exact for every maximal ideal P.
4. Let $f : M \to N$ be an R-module homomorphism. Show that f is injective [resp. surjective] if and only if f_P is injective [resp. surjective] for every prime, equivalently for every maximal, ideal P.
5. Let R be an integral domain with fraction field K. We may regard all localized rings R_P as subsets of K. Let M be the intersection of all R_P for maximal ideals P. If S is any multiplicative subset of R, show that

$$S^{-1}M \subseteq \bigcap_{P \in \max R} S^{-1}R_P$$

   where $\max R$ is the set of maximal ideals of R.
6. Continuing Problem 5, if Q is any maximal ideal of R, show that $M_Q \subseteq R_Q$.
7. Continuing Problem 6, show that the intersection of all R_P, P prime, coincides with the intersection of all R_P, P maximal, and in fact both intersections coincide with R.
8. If R is an integral domain, show that the following conditions are equivalent:

   (a) R is integrally closed;

   (b) R_P is integrally closed for every prime ideal P;

   (c) R_Q is integrally closed for every maximal ideal Q.
9. Let P be a prime ideal of R. Show that the fields R_P/PR_P and $\operatorname{Frac} R/P$ are isomorphic. Each is referred to as the *residue field* at P.

Chapter 3

Let R and S be local subrings of the field K, with maximal ideals $\mathcal{M}_R$ and $\mathcal{M}_S$ respectively. We say that S *dominates* R, and write $(R, \mathcal{M}_R) \leq (S, \mathcal{M}_S)$, if R is a subring of S and $R \cap \mathcal{M}_S = \mathcal{M}_R$.

1. If V is a valuation ring of K, show that $(V, \mathcal{M}_V)$ is maximal with respect to the partial ordering induced by domination.

   Conversely, we will show in Problems 2 and 3 that if $(V, \mathcal{M}_V)$ is maximal, then V is a valuation ring. Let k be the residue field $V/\mathcal{M}_V$, and let C be an algebraic closure of k. We define a homomorphism $h : V \to C$, by following the canonical map from V to k by the inclusion map of k into C. By (3.1.4), it suffices to show that (V, h) is a maximal extension. As in (3.1.1), if (R_1, h_1) is an extension of (V, h), we may assume R_1 local and $h_1(R_1)$ a subfield of C. Then $\ker h_1$ is the unique maximal ideal $\mathcal{M}_{R_1}$.

2. Show that $(R_1, \mathcal{M}_{R_1})$ dominates $(V, \mathcal{M}_V)$.
3. Complete the proof by showing that (V, h) is a maximal extension.
4. Show that every local subring of a field K is dominated by at least one valuation ring of K.

Chapter 4

1. Let R be the formal power series ring $k[[X_1, \dots, X_n]]$, where k is a field. Put the I-adic filtration on R, where I is the unique maximal ideal $(X_1, \dots, X_n)$. Show that the associated graded ring of R is the polynomial ring $k[X_1, \dots, X_n]$.
2. Let M and N be filtered modules over the filtered ring R. The R-homomorphism $f : M \to N$ is said to be a *homomorphism of filtered modules* if $f(M_n) \subseteq N_n$ for all $n \geq 0$. For each n, f induces a homomorphism $\overline{f}_n : M_n/M_{n+1} \to N_n/N_{n+1}$ via $\overline{f}_n(x + M_{n+1}) = f(x) + N_{n+1}$. We write $\mathrm{gr}_n(f)$ instead of $\overline{f}_n$. The $\mathrm{gr}_n(f)$ extend to a homomorphism of graded $\mathrm{gr}(R)$-modules, call it $\mathrm{gr}(f) : \mathrm{gr}(M) \to \mathrm{gr}(N)$. We write

   $$\mathrm{gr}(f) = \bigoplus_{n \geq 0} \mathrm{gr}_n(f).$$

   For the remainder of this problem and in Problems 3 and 4, we assume that $\mathrm{gr}(f)$ is injective. Show that $M_n \cap f^{-1}(N_{n+1}) \subseteq M_{n+1}$ for all $n \geq 0$.
3. Continuing Problem 2, show that $f^{-1}(N_n) \subseteq M_n$ for all $n \geq 0$.
4. Continuing Problem 3, show that if in addition we have $\cap_{n=0}^{\infty} M_n = 0$, then f is injective.
5. Show that in (4.2.10), the two filtrations $\{I^n N\}$ and $\{N \cap I^n M\}$ are equivalent.
6. If we reverse the arrows in the definition of an inverse system [see (4.2.1)], so that maps go from M_n to M_{n+1}, we get a *direct system.* The *direct limit* of such a system is the disjoint union $\coprod M_n$, with sequences x and y identified if they agree sufficiently far out in the ordering. In other words, $\theta_n(x_n) = \theta_n(y_n)$ for all sufficiently large n.

   In (4.2.6) we proved that the inverse limit functor is left exact, and exact under an additional assumption. Show that the direct limit functor is always exact. Thus if $M'_n \xrightarrow{f_n} M_n \xrightarrow{g_n} M''_n$ is exact for all n, and

   $$M = \varinjlim M_n$$

is the direct limit of the M_n (similarly for M' and M''), then the sequence

$M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact. (The maps f and g are induced by the f_n and g_n.)

7. Let M be an R-module, and let $\hat{M}$ [resp. $\hat{R}$] be the I-adic completion of M [resp. R]. Note that $\hat{M}$ is an $\hat{R}$-module via $\overline{\{a_n\}}\ \overline{\{x_n\}} = \overline{\{a_n x_n\}}$. Define an R-module homomorphism $h_M : \hat{R} \otimes_R M \to \hat{M}$ by $(\overline{r}, m) \to \overline{r}\ m$. If M is finitely generated over R, show that h_M is surjective.
8. In Problem 7, if in addition R is Noetherian, show that h_M is an isomorphism. Thus if R is complete ($R \cong \hat{R}$), then M is complete ($M \cong \hat{M}$).
9. Show that the completion of M is always complete, that is, $\hat{\hat{M}} \cong \hat{M}$.
10. Let $\hat{R}$ be the I-adic completion of the Noetherian ring R. Show that $\hat{R}$ is a flat R-module.
11. If M is complete with respect to the filtration $\{M_n\}$, show that the topology induced on M by $\{M_n\}$ must be Hausdorff.

In Problems 12-16, $\hat{R}$ is the I-adic completion of the ring R. In Problems 12-14, R is assumed Noetherian.

12. Show that $\hat{I} \cong \hat{R} \otimes_R I \cong \hat{R}I$.
13. Show that $(\hat{I})^n \cong (I^n)\hat{}$.
14. Show that $I^n/I^{n+1} \cong (\hat{I})^n/(\hat{I})^{n+1}$.
15. Show that $\hat{I}$ is contained in the Jacobson radical $J(\hat{R})$.
16. Let R be a local ring with maximal ideal $\mathcal{M}$. If $\hat{R}$ is the $\mathcal{M}$-adic completion of R, show that $\hat{R}$ is a local ring with maximal ideal $\hat{\mathcal{M}}$.

Chapter 5

1. In differential calculus, the exponential function e^x is its own derivative. What is the analogous statement in the calculus of finite differences?
2. Let M be nonzero and finitely generated over the local ring R with maximal ideal $\mathcal{M}$. Show that $V(\operatorname{ann}(M/\mathcal{M}M)) = \{\mathcal{M}\}$.
3. If I is an arbitrary ideal and P a prime ideal of R, show that $(R/I)_P \neq 0$ iff $P \supseteq I$.

In Problems 4-7, the ring S is integral over the subring R, J is an ideal of S, and $I = J \cap R$. Establish the following.

4. $\dim R = \dim S$.
5. coht $I =$ coht J.
6. ht $J \leq$ ht I.
7. If R and S are integral domains with R integrally closed, then ht $J =$ ht I.

If P is a prime ideal of R, then by definition of height, coheight and dimension, we have ht $P+$ coht $P \leq \dim R$. In Problems 8 and 9 we show that the inequality can be strict, even if R is Noetherian. Let $S = k[[X, Y, Z)]]$ be a formal power series ring over the field k, and let $R = S/I$ where $I = (XY, XZ)$. Define $\overline{X} = X + I$, $\overline{Y} = Y + I$, $\overline{Z} = Z + I$.

8. Show that the dimension of R is 2.
9. Let P be the prime ideal $(\overline{Y}, \overline{Z})$ of R. Show that P has height 0 and coheight 1, so that ht $P+$ coht $P < \dim R$.

Chapter 6

1. Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$, and suppose that the elements $a_1, \dots, a_t$ are part of a system of parameters for R. If the ideal $P = (a_1, \dots, a_t)$ is prime and has height t, show that $\operatorname{ht} P + \operatorname{coht} P = \dim R$.
2. let $S = k[[X, Y, Z]]$ be a formal power series ring over the field k, and let $R = S/I$, where $I = (XY, XZ)$. Use an overbar to denote cosets mod I, for example, $\overline{X} = X + I \in R$. Show that $\{\overline{Z}, \overline{X} + \overline{Y}\}$ is a system of parameters, but $\overline{Z}$ is a zero-divisor. On the other hand, members of a regular sequence (Section 6.2) cannot be zero-divisors.

Chapter 7

1. Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$, a free $\mathbb{Z}_4$-module. Define $f : \mathbb{Z}_4 \to \mathbb{Z}_4$ by $0 \to 0, 1 \to 2, 2 \to 0, 3 \to 2$, i.e., $f(x) = 2x \mod 4$. Let $M = 2\mathbb{Z}_4 \cong \mathbb{Z}_2$ (also a $\mathbb{Z}_4$-module), and define $g : \mathbb{Z}_4 \to M$ by $0 \to 0, 1 \to 1, 2 \to 0, 3 \to 1$, i.e., $g(x) = x \mod 2$. Show that $\cdots \longrightarrow \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{g} M \longrightarrow 0$ is a free, hence projective, resolution of M of infinite length.
2. Given an exact sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{f_{n-1}} B_{n-1} \xrightarrow{g_{n-1}} C_{n-1} \longrightarrow \cdots$$

Show that if the maps f_n are all isomorphisms, then $C_n = 0$ for all n.

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field $k = R/\mathcal{M}$. Let M be a finitely generated R-module, and define $u_M : \mathcal{M} \otimes_R M \to M$ via $u_M(a \otimes x) = ax, a \in \mathcal{M}, x \in M$. We are going to show in Problems 3,4 and 5 that if u_M is injective, then M is free. If M is generated by $x_1, \dots, x_n$, let F be a free R-module with basis $e_1, \dots, e_n$. Define a homomorphism $g : F \to M$ via $e_i \to x_i, 1 \le i \le n$. We have an exact sequence $0 \to K \to F \to M \to 0$, where $f : K \to F, g : F \to M$, and $K = \ker g$. The following diagram is commutative, with exact rows.

$$\begin{array}{ccccccccc} & & \mathcal{M} \otimes K & \longrightarrow & \mathcal{M} \otimes F & \longrightarrow & \mathcal{M} \otimes M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_K} & & \downarrow{\scriptstyle u_F} & & \downarrow{\scriptstyle u_M} & & \\ 0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & & \end{array}$$

Applying the snake lemma, we have an exact sequence

$$\ker u_M \xrightarrow{\delta} \operatorname{coker} u_K \xrightarrow{f^*} \operatorname{coker} u_F \xrightarrow{g^*} \operatorname{coker} u_M.$$

3. Show that $\operatorname{coker} u_M \cong k \otimes_R M$, and similarly $\operatorname{coker} u_F \cong k \otimes_R F$.
4. Show that $\operatorname{coker} u_K = 0$.
5. Show that g is injective. Since g is surjective by definition, it is an isomorphism, hence $M \cong F$, and M is free.
6. Let M be a finitely generated module over the Noetherian local ring R. Show that M is free if and only if M is projective, if and only if M is flat.
7. Show that in (7.2.1), M can be replaced by R/I, I an arbitrary ideal of R.
8. Show that the global dimension of a ring R is the least upper bound of $\mathrm{id}_R(R/I)$, where I ranges over all ideals of R.
9. Let $f : R \to S$ be a ring homomorphism, and let M be an R-module. Prove that the following conditions are equivalent.
   (a) $\mathrm{Tor}_1^R(M, N) = 0$ for all S-modules N.
   (b) $\mathrm{Tor}_1^R(M, S) = 0$ and $M \otimes_R S$ is a flat S-module.

Chapter 8

1. In (8.1.2), Example 4, show that $\overline{X} - 1$ and $\overline{Y} - 1$ are associates.
2. Justify the assertions made in Example 5 of (8.1.2).

   Let $(R, \mathcal{M}, k)$ be a Noetherian local ring, and let $\mathrm{gr}_{\mathcal{M}}(R)$ be the associated graded ring with respect to the $\mathcal{M}$-adic filtration [see(4.1.2)]. We can define a homomorphism of graded k-algebras $\varphi : k[X_1, \dots , X_r] \to \mathrm{gr}_{\mathcal{M}}(R)$ via $\varphi(X_i) = a_i + \mathcal{M}^2$, where the a_i generate $\mathcal{M}$. (See Chapter 4, Problem 2 for terminology.) In Problems 3-5, we are going to show that φ is an isomorphism if and only if the Hilbert polynomial $h(n) = h(\mathrm{gr}_{\mathcal{M}}(R), n)$ has degree $r - 1$. Equivalently, the Hilbert-Samuel polynomial $s_{\mathcal{M}}(R, n)$ has degree r.
3. Assume that φ is an isomorphism, and let A_n be the set of homogeneous polynomials of degree n in $k[X_1, \dots , X_r]$. Then A_n is isomorphic as a k-vector space to $I = (X_1, \dots , X_r)$. Compute the Hilbert polynomial of $\mathrm{gr}_{\mathcal{M}}(R)$ and show that it has degree $r - 1$.
4. Now assume that φ is not an isomorphism, so that its kernel B is nonzero. Then B becomes a graded ring $\oplus_{n\geq 0} B_n$ with a grading inherited from the polynomial ring $A = k[X_1, \dots , X_r]$. We have an exact sequence

   $$0 \to B_n \to A_n \to \mathcal{M}^n/\mathcal{M}^{n+1} \to 0.$$

   Show that

   $$h(n) = \binom{n + r - 1}{r - 1} - l_k(B_n).$$

5. Show that the polynomial-like functions on the right side of the above equation for $h(n)$ have the same degree and the same leading coefficient. It follows that the Hilbert polynomial has degree less than $r - 1$, completing the proof.

6. Let $(R, \mathcal{M}, k)$ be a Noetherian local ring of dimension d. Show that R is regular if and only if the associated graded ring $\mathrm{gr}_{\mathcal{M}}(R)$ is isomorphic as a graded k-algebra to $k[X_1, \dots, X_d]$.