Chapter 8

Regular Local Rings

In algebraic geometry, the local ring of an affine algebraic variety V at a point P is the set $\mathcal{O}(P, V)$ of rational functions on V that are defined at P. Then P will be a nonsingular point of V if and only if $\mathcal{O}(P, V)$ is a regular local ring.

8.1 Basic Definitions and Examples

8.1.1 Definitions and Comments

Let $(R, \mathcal{M}, k)$ be a Noetherian local ring. (The notation means that the maximal ideal is $\mathcal{M}$ and the residue field is $k = R/\mathcal{M}$.) If d is the dimension of R, then by the dimension theorem [see (5.4.1)], every generating set of $\mathcal{M}$ has at least d elements. If $\mathcal{M}$ does in fact have a generating set S of d elements, we say that R is *regular* and that S is a *regular system of parameters*. (Check the definition (6.1.1) to verify that S is indeed a system of parameters.)

8.1.2 Examples

1. If R has dimension 0, then R is regular iff $\{0\}$ is a maximal ideal, in other words, iff R is a field.

2. If R has dimension 1, then by (3.3.11), condition (3), R is regular iff R is a discrete valuation ring. Note that (3.3.11) assumes that R is an integral domain, but this is not a problem because we will prove shortly that every regular local ring is a domain.

3. Let $R = K[[X_1, \dots, X_d]]$, where K is a field. By (5.4.9), $\dim R = d$, hence R is regular and $\{X_1, \dots, X_d\}$ is a regular system of parameters.

4. Let K be a field whose characteristic is not 2 or 3, and let $R = K[X, Y]/(X^3 - Y^2)$, localized at the maximal ideal $\mathcal{M} = \{\overline{X} - 1, \overline{Y} - 1\}$. (The overbars indicate calculations mod $(X^3 - Y^2)$.) It appears that $\{\overline{X} - 1, \overline{Y} - 1\}$ is a minimal generating set for $\mathcal{M}$, but this is not the case (see Problem 1). In fact $\mathcal{M}$ is principal, hence $\dim R = 1$ and R is regular. (See Example 2 above, and note that R is a domain because $X^3 - Y^2$ is irreducible, so $(X^3 - Y^2)$ is a prime ideal.)

5. Let R be as in Example 4, except that we localize at $\mathcal{M} = (\overline{X}, \overline{Y})$ and drop the restriction on the characteristic of K. Now it takes two elements to generate $\mathcal{M}$, but $\dim R = 1$ (Problem 2). Thus R is not regular.

Here is a convenient way to express regularity.

8.1.3 Proposition

Let $(R, \mathcal{M}, k)$ be a Noetherian local ring. Then R is regular if and only if the dimension of R coincides with $\dim_k \mathcal{M}/\mathcal{M}^2$, the dimension of $\mathcal{M}/\mathcal{M}^2$ as a vector space over k. (See (3.3.11), condition (6), for a prior appearance of this vector space.)

Proof. Let d be the dimension of R. If R is regular and $a_1, \dots, a_d$ generate $\mathcal{M}$, then the $a_i + \mathcal{M}^2$ span $\mathcal{M}/\mathcal{M}^2$, so $\dim_k \mathcal{M}/\mathcal{M}^2 \le d$. But the opposite inequality always holds (even if R is not regular), by (5.4.2). Conversely, if $\{a_1 + \mathcal{M}^2, \dots, a_d + \mathcal{M}^2\}$ is a basis for $\mathcal{M}/\mathcal{M}^2$, then the a_i generate $\mathcal{M}$. (Apply (0.3.4) with $J = M = \mathcal{M}$.) Thus R is regular. ♣

8.1.4 Theorem

A regular local ring is an integral domain.

Proof. The proof of (8.1.3) shows that the associated graded ring of R, with the $\mathcal{M}$-adic filtration [see (4.1.2)], is isomorphic to the polynomial ring $k[X_1, \dots, X_d]$, and is therefore a domain (Problem 6). The isomorphism identifies a_i with X_i, $i = 1, \dots, d$. By the Krull intersection theorem, $\cap_{n=0}^{\infty} \mathcal{M}^n = 0$. (Apply (4.3.4) with $M = R$ and $I = \mathcal{M}$.) Now let a and b be nonzero elements of R, and choose m and n such that $a \in \mathcal{M}^m \setminus \mathcal{M}^{m+1}$ and $b \in \mathcal{M}^n \setminus \mathcal{M}^{n+1}$. Let $\overline{a}$ be the image of a in $\mathcal{M}^m/\mathcal{M}^{m+1}$ and let $\overline{b}$ be the image of b in $\mathcal{M}^n/\mathcal{M}^{n+1}$. Then $\overline{a}$ and $\overline{b}$ are nonzero, hence $\overline{a}\,\overline{b} \neq 0$ (because the associated graded ring is a domain). But $\overline{a}\,\overline{b} = \overline{ab}$, the image of ab in $\mathcal{M}^{m+n+1}$, and it follows that ab cannot be 0. ♣

We now examine when a sequence can be extended to a regular system of parameters.

8.1.5 Proposition

Let $(R, \mathcal{M}, k)$ be a regular local ring of dimension d, and let $a_1, \dots, a_t \in \mathcal{M}$, where $1 \le t \le d$. The following conditions are equivalent.

(1) $a_1, \dots, a_t$ can be extended to a regular system of parameters for R.
(2) $\overline{a}_1, \dots, \overline{a}_t$ are linearly independent over k, where $\overline{a}_i = a_i \mod \mathcal{M}^2$.
(3) $R/(a_1, \dots, a_t)$ is a regular local ring of dimension $d - t$.

Proof. The proof of (8.1.3) shows that (1) and (2) are equivalent. Specifically, the a_i extend to a regular system of parameters iff the $\overline{a}_i$ extend to a k-basis of $\mathcal{M}/\mathcal{M}^2$. To prove that (1) implies (3), assume that $a_1, \dots, a_t, a_{t+1}, \dots, a_d$ is a regular system of parameters for R. By (6.1.3), the dimension of $\overline{R} = R/(a_1, \dots, a_t)$ is $d - t$. But the $d - t$ elements $\overline{a}_i, i = t+1, \dots, d$, generate $\overline{\mathcal{M}} = \mathcal{M}/(a_1, \dots, a_t)$, hence $\overline{R}$ is regular.

Now assume (3), and let $a_{t+1}, \dots, a_d$ be elements of $\mathcal{M}$ whose images in $\overline{\mathcal{M}}$ form a regular system of parameters for $\overline{R}$. If $x \in \mathcal{M}$, then modulo $I = (a_1, \dots, a_t)$, we have

$x - \sum_{t+1}^{d} c_i a_i = 0$ for some $c_i \in R$. In other words, $x - \sum_{t+1}^{d} c_i a_i \in I$. It follows that $a_1, \ldots, a_t, a_{t+1}, \ldots, a_d$ generate $\mathcal{M}$. Thus R is regular (which we already know by hypothesis) and $a_1, \ldots, a_t$ extend to a regular system of parameters for R. ♣

8.1.6 Theorem

Let $(R, \mathcal{M}, k)$ be a Noetherian local ring. Then R is regular if and only if $\mathcal{M}$ can be generated by an R-sequence. The length of any such R-sequence is the dimension of R.

Proof. Assume that R is regular, with a regular system of parameters $a_1, \ldots, a_d$. If $1 \leq t \leq d$, then by (8.1.5), $\overline{R} = R/(a_1, \ldots, a_t)$ is regular and has dimension $d - t$. The maximal ideal $\overline{\mathcal{M}}$ of $\overline{R}$ can be generated by $\overline{a}_{t+1}, \ldots, \overline{a}_d$, so these elements form a regular system of parameters for $\overline{R}$. By (8.1.4), $\overline{a}_{t+1}$ is not a zero-divisor of $\overline{R}$, in other words, a_{t+1} is not a zero-divisor of $R/(a_1, \ldots, a_t)$. By induction, $a_1, \ldots, a_d$ is an R-sequence. (To start the induction, set $t = 0$ and take $(a_1, \ldots, a_t)$ to be the zero ideal.)

Now assume that $\mathcal{M}$ is generated by the R-sequence $a_1, \ldots, a_d$. By repeated applicaion of (5.4.7), we have $\dim R/\mathcal{M} = \dim R - d$. But $R/\mathcal{M}$ is the residue field k, which has dimension 0. It follows that $\dim R = d$, so R is regular. ♣

8.1.7 Corollary

A regular local ring is Cohen-Macaulay.

Proof. By (8.1.6), the maximal ideal $\mathcal{M}$ of the regular local ring R can be generated by an R-sequence $a_1, \ldots, a_d$, with (necessarily) $d = \dim R$. By definition of depth [see(6.2.5)], $d \leq$ depth R. But by (6.2.6), depth $R \leq \dim R$. Since $\dim R = d$, it follows that depth$R = \dim R$. ♣