Chapter 7

Homological Methods

We now begin to apply homological algebra to commutative ring theory. We assume as background some exposure to derived functors and basic properties of Ext and Tor. In addition, we will use standard properties of projective and injective modules. Everything we need is covered in TBGY, Chapter 10 and the supplement.

7.1 Homological Dimension: Projective and Global

Our goal is to construct a theory of dimension of a module M based on possible lengths of projective and injective resolutions of M.

7.1.1 Definitions and Comments

A projective resolution $0 \to X_n \to \cdots \to X_0 \to M \to 0$ of the R-module M is said to be of *length* n. The smallest such n is called the *projective dimension* of M, denoted by $\operatorname{pd}_R M$. (If M has no finite projective resolution, we set $\operatorname{pd}_R M = \infty$.)

7.1.2 Lemma

The projective dimension of M is 0 if and only if M is projective.

Proof. If M is projective, then $0 \to X_0 = M \to M \to 0$ is a projective resolution, where the map from M to M is the identity. Conversely, if $0 \to X_0 \to M \to 0$ is a projective resolution, then $M \cong X_0$, hence M is projective. ♣

7.1.3 Lemma

If R is a PID, then for every R-module M, $\operatorname{pd}_R M \le 1$. If M is an abelian group whose torsion subgroup is nontrivial, then $\operatorname{pd}_R M = 1$.

Proof. There is an exact sequence $0 \to X_1 \to X_0 \to M \to 0$ with X_0 free and X_1, a submodule of a free module over a PID, also free. Thus $\operatorname{pd}_R M \le 1$. If $\operatorname{pd}_R M = 0$, then by (7.1.2), M is projective, hence free because R is a PID. Since a free module has zero torsion, the second assertion follows. ♣

7.1.4 Definition

The *global dimension* of a ring R, denoted by $\operatorname{gldim} R$, is the least upper bound of $\operatorname{pd}_R M$ as M ranges over all R-modules.

7.1.5 Remarks

If R is a field, then every R-module is free, so $\operatorname{gldim} R = 0$. By (7.1.3), a PID has global dimension at most 1. Since an abelian group with nonzero torsion has projective dimension 1, $\operatorname{gldim} \mathbb{Z} = 1$.

We will need the following result from homological algebra; for a proof, see TBGY, subsection S5.7.

7.1.6 Proposition

If M is an R-module, the following conditions are equivalent.

(i) M is projective;

(ii) $\operatorname{Ext}_R^n(M, N) = 0$ for all $n \geq 1$ and all R-modules N;

(iii) $\operatorname{Ext}_R^1(M, N) = 0$ for all R-modules N.

We can now characterize projective dimension in terms of the Ext functor.

7.1.7 Theorem

If M is an R-module and n is a positive integer, the following conditions are equivalent.

1. $\operatorname{pd}_R M \leq n$.
2. $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > n$ and every R-module N.
3. $\operatorname{Ext}_R^{n+1}(M, N) = 0$ for every R-module N.
4. If $0 \to K_{n-1} \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ is an exact sequence with all X_i projective, then K_{n-1} is projective.

Proof. To show that (1) implies (2), observe that by hypothesis, there is a projective resolution $0 \to X_n \to \cdots \to X_0 \to M \to 0$. Use this resolution to compute Ext, and conclude that (2) holds. Since (3) is a special case of (2), we have (2) implies (3). If (4) holds, construct a projective resolution of M in the usual way, but pause at X_{n-1} and terminate the sequence with $0 \to K_{n-1} \to X_{n-1}$. By hypothesis, K_{n-1} is projective, and this gives (4) implies (1). The main effort goes into proving that (3) implies (4). We break the exact sequence given in (4) into short exact sequences. The procedure is a bit different from the decomposition of (5.2.3). Here we are proceeding from right to left, and our first short exact sequence is

$$0 \longrightarrow K_0 \xrightarrow{i_0} X_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

where K_0 is the kernel of ϵ. The induced long exact sequence is

$$\cdots \to \operatorname{Ext}_R^n(X_0, N) \to \operatorname{Ext}_R^n(K_0, N) \to \operatorname{Ext}_R^{n+1}(M, N) \to \operatorname{Ext}_R^{n+1}(X_0, N) \to \cdots$$

Now if every third term in an exact sequence is 0, then the maps in the middle are both injective and surjective, hence isomorphisms. This is precisely what we have here, because X_0 is projective and (7.1.6) applies. Thus $\operatorname{Ext}_R^{n+1}(M, N) \cong \operatorname{Ext}_R^n(K_0, N)$, so as we slide from right to left through the exact sequence, the upper index decreases by 1. This technique is referred to as **dimension shifting**.

Now the second short exact sequence is

$$0 \longrightarrow K_1 \xrightarrow{i_1} X_1 \xrightarrow{d_1} K_0 \longrightarrow 0.$$

We can replace X_0 by K_0 because $\operatorname{im} d_1 = \ker \epsilon = K_0$. The associated long exact sequence is

$$\cdots \to \operatorname{Ext}_R^n(X_1, N) \to \operatorname{Ext}_R^n(K_1, N) \to \operatorname{Ext}_R^{n+1}(K_0, N) \to \operatorname{Ext}_R^{n+1}(X_1, N) \to \cdots$$

and dimension shifting gives $\operatorname{Ext}_R^n(K_0, N) \cong \operatorname{Ext}_R^{n-1}(K_1, N)$. Iterating this procedure, we get $\operatorname{Ext}_R^{n+1}(M, N) \cong \operatorname{Ext}_R^1(K_{n-1}, N)$, hence by the hypothesis of (3), $\operatorname{Ext}_R^1(K_{n-1}, N) = 0$. By (7.1.6), K_{n-1} is projective. ♣

7.1.8 Corollary

gldim $R \le n$ if and only if $\operatorname{Ext}_R^{n+1}(M, N) = 0$ for all R-modules M and N.

Proof. By the definition (7.1.4) of global dimension, gldim $R \le n$ iff $\operatorname{pd}_R M$ for all M iff (by (1) implies (3) of (7.1.7)) $\operatorname{Ext}_R^{n+1}(M, N) = 0$ for all M and N. ♣

7.2 Injective Dimension

As you might expect, projective dimension has a dual notion. To develop it, we will need the analog of (7.1.6) for injective modules. A proof is given in TBGY, subsection S5.8.

7.2.1 Proposition

If N is an R-module, the following conditions are equivalent.

(i) N is injective;

(ii) $\operatorname{Ext}_R^n(M, N) = 0$ for all $n \ge 1$ and all R-modules M;

(iii) $\operatorname{Ext}_R^1(M, N) = 0$ for all R-modules M.

We are going to dualize (7.1.7), and the technique of **dimension shifting** is again useful.

7.2.2 Proposition

Let $0 \to M' \to E \to M'' \to 0$ be an exact sequence, with E injective. Then for all $n \ge 1$ and all R-modules M, we have $\operatorname{Ext}_R^{n+1}(M, M') \cong \operatorname{Ext}_R^n(M, M'')$. Thus as we slide through the exact sequence from left to right, the index of Ext drops by 1.

Proof. The given short exact sequence induces the following long exact sequence:

$$\cdots \to \operatorname{Ext}_R^n(M, E) \to \operatorname{Ext}_R^n(M, M'') \to \operatorname{Ext}_R^{n+1}(M, M') \to \operatorname{Ext}_R^{n+1}(M, E) \to \cdots$$

By (7.2.1), the outer terms are 0 for $n \ge 1$, hence as in the proof of (7.1.7), the map in the middle is an isomorphism. ♣

7.2.3 Definitions and Comments

An injective resolution $0 \to N \to X_0 \to \cdots \to \cdots X_n \to 0$ of the R-module N is said to be of *length* n. The smallest such n is called the *injective dimension* of M, denoted by $\operatorname{id}_R M$. (If N has no finite injective resolution, we set $\operatorname{id}_R M = \infty$.) Just as in (7.1.2), $\operatorname{id}_R N = 0$ if and only if N is injective.

7.2.4 Proposition

If N is an R-module and n is a positive integer, the following conditions are equivalent.

1. $\operatorname{id}_R N \le n$.
2. $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > n$ and every R-module M.
3. $\operatorname{Ext}^{n+1}_R(M,N) = 0$ for every R-module M.
4. If $0 \to N \to X_0 \to \cdots \to X_{n-1} \to C_{n-1} \to 0$ is an exact sequence with all X_i injective, then C_{n-1} is injective.

Proof. If (1) is satisfied, we have an exact sequence $0 \to N \to X_0 \to \cdots \to X_n \to 0$, with the X_i injective. Use this sequence to compute Ext, and conclude that (2) holds. If we have (2), then we have the special case (3). If (4) holds, construct an injective resolution of N, but pause at step $n-1$ and terminate the sequence by $X_{n-1} \to C_{n-1} \to 0$. By hypothesis, C_{n-1} is injective, proving that (4) implies (1). To prove that (3) implies (4), we decompose the exact sequence of (4) into short exact sequences. The process is similar to that of (5.2.3), but with emphasis on kernels rather than cokernels. The decomposition is given below.

$$0 \to N \to X_0 \to K_0 \to 0, \quad 0 \to K_0 \to X_1 \to K_1 \to 0, \dots ,$$

$$0 \to K_{n-2} \to X_{n-1} \to C_{n-1} \to 0$$

We now apply the dimension shifting result (7.2.2) to each short exact sequence. If the index of Ext starts at $n+1$, it drops by 1 as we go through each of the n sequences, and it ends at 1. More precisely,

$$\operatorname{Ext}^{n+1}_R(M,N) \cong \operatorname{Ext}^1_R(M, C_{n-1})$$

for any M. The left side is 0 by hypothesis, so the right side is also 0. By (7.2.1), C_{n-1} is injective. ♣

7.2.5 Corollary

The global dimension of R is the least upper bound of $\operatorname{id}_R N$ over all R-modules N.

Proof. By the definition (7.1.4) of global dimension, gldim $R \le n$ iff $\operatorname{pd}_R M \le n$ for all M. Equivalently, by (7.1.7), $\operatorname{Ext}^{n+1}_R(M,N) = 0$ for all M and N. By (7.2.4), this happens iff $\operatorname{id}_R N \le n$ for all N. ♣

7.3 Tor and Dimension

We have observed the interaction between homological dimension and the Ext functor, and this suggests that it would be profitable to bring in the Tor functor as well. We will need the following result, which is proved in TBGY, subsection S5.6.

7.3.1 Proposition

If M is an R-module, the following conditions are equivalent.

(i) M is flat.

(ii) $\operatorname{Tor}_n^R(M, N) = 0$ for all $n \geq 1$ and all R-modules N.

(iii) $\operatorname{Tor}_1^R(M, N) = 0$ for all R-modules N.

In addition, if R is a Noetherian local ring and M is finitely generated over R, then M is free if and only if M is projective, if and only if M is flat. See Problems 3-6 for all the details.

7.3.2 Proposition

Let R be Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field k. Let M be a finitely generated R-module. Then M is free ($\Longleftrightarrow$ projective $\Longleftrightarrow$ flat) if and only if $\operatorname{Tor}_1^R(M, k) = 0$.

Proof. The "only if" part follows from (7.3.1). To prove the "if" part, let $\{x_1, \dots, x_n\}$ be a minimal set of generators for M. Take a free module F with basis $\{e_1, \dots, e_n\}$ and define an R-module homomorphism $f : F \to M$ via $f(e_i) = x_i, i = 1, \dots, n$. If K is the kernel of f, we have the short exact sequence $0 \to K \to F \to M \to 0$. Since $\operatorname{Tor}_1^R(M, k) = 0$, we can truncate the associated long exact sequence:

$$0 = \operatorname{Tor}_1^R(M, k) \to K \otimes_R k \to F \otimes_R k \to M \otimes_R k \to 0$$

where the map $\overline{f} : F \otimes_R k \to M \otimes_R k$ is induced by f. Now $\overline{f}$ is surjective by construction, and is injective by minimality of the generating set [see (0.3.4) and the base change device below]. Thus $K \otimes_R k = \ker \overline{f} = 0$. But (TBGY, subsection S7.1)

$$K \otimes_R k = K \otimes_R (R/\mathcal{M}) \cong K/\mathcal{M}K$$

so $K = \mathcal{M}K$. By NAK, $K = 0$. Therefore f is an isomorphism of F and M, hence M is free. ♣

7.3.3 Theorem

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field k. If M is a finitely generated R-module, the following conditions are equivalent.

(i) $\operatorname{pd}_R M \leq n$.

(ii) $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > n$ and every R-module N.

(iii) $\text{Tor}^R_{n+1}(M,N) = 0$ for every R-module N.

(iv) $\text{Tor}^R_{n+1}(M,k) = 0$.

Proof. If (i) holds, then M has a projective resolution of length n, and if we use this resolution to compute Tor, we get (ii). There is no difficulty with (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv), so it remains to prove (iv) $\Longrightarrow$ (i). Let $0 \to K_{n-1} \to X_{n-1} \to \cdots \to X_0 \to M \to 0$ be an exact sequence with all X_i projective. By (7.1.7), it suffices to show that K_{n-1} is projective. Now we apply dimension shifting as in the proof of (7.1.7). For example, the short exact sequence $0 \to K_1 \to X_1 \to K_0 \to 0$ [see(7.1.7)] induces the long exact sequence $\cdots \to \text{Tor}^R_n(X_1,k) \to \text{Tor}^R_n(K_0,k) \to \text{Tor}^R_{n-1}(K_1,k) \to \text{Tor}^R_{n-1}(X_1,k) \to \cdots$ and as before, the outer terms are 0, which implies that the map in the middle is an isomorphism. Iterating, we have $\text{Tor}^R_1(K_{n-1},k) \cong \text{Tor}^R_{n+1}(M,k) = 0$ by hypothesis. By (7.3.2), K_{n-1} is projective. ♣

7.3.4 Corollary

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field k. For any positive integer n, the following conditions are equivalent.

(1) $\text{gldim}\, R \le n$.

(2) $\text{Tor}^R_{n+1}(M,N) = 0$ for all finitely generated R-modules M and N.

(3) $\text{Tor}^R_{n+1}(k,k) = 0$.

Proof. If (1) holds, then $\text{pd}_R\, M \le n$ for all M, and (2) follows from (7.3.3). Since (3) is a special case of (2), it remains to prove that (3) implies (1). Assuming (3), (7.3.3) gives $\text{Tor}^R_{n+1}(k,N) = \text{Tor}^R_{n+1}(N,k) = 0$ for all R-modules N. Again by (7.3.3), the projective dimension of any R-module N is at most n, hence $\text{gldim}\, R \le n$. ♣

7.4 Application

As promised in (6.2.5), we will prove that under a mild hypothesis, all maximal M-sequences have the same length.

7.4.1 Lemma

Let M and N be R-modules, and let $a_1, \dots, a_n$ be an M-sequence. If a_n annihilates N, then the only R-homomorphism h from N to $M' = M/(a_1, \dots, a_{n-1})M$ is the zero map.

Proof. If x is any element of N, then $a_n h(x) = h(a_n x) = h(0) = 0$. Since a_n is not a zero-divisor of M', the result follows. ♣

7.4.2 Proposition

Strengthen the hypothesis of (7.4.1) so that each $a_i, i = 1, \dots, n$ annihilates N. Then $\text{Ext}^n_R(N,M) \cong \text{hom}_R(N, M/(a_1, \dots, a_n)M)$.

Proof. The short exact sequence $0 \to M \to M \to M/a_1M \to 0$, with the map from M to M given by multiplication by a_1, induces the following long exact sequence:

$$\mathrm{Ext}_R^{n-1}(N, M) \longrightarrow \mathrm{Ext}_R^{n-1}(N, M/a_1M) \xrightarrow{\delta} \mathrm{Ext}_R^n(N, M) \xrightarrow{a_1} \mathrm{Ext}_R^n(N, M)$$

where the label a_1 indicates multiplication by a_1. In fact this map is zero, because a_1 annihilates N; hence δ is surjective. By induction hypothesis, $\mathrm{Ext}_R^{n-1}(N, M)$ is isomorphic to $\hom_R(N, M/(a_1, \dots, a_{n-1})M = 0$ by (7.4.1). (The result is vacuously true for $n = 1$.) Thus δ is injective, hence an isomorphism. Consequently, if $\overline{M} = M/a_1M$, we have $\mathrm{Ext}_R^{n-1}(N, \overline{M}) \cong \mathrm{Ext}_R^n(N, M)$. Again using the induction hypothesis, we have $\mathrm{Ext}_R^{n-1}(N, \overline{M}) \cong \hom_R(N, \overline{M}/(a_2, \dots, a_n)\overline{M} = \hom_R(N, M/(a_1, \dots, a_n)M)$. ♣

We prove a technical lemma to prepare for the main theorem.

7.4.3 Lemma

Let M_0 be an R-module, and I an ideal of R. Then $\hom_R(R/I, M_0) \neq 0$ if and only if there is a nonzero element of M_0 annihilated by I. Equivalently, by (1.3.1), I is contained in some associated prime of M_0. (If there are only finitely many associated primes, for example if R is Noetherian [see (1.3.9)], then by (0.1.1), another equivalent condition is that I is contained in the union of the associated primes of M_0.)

Proof. If there is a nonzero homomorphism from R/I to M_0, it will map $1+I$ to a nonzero element $x \in M_0$. If $a \in I$, then $a + I$ is mapped to ax. But $a + I = 0 + I$ since $a \in I$, so ax must be 0. Conversely, if x is a nonzero element of M_0 annihilated by I, then we can construct a nonzero homomorphism by mapping $1 + I$ to x, and in general, $r + I$ to rx. We must check that the map is well defined, but this follows because I annihilates x. ♣

7.4.4 Theorem

Let M be a finitely generated module over the Noetherian ring R, and I an ideal of R such that $IM \neq M$. Then any two maximal M-sequences in I have the same length, namely the smallest nonnegative integer n such that $\mathrm{Ext}_R^n(R/I, M) \neq 0$.

Proof. In (7.4.2), take $N = R/I$ and let $\{a_1, \dots, a_n\}$ be a set of generators of I. Then

$$\mathrm{Ext}_R^n(R/I, M) \cong \hom_R(R/I, M_0)$$

where $M_0 = M/(a_1, \dots, a_n)M$. By (7.4.3), $\mathrm{Ext}_R^n(R/I, M) = 0$ if and only if I is not contained in the union of all associated primes of M_0. In view of (1.3.6), this says that if $a_1, \dots, a_n$ is an M-sequence in I, it can be extended to some $a_{n+1} \in I$ as long as $\mathrm{Ext}_R^n(R/I, M) = 0$. This is precisely the statement of the theorem. ♣

7.4.5 Remarks

Under the hypothesis of (7.4.4), we call the maximum length of an M-sequence in I the *grade of I on M*. If R is a Noetherian local ring with maximal ideal $\mathcal{M}$, then by (6.2.2), the elements a_i of an M-sequence are nonunits, hence belong to $\mathcal{M}$. Thus the depth of M, as defined in (6.2.5), coincides with the grade of $\mathcal{M}$ on M.

Again let M be finitely generated over the Noetherian local ring R. By (7.4.4), the depth of M is 0 if and only if $\hom_R(R/\mathcal{M}, M) \neq 0$. By (7.4.3) and the maximality of $\mathcal{M}$, this happens iff $\mathcal{M}$ is an associated prime of M. Note also that by (6.1.1) and (6.2.3), if $a_1, \ldots, a_r$ is an M-sequence of maximal length, then the module $M/(a_1, \ldots, a_r)M$ has finite length.