Chapter 6

Depth

6.1 Systems of Parameters

We prepare for the study of regular local rings, which play an important role in algebraic geometry.

6.1.1 Definition

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$, and let M be a finitely generated R-module of dimension n. A *system of parameters* for M is a set $\{a_1, \dots, a_n\}$ of elements of $\mathcal{M}$ such that $M/(a_1, \dots, a_n)M$ has finite length. The finiteness of the Chevalley dimension (see (5.3.2) and (5.3.3) guarantees the existence of such a system.

6.1.2 Example

Let R be a Noetherian local ring of dimension d. Then any set $\{a_1, \dots, a_d\}$ that generates an ideal of definition is a system of parameters for R, by (5.4.1). In particular, if $R = k[[X_1, \dots, X_n]]$ is a formal power series ring over a field, then $X_1, \dots, X_n$ form a system of parameters, since they generate the maximal ideal.

6.1.3 Proposition

Let M be finitely generated and of dimension n over the Noetherian local ring R, and let $a_1, \dots, a_t$ be arbitrary elements of the maximal ideal $\mathcal{M}$. Then $\dim(M/(a_1, \dots, a_t)M) \geq n - t$, with equality if and only if the a_i can be extended to a system of parameters for M.

Proof. Let a be any element of $\mathcal{M}$, and let $N = M/aM$. Choose $b_1, \dots, b_r \in \mathcal{M}$ such that $N/(b_1, \dots, b_r)N$ has finite length, with r as small as possible. Then $M/(a, b_1, \dots, b_r)M$ also has finite length, because

$$(M/aM)/(b_1, \dots, b_r)(M/aM) \cong M/(a, b_1, \dots, b_r)M.$$

It follows that the Chevalley dimension of M is at most $r+1$, in other words,

$$\delta(M/aM) \geq \delta(M) - 1.$$

The proof will be by induction on t, and we have just taken care of $t=1$ as well as the key step in the induction, namely

$$\dim(M/(a_1, \dots, a_t)M) = \dim(N/a_1N)$$

where $N = M/(a_2, \dots, a_t)M$. By the $t=1$ case and the induction hypothesis,

$$\dim(N/a_1N) \geq \dim N - 1 \geq \dim M - (t-1) - 1 = \dim M - t$$

as asserted. If $\dim(M/(a_1, \dots, a_t)M) = n - t$ with $n = \dim M$, then we can choose a system of parameters $a_{t+1}, \dots, a_n$ for $N = M/(a_1, \dots, a_t)M$. Then

$$N/(a_{t+1}, \dots, a_n)N \cong M/(a_1, \dots, a_t, a_{t+1}, \dots, a_n)M$$

has finite length. Thus $a_1, \dots, a_n$ form a system of parameters for M. Conversely, if $a_1, \dots, a_t$ can be extended to a system of parameters $a_1, \dots, a_n$ for M, define $N = M/(a_1, \dots, a_t)M$. Then $N/(a_{t+1}, \dots, a_n)N \cong M/(a_1, \dots, a_n)M$ has finite length, hence $\dim N \leq n - t$. But $\dim N \geq n - t$ by the main assertion, and the proof is complete. ♣

6.2 Regular Sequences

We introduce sequences that are guaranteed to be extendable to a system of parameters.

6.2.1 Definition

Let M be an R-module. The sequence $a_1, \dots, a_t$ of nonzero elements of R is an *M-sequence*, also called a *regular sequence for M* or an *M-regular sequence*, if $(a_1, \dots, a_t)M \neq M$ and for each $i = 1, \dots, t$, a_i is not a zero-divisor of $M/(a_1, \dots, a_{i-1})M$.

6.2.2 Comments and Examples

We interpret the case $i=1$ as saying that a_1 is not a zero-divisor of M, that is, if $x \in M$ and $a_1x = 0$, then $x = 0$. Since $(a_1, \dots, a_t)M \neq M$, $M \neq 0$ and the a_i are nonunits.

It follows from the definition that the elements $a_1, \dots, a_t$ form an M-sequence if and only if for all $i = 1, \dots, t$, $a_1, \dots, a_i$ is an M-sequence and $a_{i+1}, \dots, a_t$ is an $M/(a_1, \dots, a_i)M$-sequence.

1. If $R = k[X_1, \dots, X_n]$ with k a field, then $X_1, \dots, X_n$ is an R-sequence.

2. (A tricky point) A permutation of a regular sequence need not be regular. For example, let $R = k[X, Y, Z]$, where k is a field. Then $X, Y(1-X), Z(1-X)$ is an R-sequence, but $Y(1-X), Z(1-X), X$ is not, because the image of $Z(1-X)Y$ is zero in $R/(Y(1-X))$.

6.2.3 Theorem

Let M be a finitely generated module over the Noetherian local ring R. If $a_1, \dots, a_t$ is an M-sequence, then $\{a_1, \dots, a_t\}$ can be extended to a system of parameters for M.

Proof. We argue by induction on t. Since a_1 is not a zero-divisor of M, we have $\dim M/a_1M = \dim M - 1$ by (5.4.7). (Remember that the a_i are nonunits (see (6.2.2)) and therefore belong to the maximal ideal of R.) By (6.1.3), a_1 is part of a system of parameters for M. If $t > 1$, the induction hypothesis says that $a_1, \dots, a_{t-1}$ is part of a system of parameters for M. By (6.1.3), $\dim M/(a_1, \dots, a_{t-1})M = n - (t-1)$, where $n = \dim M$. Since a_t is not a zero-divisor of $N = M/(a_1, \dots, a_{t-1})M$, we have, as in the $t = 1$ case, $\dim N/a_tN = \dim N - 1$. But, as in the proof of (6.1.3),

$$N/a_tN \cong M/(a_1, \dots, a_t)M,$$

hence

$$\dim M/(a_1, \dots, a_t)M = \dim N/a_tN = \dim N - 1 = n - (t-1) - 1 = n - t.$$

By (6.1.3), $a_1, \dots, a_t$ extend to a system of parameters for M. ♣

6.2.4 Corollary

If R is a Noetherian local ring, then every R-sequence can be extended to a system of parameters for R.

Proof. Take $M = R$ in (6.2.3). ♣

6.2.5 Definition

Let M be a nonzero finitely generated module over the Noetherian local ring R. The *depth* of M over R, written $\text{depth}_R M$ or simply depth M, is the maximum length of an M-sequence. We will see in the next chapter that any two maximal M-sequences have the same length.

6.2.6 Theorem

Let M be a nonzero finitely generated module over the Noetherian local ring R. Then depth $M \leq \dim M$.

Proof. Since $\dim M$ is the number of elements in a system of parameters, the result follows from (6.2.3). ♣

6.2.7 Proposition

Let M be a finitely generated module over the Noetherian ring R, and let $a_1, \dots, a_n$ be an M-sequence with all a_i belonging to the Jacobson radical $J(R)$. Then any permutation of the a_i is also an M-sequence.

Proof. It suffices to consider the transposition that interchanges a_1 and a_2. First let us show that a_1 is not a zero-divisor of M/a_2M. Suppose $a_1\overline{x} = 0$, where $\overline{x}$ belongs to M/a_2M. Then a_1x belongs to a_2M, so we may write $a_1x = a_2y$ with $y \in M$. By hypothesis, a_2 is not a zero-divisor of M/a_1M, so y belongs to a_1M. Therefore $y = a_1z$ for some $z \in M$. Then $a_1x = a_2y = a_2a_1z$. By hypothesis, a_1 is not a zero-divisor of M, so $x = a_2z$, and consequently $\overline{x} = 0$.

To complete the proof, we must show that a_2 is not a zero-divisor of M. If N is the submodule of M annihilated by a_2, we will show that $N = a_1N$. Since $a_1 \in J(R)$, we can invoke NAK (0.3.3) to conclude that $N = 0$, as desired. It suffices to show that $N \subseteq a_1N$, so let $x \in N$. By definition of N we have $a_2x = 0$. Since a_2 is not a zero-divisor of M/a_1M, x must belong to a_1M, say $x = a_1y$ with $y \in M$. Thus $a_2x = a_2a_1y = 0$. But a_1 is not a zero-divisor of M, hence $a_2y = 0$ and therefore $y \in N$. But $x = a_1y$, so $x \in a_1N$, and we are finished. ♣

6.2.8 Corollary

Let M be a finitely generated module over the Noetherian local ring R. Then any permutation of an M-sequence is also an M-sequence.

Proof. By (6.2.2), the members of the sequence are nonunits, hence they belong to the maximal ideal, which coincides with the Jacobson radical. ♣

6.2.9 Definitions and Comments

Let M be a nonzero finitely generated module over a Noetherian local ring R. If the depth of M coincides with its dimension, we call M a *Cohen-Macaulay module*. We say that R is a *Cohen-Macaulay ring* if it is a Cohen-Macaulay module over itself. To study these rings and modules, we need some results from homological algebra. The required tools will be developed in Chapter 7.