Chapter 5

Dimension Theory

The geometric notion of the dimension of an affine algebraic variety V is closely related to algebraic properties of the coordinate ring of the variety, that is, the ring of polynomial functions on V. This relationship suggests that we look for various ways of defining the dimension of an arbitrary commutative ring. We will see that under appropriate hypotheses, several concepts of dimension are equivalent. Later, we will connect the algebraic and geometric ideas.

5.1 The Calculus of Finite Differences

Regrettably, this charming subject is rarely taught these days, except in actuarial programs. It turns out to be needed in studying Hilbert and Hilbert-Samuel polynomials in the next section.

5.1.1 Lemma

Let g and G be real-valued functions on the nonnegative integers, and assume that $\Delta G = g$, that is, $G(k+1) - G(k) = g(k)$ for all $k \geq 0$. (We call ΔG the *difference* of G.) Then

$$\sum_{k=a}^{b} g(k) = G(k)|_a^{b+1} = G(b+1) - G(a).$$

Proof. Add the equations $G(a+1) - G(a) = g(a)$, $G(a+2) - G(a+1) = g(a+1)$, ..., $G(b+1) - G(b) = g(b)$. ♣

5.1.2 Lemma

If r is a positive integer, define $k^{(r)} = k(k-1)(k-2)\cdots(k-r+1)$. Then $\Delta k^{(r)} = rk^{(r-1)}$.

Proof. Just compute:

$$\begin{aligned}\Delta k^{(r)} &= (k+1)^{(r)} - k^{(r)} = (k+1)k(k-1)\cdots(k-r+2) - k(k-1)\cdots(k-r+1)\\ &= k(k-1)\cdots(k-r+2)[k+1-(k-r+1)] = rk^{(r-1)}.\end{aligned}$$ ♣

5.1.3 Examples

$\Delta k^{(2)} = 2k^{(1)}$, so $\sum_{k=1}^{n} k = [k^{(2)}/2]|_1^{n+1} = (n+1)n/2$.
$k^2 = k(k-1) + k = k^{(2)} + k^{(1)}$, so

$$\sum_{k=1}^{n} k^2 = [k^{(3)}/3]|_1^{n+1} + (n+1)n/2 = (n+1)n(n-1)/3 + (n+1)n/2$$
$$= n(n+1)(2n+1)/6.$$

$k^{(3)} = k(k-1)(k-2) = k^3 - 3k^2 + 2k$, so $k^3 = k^{(3)} + 3k^2 - 2k$. Therefore

$$\sum_{k=1}^{n} k^3 = [k^{(4)}/4]|_1^{n+1} + 3n(n+1)(2n+1)/6 - 2n(n+1)/2.$$

The first term on the right is $(n+1)n(n-1)(n-2)/4$, so the sum of the first n cubes is

$$[n(n+1)/4][n^2 - 3n + 2 + 2(2n+1) - 4]$$

which simplifies to $[n(n+1)/2]^2$.

In a similar fashion we can find $\sum_{k=1}^{n} k^s$ for any positive integer s.

5.1.4 Definitions and Comments

A *polynomial-like function* is a function f from the natural numbers (nonnegative integers) $\mathbb{N}$ to the rational numbers $\mathbb{Q}$, such that f eventually agrees with a polynomial $g \in \mathbb{Q}[X]$. In other words, $f(n) = g(n)$ for all sufficiently large n (abbreviated $n >> 0$). The *degree* of f is taken to be the degree of g.

5.1.5 Lemma

Let $f : \mathbb{N} \to \mathbb{Q}$. Then f is a polynomial-like function of degree r if and only if Δf is a polynomial-like function of degree $r-1$. (We can allow $r = 0$ if we take the degree of the zero polynomial to be -1.)

Proof. This follows from (5.1.1) and (5.1.2), along with the observation that a function whose difference is zero is constant. (The analogous result from differential calculus that a function with zero derivative is constant is harder to prove, and is usually accomplished via the mean value theorem.) ♣

5.2 Hilbert and Hilbert-Samuel Polynomials

There will be two polynomial-like functions of interest, and we begin preparing for their arrival.

5.2.1 Proposition

Let $R = \oplus_{n\geq 0} R_n$ be a graded ring. Assume that R_0 is Artinian and R is finitely generated as an algebra over R_0. If $M = \oplus_{n\geq 0} M_n$ is a finitely generated graded R-module, then each M_n is a finitely generated R_0-module.

Proof. By (4.1.3) and (1.6.13), R is a Noetherian ring, hence M is a Noetherian R-module. Let N_n be the direct sum of the $M_m, m \geq n$. Since M is Noetherian, N_n is finitely generated over R, say by $x_1, \dots , x_t$. Since $N_n = M_n \oplus \oplus_{m>n} M_m$, we can write $x_i = y_i + z_i$ with $y_i \in M_n$ and $z_i \in \oplus_{m>n} M_m$. It suffices to prove that $y_1, \dots , y_t$ generate M_n over R_0. If $y \in M_n$, then y is of the form $\sum_{i=1}^t a_i x_i$ with $a_i \in R$. But just as we decomposed x_i above, we can write $a_i = b_i + c_i$ where $b_i \in R_0$ and $c_i \in \oplus_{j>0} R_j$. Thus

$$y = \sum_{i=1}^{t} (b_i + c_i)(y_i + z_i) = \sum_{i=1}^{t} b_i y_i$$

because the elements $b_i z_i, c_i y_i$ and $c_i z_i$ must belong to $\oplus_{m>n} M_m$. ♣

5.2.2 Corollary

In (5.2.1), the length $l_{R_o}(M_n)$ of the R_0-module M_n is finite for all $n \geq 0$.

Proof. Apply (5.2.1) and (1.6.14). ♣

We will need the following basic property of composition length.

5.2.3 Additivity of Length

Suppose we have an exact sequence of R-modules $0 \to A_1 \to A_2 \to \cdots \to A_n \to 0$, all with finite length. Then we have *additivity of length*, that is,

$$l(A_1) - l(A_2) + \cdots + (-1)^{n-1} l(A_n) = 0.$$

This is probably familiar for a short exact sequence $0 \to N \to M \to M/N \to 0$, where the additivity property can be expressed as $l(M) = l(N) + l(M/N)$. (See TBGY, Section 7.5, Problem 5.) The general result is accomplished by decomposing a long exact sequence into short exact sequences. ("Long" means longer than short.) To see how the process works, consider an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E \longrightarrow 0.$$

Our first short exact sequence is

$$0 \to A \to B \to \operatorname{coker} f \to 0.$$

Now $\operatorname{coker} f = B/\operatorname{im} f = B/\ker g \cong \operatorname{im} g$ $(= \ker h)$, so our second short exact sequence is

$$0 \to \operatorname{im} g \to C \to \operatorname{coker} g \to 0.$$

As above, $\operatorname{coker} g \cong \operatorname{im} h$ $(= \ker i)$, so the third short exact sequence is

$$0 \to \operatorname{im} h \to D \to \operatorname{coker} h \to 0.$$

But $\operatorname{coker} h \cong \operatorname{im} i = E$, so we may replace the third short exact sequence by

$$0 \to \operatorname{im} h \to D \to E \to 0.$$

Applying additivity for short exact sequences, we have

$$l(A) - l(B) + l(\operatorname{coker} f) - l(\operatorname{im} g) + l(C) - l(\operatorname{coker} g) + l(\operatorname{im} h) - l(D) + l(E) = 0.$$

After cancellation, this becomes

$$l(A) - l(B) + l(C) - l(D) + l(E) = 0$$

as desired.

5.2.4 Theorem

Let $R = \oplus_{n \geq 0} R_n$ be a graded ring. Assume that R_0 is Artinian and R is finitely generated as an algebra over R_0, with all generators $a_1, \dots, a_r$ belonging to R_1. If M is a finitely generated graded R-module, define $h(M,n) = l_{R_0}(M_n), n \in \mathbb{N}$. Then h, as a function of n with M fixed, is polynomial-like of degree at most $r-1$. Using slightly loose language, we call h the *Hilbert polynomial* of M.

Proof. We argue by induction on r. If $r = 0$, then $R = R_0$. Choose a finite set of homogeneous generators for M over R. If d is the maximum of the degrees of the generators, then $M_n = 0$ for $n > d$, and therefore $h(M,n) = 0$ for $n >> 0$. Now assume $r > 0$, and let λ_r be the endomorphism of M given by multiplication by a_r. By hypothesis, $a_r \in R_1$, so $\lambda_r(M_n) \subseteq M_{n+1}$. If K_n is the kernel, and C_n the cokernel, of $\lambda_r : M_n \to M_{n+1}$, we have the exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{\lambda_r} M_{n+1} \longrightarrow C_n \longrightarrow 0.$$

Let K be the direct sum of the K_n and C the direct sum of the C_n, $n \geq 0$. Then K is a submodule of M and C a quotient of M. Thus K and C are finitely generated Noetherian graded R-modules, so by (5.2.1) and (5.2.2), $h(K,n)$ and $h(C,n)$ are defined and finite. By (5.2.3),

$$h(K,n) - h(M,n) + h(M,n+1) - h(C,n) = 0$$

hence $\Delta h(M,n) = h(C,n) - h(K,n)$. Now a_r annihilates K and C, so K and C are finitely generated T-modules, where T is the graded subring of R generated over R_0 by $a_1, \dots, a_{r-1}$. (If an ideal I annihilates an R-module M, then M is an R/I-module; see TBGY, Section 4.2, Problem 6.) By induction hypothesis, $h(K,n)$ and $h(C,n)$ are polynomial-like of degree at most $r-2$, hence so is $\Delta h(M,n)$. By (5.1.5), $h(M,n)$ is polynomial-like of degree at most $r-1$. ♣

5.2.5 Definitions and Comments

Let R be any Noetherian local ring with maximal ideal $\mathcal{M}$. An ideal I of R is said to be an *ideal of definition* if $\mathcal{M}^n \subseteq I \subseteq \mathcal{M}$ for some $n \geq 1$. Equivalently, R/I is an Artinian ring. [See (3.3.10), and note that $\sqrt{I} = \mathcal{M}$ if and only if every prime ideal containing I is maximal, so (1.6.11) applies.]

5.2.6 The Hilbert-Samuel Polynomial

Let I be an ideal of definition of the Noetherian local ring R. If M is a finitely generated R-module, then M/IM is a finitely generated module over the Artinian ring R/I. Thus M/IM is Artinian (as well as Noetherian), hence has finite length over R/I. With the I-adic filtrations on R and M, the associated graded ring and the associated graded module [see (4.1.2)] are given by

$$\mathrm{gr}_I(R) = \oplus_{n\geq 0}(I^n/I^{n+1}), \quad \mathrm{gr}_I(M) = \oplus_{n\geq 0}(I^nM/I^{n+1}M).$$

If I is generated over R by $a_1, \dots, a_r$, then the images $\overline{a}_1, \dots, \overline{a}_r$ in I/I^2 generate $\mathrm{gr}_I(R)$ over R/I. (Note that by definition of a graded ring, $R_iR_j \subseteq R_{i+j}$, which allows us to produce elements in R_t for arbitrarily large t.) By (5.2.4),

$$h(\mathrm{gr}_I(M), n) = l_{R/I}(I^nM/I^{n+1}M) < \infty.$$

Again recall that if N is an R-module and the ideal I annihilates N, then N becomes an R/I-module via $(a+I)x = ax$. It follows that we may replace $l_{R/I}$ by l_R in the above formula. We define the *Hilbert-Samuel polynomial* by

$$s_I(M, n) = l_R(M/I^nM).$$

Now the sequence

$$0 \to I^nM/I^{n+1}M \to M/I^{n+1}M \to M/I^nM \to 0$$

is exact by the third isomorphism theorem. An induction argument using additivity of length shows that $s_I(M, n)$ is finite. Consequently

$$\Delta s_I(M, n) = s_I(M, n+1) - s_I(M, n) = h(\mathrm{gr}_I(M), n).$$

By (5.2.4), $h(\mathrm{gr}_I(M), n)$ is polynomial-like of degree at most $r-1$, so by (5.1.5), $s_I(M, n)$ is polynomial-like of degree at most r.

The Hilbert-Samuel polynomial $s_I(M, n)$ depends on the particular ideal of definition I, but the degree $d(M)$ of $s_I(M, n)$ is the same for all possible choices. To see this, let t be a positive integer such that $\mathcal{M}^t \subseteq I \subseteq \mathcal{M}$. Then for every $n \geq 1$ we have $\mathcal{M}^{tn} \subseteq I^n \subseteq \mathcal{M}^n$, so $s_{\mathcal{M}}(M, tn) \geq s_I(M, n) \geq s_{\mathcal{M}}(M, n)$. If the degrees of these polynomial are, from right to left, d_1, d_2 and d_3, we have $O(d_1^n) \leq O(d_2^n) \leq O(d_3^n)$, with $d_3 = d_1$. Therefore all three degrees coincide.

The Hilbert-Samuel polynomial satisfies a property analogous to (5.2.3), the additivity of length.

5.2.7 Theorem

Let I be an ideal of definition of the Noetherian local ring R, and suppose we have an exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated R-modules. Then

$$s_I(M', n) + s_I(M'', n) = s_I(M, n) + r(n)$$

where $r(n)$ is polynomial-like of degree less than $d(M)$, and the leading coefficient of $r(n)$ is nonnegative.

Proof. The following sequence is exact:

$$0 \to M'/(M' \cap I^n M) \to M/I^n M \to M''/I^n M'' \to 0.$$

Set $M'_n = M' \cap I^n M$. Then by additivity of length,

$$s_I(M, n) - s_I(M'', n) = l_R(M'/M'_n)$$

hence $l_R(M'/M'_n)$ is polynomial-like. By the Artin-Rees lemma (4.1.7), the filtration $\{M'_n\}$ is I-stable, so $IM'_n = M'_{n+1}$ for sufficiently large n, say, $n \geq m$. Thus for every $n \geq 0$ we have $M'_{n+m} = M' \cap I^{n+m} M \supseteq I^{n+m} M'$, and consequently

$$I^{n+m} M' \subseteq M'_{n+m} = I^n M'_m \subseteq I^n M',$$

which implies that

$$l_R(M'/I^{n+m} M') \geq l_R(M'/M'_{n+m}) \geq l_R(M'/I^n M').$$

The left and right hand terms of this inequality are $s_I(M', n+m)$ and $s_I(M', n)$ respectively, and it follows that $s_I(M', n)$ and $l_R(M'/M'_n)$ have the same degree and the same leading coefficient. Moreover, $s_I(M', n) - l_R(M'/M'_n) = r(n)$ is polynomial-like of degree less than $\deg l_R(M'/M'_n) \leq \deg s_I(M, n)$, as well as nonnegative for $n >> 0$. The result now follows upon adding the equations

$$s_I(M, n) - s_I(M'', n) = l_R(M'/M'_n)$$

and

$$r(n) = s_I(M', n) - l_R(M'/M'_n). \clubsuit$$

5.2.8 Corollary

Let M' be a submodule of M, where M is a finitely generated module over the Noetherian local ring R. Then $d(M') \leq d(M)$.

Proof. Apply (5.2.7), noting that we can ignore $r(n)$ because it is of lower degree than $s_I(M, n)$. $\clubsuit$

5.3 The Dimension Theorem

5.3.1 Definitions and Comments

The *dimension* of a ring R, denoted by $\dim R$, will be taken as its *Krull dimension*, the maximum length n of a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals of R. If there is no upper bound on the length of such a chain, we take $n = \infty$. An example of an infinite-dimensional ring is the non-Noetherian ring $k[X_1, X_2, \dots]$, where k is a field. We have the infinite chain of prime ideals $(X_1) \subset (X_1, X_2) \subset (X_1, X_2, X_3) \subset \cdots$. At the other extreme, a field, and more generally an Artinian ring, has dimension 0 by (1.6.4).

A *Dedekind domain* is a Noetherian, integrally closed integral domain in which every nonzero prime ideal is maximal. A Dedekind domain that is not a field has dimension 1. Algebraic number theory provides many examples, because the ring of algebraic integers of a number field is a Dedekind domain.

There are several other ideas that arise from the study of chains of prime ideals. We define the *height* of a prime ideal P (notation $\operatorname{ht} P$) as the maximum length n of a chain of prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n = P$. By (0.4.6), the height of P is the dimension of the localized ring R_P.

The *coheight* of the prime ideal P (notation $\operatorname{coht} P$) is the maximum length n of a chain of prime ideals $P = P_0 \subset P_1 \subset \cdots \subset P_n$. It follows from the correspondence theorem and the third isomorphism theorem that the coheight of P is the dimension of the quotient ring R/P. (If I and J are ideals of R with $I \subseteq J$, and $S = (R/I)/(J/I)$, then $S \cong R/J$, so S is an integral domain iff R/J is an integral domain, and J/I is a prime ideal of R/I iff J is a prime ideal of R.)

If I is an arbitrary ideal of R, we define the height of I as the infimum of the heights of prime ideals $P \supseteq I$, and the coheight of I as the supremum of the coheights of prime ideals $P \supseteq I$.

5.3.2 The Dimension of a Module

Intuitively, the *dimension* of an R-module M, denoted by $\dim M$, will be measured by length of chains of prime ideals, provided that the prime ideals in the chain contribute to M in the sense that they belong to the support of M. Formally, we define $\dim M = \dim(R/\operatorname{ann} M)$ if $M \neq 0$, and we take the dimension of the zero module to be -1.

We now assume that M is nonzero and finitely generated over the Noetherian ring R. By (1.3.3), M has at least one associated prime. By (1.5.5), $P \supseteq \operatorname{ann} M$ iff $P \in \operatorname{Supp} M$, and by (1.5.9), the minimal elements of $\operatorname{AP}(M)$ and $\operatorname{Supp} M$ are the same. Thus

$$\dim M = \sup\{\operatorname{coht} P : P \in \operatorname{Supp} M\} = \sup\{\operatorname{coht} P : P \in \operatorname{AP}(M)\}.$$

By (1.6.9), the following conditions are equivalent.

1. $\dim M = 0$;
2. Every prime ideal in $\operatorname{Supp} M$ is maximal;
3. Every associated prime ideal of M is maximal;
4. The length of M as an R-module is finite.

We make the additional assumption that R is a local ring with maximal ideal $\mathcal{M}$. Then by (1.5.5),

$$\operatorname{Supp}(M/\mathcal{M}M) = V(\operatorname{ann}(M/\mathcal{M}M))$$

which coincides with $\{\mathcal{M}\}$ by Problem 2. By the above equivalent conditions, $l_R(M/\mathcal{M}M)$ is finite. Since $\mathcal{M}$ is finitely generated, we can assert that there is a smallest positive integer r, called the *Chevalley dimension* $\delta(M)$, such that for some elements $a_1, \dots, a_r$ belonging to $\mathcal{M}$ we have $l_R(M/(a_1, \dots, a_r)M < \infty$. If $M = 0$ we take $\delta(M) = -1$.

5.3.3 Dimension Theorem

Let M be a finitely generated module over the Noetherian local ring R. The following quantities are equal:
1. The dimension $\dim M$ of the module M;
2. The degree $d(M)$ of the Hilbert-Samuel polynomial $s_I(M,n)$, where I is any ideal of definition of R. (For convenience we take $I = \mathcal{M}$, the maximal ideal of R, and we specify that the degree of the zero polynomial is -1.);
3. The Chevalley dimension $\delta(M)$.

Proof. We divide the proof into three parts.

1. $\dim M \le d(M)$, hence $\dim M$ is finite.

If $d(M) = -1$, then $s_{\mathcal{M}}(M,n) = l_R(M/\mathcal{M}^n M) = 0$ for $n >> 0$. By NAK, $M = 0$ so $\dim M = -1$. Thus assume $d(M) \ge 0$. By (1.3.9) or (1.5.10), M has only finitely many associated primes. It follows from (5.3.2) and (5.3.1) that for some associated prime P we have $\dim M = \operatorname{coht} P = \dim R/P$. By (1.3.2) there is an injective homomorphism from R/P to M, so by (5.2.8) we have $d(R/P) \le d(M)$. If we can show that $\dim R/P \le d(R/P)$, it will follow that $\dim M = \dim R/P \le d(R/P) \le d(M)$.

It suffices to show that for any chain of prime ideals $P = P_0 \subset \cdots \subset P_t$ in R, the length t of the chain is at most $d(R/P)$. If $t = 0$, then $R/P \ne 0$ (because P is prime), hence $d(R/P) \ne -1$ and we are finished. Thus assume $t \ge 1$, and assume that the result holds up to $t-1$. Choose $a \in P_1 \setminus P$, and consider prime ideals Q such that $Ra + P \subseteq Q \subseteq P_1$. We can pick a Q belonging to $\operatorname{AP}(R/(Ra+P))$. [Since $Ra + P \subseteq Q$, we have $(R/(Ra+P))_Q \ne 0$; see Problem 3. Then choose Q to be a minimal element in the support of $R/(Ra+P)$, and apply (1.5.9).] By (1.3.2) there is an injective homomorphism from R/Q to $R/(Ra+P)$, so by (5.2.8) we have $d(R/Q) \le d(R/(Ra+P))$. Since the chain $Q \subset P_2 \subset \cdots \subset P_t$ is of length $t-1$, the induction hypothesis implies that $t-1 \le d(R/Q)$, hence $t-1 \le d(R/(Ra+P))$. Now the sequence

$$0 \to R/P \to R/P \to R/(Ra+P) \to 0$$

is exact, where the map from R/P to itself is multiplication by a. (The image of the map is $Ra + P$.) By (5.2.7),

$$s_{\mathcal{M}}(R/P,n) + s_{\mathcal{M}}(R/(Ra+P),n) = s_{\mathcal{M}}(R/P,n) + r(n)$$

where $r(n)$ is polynomial-like of degree less than $d(R/P)$. Thus $d(R/(Ra+P)) < d(R/P)$, and consequently $t-1 < d(R/P)$. Therefore $t \le d(R/P)$, as desired.

2. $d(M) \le \delta(M)$.

If $\delta(M) = -1$, then $M = 0$ and $d(M) = -1$. Assume $M \ne 0$ and $r = \delta(M) \ge 0$, and let $a_1, \dots, a_r$ be elements of $\mathcal{M}$ such that $M/(a_1, \dots, a_r)M$ has finite length. Let I be the ideal $(a_1, \dots, a_r)$ and let P be the annihilator of M; set $Q = I + P$.

We claim that the support of R/Q is $\{\mathcal{M}\}$. To prove this, first note that $M/IM \cong M \otimes_R R/I$. (TBGY, subsection S7.1 of the supplement.) By Problem 9 of Chapter 1, $\operatorname{Supp} M/IM = \operatorname{Supp} M \cap \operatorname{Supp} R/I$, which by (1.5.5) is $V(P) \cap V(I) = V(Q) = \operatorname{Supp} R/Q$. (Note that the annihilator of R/I is I and the annihilator of R/Q is Q.) But the support of M/IM is $\{\mathcal{M}\}$ by (1.6.9), proving the claim. (If $M/IM = 0$, then $M = 0$ by NAK, contradicting our assumption.)

Again by (1.6.9), $\text{AP}(R/Q) = \{\mathcal{M}\}$, so by (1.3.11), Q is $\mathcal{M}$-primary. By (5.2.5) and (3.3.10), Q is an ideal of definition of R.

Let $\overline{R} = R/P$, $\overline{Q} = Q/P$, and consider M as an $\overline{R}$-module. Then $\overline{R}$ is a Noetherian local ring and $\overline{Q}$ is an ideal of definition of $\overline{R}$ generated by $\overline{a}_1, \dots, \overline{a}_r$, where $\overline{a}_i = a_i + P$. By (5.2.6), the degree of the Hilbert-Samuel polynomial $s_{\overline{Q}}(M, n)$ is at most r. But by the correspondence theorem, $l_{\overline{R}}(M/\overline{Q}^n M) = l_R(M/Q^n M)$, hence $s_{\overline{Q}}(M, n) = s_Q(M, n)$. It follows that $d(M) \le r$.

3. $\delta(M) \le \dim M$.

If $\dim M = -1$ then $M = 0$ and $\delta(M) = -1$, so assume $M \ne 0$. If $\dim M = 0$, then by (5.3.2), M has finite length, so $\delta(M) = 0$.

Now assume $\dim M > 0$. (We have already noted in part 1 that $\dim M$ is finite.) Let $P_1, \dots, P_t$ be the associated primes of M whose coheight is as large as it can be, that is, $\text{coht}\, P_i = \dim M$ for all $i = 1, \dots, t$. Since the dimension of M is greater than 0, $P_i \subset \mathcal{M}$ for every i, so by the prime avoidance lemma (0.1.1),

$$\mathcal{M} \not\subseteq \cup_{1 \le i \le t} P_i.$$

Choose an element a in $\mathcal{M}$ such that a belongs to none of the P_i, and let $N = M/aM$. Then

$$\text{Supp}\, N \subseteq \text{Supp}\, M \setminus \{P_1, \dots, P_t\}.$$

To see this, note that if $N_P \ne 0$, then $M_P \ne 0$; also, $N_{P_i} = 0$ for all i because $a \notin P_i$, hence division by a is allowed. Thus $\dim N < \dim M$, because $\dim M = \text{coht}\, P_i$ and we are removing all the P_i. Let $r = \delta(N)$, and let $a_1, \dots, a_r$ be elements of $\mathcal{M}$ such that $N/(a_1, \dots, a_r)N$ has finite length. But

$$M/(a, a_1, \dots, a_r)M \cong N/(a_1, \dots, a_r)N$$

(apply the first isomorphism theorem) , so $M/(a, a_1, \dots, a_r)M$ also has finite length. Thus $\delta(M) \le r + 1$. By the induction hypothesis, $\delta(N) \le \dim N$. In summary,

$$\delta(M) \le r + 1 = \delta(N) + 1 \le \dim N + 1 \le \dim M. \clubsuit$$

5.4 Consequences of the Dimension Theorem

In this section we will see many applications of the dimension theorem (5.3.3).

5.4.1 Proposition

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$. If M is a finitely generated R-module, then $\dim M < \infty$; in particular, $\dim R < \infty$. Moreover, the dimension of R is the minimum, over all ideals I of definition of R, of the number of generators of I.

Proof. Finiteness of dimension follows from (5.3.3). The last assertion follows from the definition of Chevalley dimension in (5.3.2). In more detail, R/I has finite length iff (by the Noetherian hypothesis) R/I is Artinian iff [by (5.2.5)] I is an ideal of definition. ♣

5.4.2 Proposition

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$ and residue field $k = R/\mathcal{M}$. Then $\dim R \leq \dim_k(\mathcal{M}/\mathcal{M}^2)$.

Proof. Let $a_1, \dots, a_r$ be elements of $\mathcal{M}$ such that $\overline{a}_1, \dots, \overline{a}_r$ form a k-basis of $\mathcal{M}/\mathcal{M}^2$, where $\overline{a}_i = a_i + \mathcal{M}$. Then by (0.3.4), $a_1, \dots, a_r$ generate $\mathcal{M}$. Since $\mathcal{M}$ itself is an ideal of definition (see (5.2.5)), we have $\dim R \leq r$ by (5.4.1). (Alternatively, by (5.4.5), the height of $\mathcal{M}$ is at most r. Since ht $\mathcal{M} = \dim R$, the result follows.) ♣

5.4.3 Proposition

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$, and $\hat{R}$ its $\mathcal{M}$-adic completion. Then $\dim R = \dim \hat{R}$.

Proof. By (4.2.9), $R/\mathcal{M}^n \cong \hat{R}/\hat{\mathcal{M}}^n$. By (5.2.6), $s_{\mathcal{M}}(R, n) = s_{\hat{\mathcal{M}}}(\hat{R}, n)$. In particular, the two Hilbert-Samuel polynomials must have the same degree. Therefore $d(R) = d(\hat{R})$, and the result follows from (5.3.3). ♣

[Note that $\hat{R}$ is local by Problem 16 of Chapter 4, and $\hat{R}$ is also Noetherian. (Sketch: If R is Noetherian, then so is gr(R), the associated graded ring of R. But gr$(R) \cong$ gr$(\hat{R})$, so gr$(\hat{R})$ is Noetherian, which in turn implies that $\hat{R}$ is Noetherian.)]

5.4.4 Theorem

If R is a Noetherian ring, then the prime ideals of R satisfy the descending chain condition.

Proof. We may assume without loss of generality that R is a local ring. Explicitly, if P_0 is a prime ideal of R, let A be the localized ring R_{P_0}. Then the chain $P_0 \supset P_1 \supset P_2 \supset \dots$ of prime ideals of R will stabilize if and only if the chain $AP_0 \supset AP_1 \supset AP_2 \supset \cdots$ of prime ideals of A stabilizes. But if R is local as well as Noetherian, the result is immediate because $\dim R < \infty$. ♣

5.4.5 Generalization of Krull's Principal Ideal Theorem

Let P be a prime ideal of the Noetherian ring R. The following conditions are equivalent:
(a) ht $P \leq n$;
(b) There is an ideal I of R that is generated by n elements, such that P is a minimal prime ideal over I. (In other words, P is minimal subject to $P \supseteq I$.)

Proof. If (b) holds, then IR_P is an ideal of definition of R_P that is generated by n elements. (See (3.3.10), and note that if P is minimal over I iff $\sqrt{I} = P$.) By (5.3.1) and (5.4.1), ht $P = \dim R_P \leq n$. Conversely, if (a) holds then $\dim R_P \leq n$, so by (5.4.1) there is an ideal of definition J of R_P generated by n elements $a_1/s, \dots, a_n/s$ with $s \in R \setminus P$. The elements a_i must belong to P, else the a_i/s will generate R_P, which is a contradiction because J must be proper; see (5.2.5). Take I to be the ideal of R generated by $a_1, \dots, a_n$. Invoking (3.3.10) as in the first part of the proof, we conclude that I satisfies (b). ♣

5.4.6 Krull's Principal Ideal Theorem

Let a be a nonzero element of the Noetherian ring R. If a is neither a unit nor a zero-divisor, then every minimal prime ideal P over (a) has height 1.

Proof. It follows from (5.4.5) that ht $P \leq 1$. Thus assume ht $P = \dim R_P = 0$. We claim that $R_P \neq 0$, hence $P \in \operatorname{Supp} R$. For if $a/1 = 0$, then for some $s \in R \setminus P$ we have $sa = 0$, which contradicts the hypothesis that a is not a zero-divisor. We may assume that P is minimal in the support of R, because otherwise P has height 1 and we are finished. By (1.5.9), P is an associated prime of R, so by (1.3.6), P consists entirely of zero-divisors, a contradiction. ♣

The hypothesis that a is not a unit is never used, but nothing is gained by dropping it. If a is a unit, then a cannot belong to the prime ideal P and the theorem is vacuously true.

5.4.7 Theorem

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$, and let $a \in \mathcal{M}$ be a non zero-divisor. Then $\dim R/(a) = \dim R - 1$.

Proof. We have $\dim R > 0$, for if $\dim R = 0$, then $\mathcal{M}$ is the only prime ideal, and as in the proof of (5.4.6), $\mathcal{M}$ consists entirely of zero-divisors, a contradiction. In the proof of part 3 of the dimension theorem (5.3.3), take $M = R$ and $N = R/(a)$ to conclude that $\dim R/(a) < \dim R$, hence $\dim R/(a) \leq \dim R - 1$. To prove equality, we appeal to part 2 of the proof of (5.3.3). This allows us to find elements $a_1, \dots, a_s \in \mathcal{M}$, with $s = \dim R/(a)$, such that the images $\overline{a}_i$ in $R/(a)$ generate an $\mathcal{M}/(a)$-primary ideal of $R/(a)$. Then $a, a_1, \dots, a_s$ generate an $\mathcal{M}$-primary ideal of R, so by (5.4.1) and (3.3.10), $\dim R \leq 1 + s = 1 + \dim R/(a)$. ♣

5.4.8 Corollary

Let a be a non zero-divisor belonging to the prime ideal P of the Noetherian ring R. Then ht $P/(a) =$ ht $P - 1$.

Proof. In (5.4.7), replace R by R_P and $R/(a)$ by $(R_P)_Q$, where Q is a minimal prime ideal over (a). ♣

5.4.9 Theorem

Let $R = k[[X_1, \dots, X_n]]$ be a formal power series ring in n variables over the field k. Then $\dim R = n$.

Proof. The unique maximal ideal is $(X_1, \dots, X_n)$, so the dimension of R is at most n. On the other hand, the dimension is at least n because of the chain

$$(0) \subset (X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, \dots, X_n)$$

of prime ideals. ♣

5.5 Strengthening of Noether's Normalization Lemma

5.5.1 Definition

An *affine k-algebra* is an integral domain that is also a finite-dimensional algebra over a field k.

Affine algebras are of great interest in algebraic geometry because they are the coordinate rings of affine algebraic varieties. To study them we will need a stronger version of Noether's normalization lemma. In this section we will give the statement and proof, following Serre's Local Algebra, page 42.

5.5.2 Theorem

Let A be a finitely generated k-algebra, and $I_1 \subset \cdots \subset I_r$ a chain of nonzero proper ideals of A. There exists a nonnegative integer n and elements $x_1, \dots, x_n \in A$ algebraically independent over k such that the following conditions are satisfied.

1. A is integral over $B = k[x_1, \dots, x_n]$. (This is the standard normalization lemma.)

2. For each $i = 1, \dots, r$, there is a positive integer $h(i)$ such that $I_i \cap B$ is generated (as an ideal of B) by $x_1, \dots, x_{h(i)}$.

Proof. It suffices to let A be a polynomial ring $k[Y_1, \dots, Y_m]$. For we may write $A = A'/I'_0$ where $A' = k[Y_1, \dots, Y_m]$. If I'_i is the preimage of I_i under the canonical map $A' \to A'/I'_0$, and we find elements $x'_0, \dots, x'_n \in A'$, relative to the ideals $I'_0 \subset I'_1 \subset \cdots \subset I'_r$, then the images of $x'_{i-h(0)}$ in A, $i > h(0)$, satisfy the required conditions. The proof is by induction on r.

Assume $r = 1$. We first consider the case in which I_1 is a principal ideal $(x_1) = x_1 A$ with $x_1 \notin k$. By our assumption that A is a polynomial ring, we have $x_1 = g(Y_1, \dots, Y_m)$ for some nonconstant polynomial g with coefficients in k. We claim that there are positive integers $r_i (i = 2, \dots, m)$ such that A is integral over $B = k[x_1, \dots, x_m]$, where

$$x_i = Y_i - Y_1^{r_i}, \quad i = 2, \dots, m.$$

If we can show that Y_1 is integral over B, then (since the x_i belong to B, hence are integral over B) all Y_i are integral over B, and therefore A is integral over B. Now Y_1 satisfies the equation $x_1 = g(Y_1, \dots, Y_m)$, so

$$g(Y_1, x_2 + Y_1^{r_2}, \dots, x_m + Y_1^{r_m}) - x_1 = 0.$$

If we write the polynomial g as a sum of monomials $\sum c_\alpha Y^\alpha, \alpha = (a_1, \dots, a_m), c_\alpha \neq 0$, the above equation becomes

$$\sum c_\alpha Y_1^{a_1} (x_2 + Y_1^{r_2})^{a_2} \cdots (x_m + Y_1^{r_m})^{a_m} - x_1 = 0.$$

To produce the desired r_i, let $f(\alpha) = a_1 + a_2 r_2 + \cdots + a_m r_m$, and pick the r_i so that all the $f(\alpha)$ are distinct. For example, take $r_i = s^i$, where s is greater than the maximum of the a_j. Then there will be a unique α that maximizes f, say $\alpha = \beta$, and we have

$$c_\beta Y_1^{f(\beta)} + \sum_{j < f(\beta)} p_j(x_1, \dots, x_m) Y_1^j = 0$$

so Y_1 is integral over B, and as we noted above, $A = k[Y_1, \dots, Y_m]$ is integral over $B = k[x_1, \dots, x_m]$. Since A has transcendence degree m over k and an integral extension must be algebraic, it follows that $x_1, \dots, x_m$ are algebraically independent over k. Thus the first assertion of the theorem holds (in this first case, where I_1 is principal). If we can show that $I_1 \cap B = (x_1) = x_1 B$, the second assertion will also hold. The right-to-left inclusion follows from our assumptions about x_1, so let t belong to $I_1 \cap B$. Then $t = x_1 u$ with $u \in A$, hence, dividing by x_1, $u \in A \cap k(x_1, \dots, x_m)$. Since B is isomorphic to a polynomial ring, it is a unique factorization domain and therefore integrally closed. Since A is integral over B, we have $u \in B$. Thus $x_1 A \cap B = x_1 B$, and the proof of the first case is complete. Note that we have also shown that $A \cap k(x_1, \dots, x_m) = B = k[x_1, \dots, x_m]$.

Still assuming $r = 1$, we now consider the general case by induction on m. If $m = 0$ there is nothing to prove, and we have already taken care of $m = 1$ (because A is then a PID). Let x_1 be a nonzero element of I_1, and note that $x_1 \notin k$ because I_1 is proper. By what we have just proved, there are elements $t_2, \dots, t_m \in A$ such that $x_1, t_2, \dots, t_m$ are algebraically independent over k, A is integral over the polynomial ring $C = k[x_1, t_2, \dots, t_m]$, and $x_1 A \cap C = x_1 C$. By the induction hypothesis, there are elements $x_2, \dots, x_m$ satisfying the conditions of the theorem for $k[t_2, \dots, t_m]$ and the ideal $I_1 \cap k[t_2, \dots, t_m]$. Then $x_1, \dots, x_m$ satisfy the desired conditions.

Finally, we take the inductive step from $r-1$ to r. let $t_1, \dots, t_m$ satisfy the conditions of the theorem for the chain of ideals $I_1 \subset \cdots \subset I_{r-1}$, and let $s = h(r-1)$. By the argument of the previous paragraph, there are elements $x_{s+1}, \dots, x_m \in k[t_{s+1}, \dots, t_m]$ satisfying the conditions for the ideal $I_r \cap k[t_{s+1}, \dots, t_m]$. Take $x_i = t_i$ for $1 \le i \le s$. ♣

5.6 Properties of Affine k-algebras

We will look at height, coheight and dimension of affine algebras.

5.6.1 Proposition

Let $S = R[X]$ where R is an arbitrary ring. If $Q \subset Q'$, where Q and Q' are prime ideals of S both lying above the same prime ideal P of R, then $Q = PS$.

Proof. Since R/P can be regarded as a subring of S/Q, we may assume without loss of generality that $P = 0$. By localizing with respect to the multiplicative set $R \setminus \{0\}$, we may assume that R is a field. But then every nonzero prime ideal of S is maximal, hence $Q = 0$. Since PS is also 0, the result follows. ♣

5.6.2 Corollary

Let I be an ideal of the Noetherian ring R, and let P be a prime ideal of R with $I \subseteq P$. Let S be the polynomial ring $R[X]$, and take $J = IS$ and $Q = PS$. If P is a minimal prime ideal over I, then Q is a minimal prime ideal over J.

Proof. To verify that Q is prime, note that $R[X]/PR[X] \cong R[X]/P[X] \cong (R/P)[X]$, an integral domain. By modding out I, we may assume that $I = 0$. Suppose that the prime ideal Q_1 of S is properly contained in Q. Then $Q_1 \cap R \subseteq Q \cap R = PS \cap R = P$. (A polynomial belonging to R coincides with its constant term.) By minimality, $Q_1 \cap R = P$. By (5.6.1), $Q_1 = PS = Q$, a contradiction. ♣

5.6.3 Proposition

As above, let $S = R[X]$, R Noetherian, P a prime ideal of R, $Q = PS$. Then $\operatorname{ht} P = \operatorname{ht} Q$.

Proof. Let n be the height of P. By the generalized Krull principal ideal theorem (5.4.5), there is an ideal I of R generated by n elements such that P is a minimal prime ideal over I. By (5.6.2), $Q = PS$ is a minimal prime ideal over $J = IS$. But J is also generated over S by n elements, so again by (5.4.5), $\operatorname{ht} Q \leq \operatorname{ht} P$. Conversely, if $P_0 \subset P_1 \subset \cdots \subset P_n = P \subset R$ and $Q_i = P_i[X]$, then $Q_0 \subset Q_1 \subset \cdots \subset Q_n = Q$, so $\operatorname{ht} Q \geq \operatorname{ht} P$. ♣

We may now prove a major result on the dimension of a polynomial ring.

5.6.4 Theorem

Let $S = R[X]$, where R is a Noetherian ring. Then $\dim S = 1 + \dim R$.

Proof. Let $P_0 \subset P_1 \subset \cdots \subset P_n$ be a chain of prime ideals of R. If $Q_n = P_n S$, then by (5.6.3), $\operatorname{ht} Q_n = \operatorname{ht} P_n$. But the Q sequence can be extended via $Q_n \subset Q_{n+1} = Q_n + (X)$. (Note that X cannot belong to Q_n because $1 \notin P_n$.) It follows that $\dim S \geq 1 + \dim R$. Now consider a chain $Q_0 \subset Q_1 \subset \cdots \subset Q_n$ of prime ideals of S, and let $P_i = Q_i \cap R$ for every $i = 0, 1, \ldots, n$. We may assume that the P_i are not all distinct (otherwise $\dim R \geq \dim S \geq \dim S - 1$). Let j be the largest index i such that $P_i = P_{i+1}$. By (5.6.1), $Q_j = P_j S$, and by (5.6.3), $\operatorname{ht} P_j = \operatorname{ht} Q_j \geq j$. But by choice of j,

$$P_j = P_{j+1} \subset P_{j+2} \subset \cdots \subset P_n$$

so $\operatorname{ht} P_j + n - j - 1 \leq \dim R$. Since the height of P_j is at least j, we have $n - 1 \leq \dim R$, hence $\dim S \leq 1 + \dim R$. ♣

5.6.5 Corollary

If R is a Noetherian ring, then $\dim R[X_1, \ldots, X_n] = n + \dim R$. In particular, if K is a field then $\dim K[X_1, \ldots, X_n] = n$.

Proof. This follows from (5.6.4) by induction. ♣

5.6.6 Corollary

Let $R = K[X_1, \ldots, X_n]$, where K is a field. Then $\operatorname{ht}(X_1, \ldots, X_i) = i,\ 1 \leq i \leq n$.

Proof. First consider $i = n$. The height of $(X_1, \ldots, X_n)$ is at most n, the dimension of R, and in fact the height is n, in view of the chain $(X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, \ldots, X_n)$. The general result now follows by induction, using (5.4.8). ♣

If X is an affine algebraic variety over the field k, its (geometric) dimension is the transcendence degree over k of the function field (the fraction field of the coordinate ring). We now show that the geometric dimension coincides with the algebraic (Krull) dimension. We abbreviate transcendence degree by tr deg.

5.6.7 Theorem

If R is an affine k-algebra, then $\dim R = \text{tr deg}_k \text{ Frac } R$.

Proof. By Noether's normalization lemma, there are elements $x_1, \dots, x_n \in R$, algebraically independent over k, such that R is integral over $k[x_1, \dots, x_n]$. Since an integral extension cannot increase dimension (see Problem 4), $\dim R = \dim k[x_1, \dots, x_n] = n$ by (5.6.5). Let $F = \text{Frac } R$ and $L = \text{Frac } k[x_1, \dots, x_n]$. Then F is an algebraic extension of L, and since an algebraic extension cannot increase transcendence degree, we therefore have $\text{tr deg}_k F = \text{tr deg}_k L = n = \dim R$. ♣

It follows from the definitions that if P is a prime ideal of R, then $\text{ht}\, P + \text{coht}\, P \leq \dim R$. If R is an affine k-algebra, there is equality.

5.6.8 Theorem

If P is a prime ideal of the affine k-algebra R, then $\text{ht}\, P + \text{coht}\, P = \dim R$.

Proof. By Noether's normalization lemma, R is integral over a polynomial algebra. We can assume that $R = k[X_1, \dots, X_n]$ with $\text{ht}\, P = h$. (See Problems 4,5 and 6. An integral extension preserves dimension and coheight, and does not increase height. So if height plus coheight equals dimension in the smaller ring, the same must be true in the larger ring.) By the strong form (5.5.2) of Noether's normalization lemma, along with (5.6.6), there are elements $y_1, \dots, y_n$ algebraically independent over k such that R is integral over $k[y_1, \dots, y_n]$ and $Q = P \cap k[y_1, \dots, y_n] = (y_1, \dots, y_h)$ Since $k[y_1, \dots, y_n]/Q \cong k[y_{h+1}, \dots, y_n]$, it follows from (5.3.1) and (5.6.5) that $\text{coht}\, Q = \dim k[y_{h+1}, \dots, y_n] = n - h$. But $\text{coht}\, Q = \text{coht}\, P$ (Problem 5), so $\text{ht}\, P + \text{coht}\, P = h + (n - h) = n = \dim R$. ♣