Chapter 1

Primary Decomposition and Associated Primes

1.1 Primary Submodules and Ideals

1.1.1 Definitions and Comments

If N is a submodule of the R-module M, and $a \in R$, let $\lambda_a : M/N \to M/N$ be multiplication by a. We say that N is a *primary submodule* of M if N is proper and for every a, λ_a is either injective or nilpotent. Injectivity means that for all $x \in M$, we have $ax \in N \Rightarrow x \in N$. Nilpotence means that for some positive integer n, $a^n M \subseteq N$, that is, a^n belongs to the annihilator of M/N, denoted by $\operatorname{ann}(M/N)$. Equivalently, a belongs to the radical of the annihilator of M/N, denoted by $r_M(N)$.

Note that λ_a cannot be both injective and nilpotent. If so, nilpotence gives $a^n M = a(a^{n-1}M) \subseteq N$, and injectivity gives $a^{n-1}M \subseteq N$. Inductively, $M \subseteq N$, so $M = N$, contradicting the assumption that N is proper. Thus if N is a primary submodule of M, then $r_M(N)$ is the set of all $a \in R$ such that λ_a is not injective. Since $r_M(N)$ is the radical of an ideal, it is an ideal of R, and in fact it is a prime ideal. For if λ_a and λ_b are injective, so is $\lambda_{ab} = \lambda_a \circ \lambda_b$. (Note that $r_M(N)$ is proper because λ_1 is injective.) If $P = r_M(N)$, we say that N is *P-primary*.

If I is any ideal of R, then $r_R(I) = \sqrt{I}$, because $\operatorname{ann}(R/I) = I$. (Note that $a \in \operatorname{ann}(R/I)$ iff $aR \subseteq I$ iff $a = a1 \in I$.)

Specializing to $M = R$ and replacing a by y, we define a *primary ideal* in a ring R as a proper ideal Q such that if $xy \in Q$, then either $x \in Q$ or $y^n \in Q$ for some $n \geq 1$. Equivalently, $R/Q \neq 0$ and every zero-divisor in R/Q is nilpotent.

A useful observation is that if P is a prime ideal, then $\sqrt{P^n} = P$ for all $n \geq 1$. (The radical of P^n is the intersection of all prime ideals containing P^n, one of which is P. Thus $\sqrt{P^n} \subseteq P$. Conversely, if $x \in P$, then $x^n \in P^n$, so $x \in \sqrt{P^n}$.)

1.1.2 Lemma

If $\sqrt{I}$ is a maximal ideal $\mathcal{M}$, then I is $\mathcal{M}$-primary.

Proof. Suppose that $ab \in I$ and b does not belong to $\sqrt{I} = \mathcal{M}$. Then by maximality of $\mathcal{M}$, it follows that $\mathcal{M} + Rb = R$, so for some $m \in \mathcal{M}$ and $r \in R$ we have $m + rb = 1$. Now $m \in \mathcal{M} = \sqrt{I}$, hence $m^k \in I$ for some $k \geq 1$. Thus $1 = 1^k = (m + rb)^k = m^k + sb$ for some $s \in R$. Multiply by a to get $a = am^k + sab \in I$. ♣

1.1.3 Corollary

If $\mathcal{M}$ is a maximal ideal, then $\mathcal{M}^n$ is $\mathcal{M}$-primary for every $n \geq 1$.

Proof. As we observed in (1.1.1), $\sqrt{\mathcal{M}^n} = \mathcal{M}$, and the result follows from (1.1.2). ♣

1.2 Primary Decomposition

1.2.1 Definitions and Comments

A *primary decomposition* of the submodule N of M is given by $N = \cap_{i=1}^r N_i$, where the N_i are P_i-primary submodules. The decomposition is *reduced* if the P_i are distinct and N cannot be expressed as the intersection of a proper subcollection of the N_i.

We can always extract a reduced primary decomposition from an unreduced one, by discarding those N_i that contain $\cap_{j \neq i} N_j$ and intersecting those N_i that are P-primary for the same P. The following result justifies this process.

1.2.2 Lemma

If $N_1, \dots, N_k$ are P-primary, then $\cap_{i=1}^k N_i$ is P-primary.

Proof. We may assume that $k = 2$; an induction argument takes care of larger values. Let $N = N_1 \cap N_2$ and $r_M(N_1) = r_M(N_2) = P$. Assume for the moment that $r_M(N) = P$. If $a \in R,\ x \in M, ax \in N$, and $a \notin r_M(N)$, then since N_1 and N_2 are P-primary, we have $x \in N_1 \cap N_2 = N$. It remains to show that $r_M(N) = P$. If $a \in P$, then there are positive integers n_1 and n_2 such that $a^{n_1} M \subseteq N_1$ and $a^{n_2} M \subseteq N_2$. Therefore $a^{n_1+n_2} M \subseteq N$, so $a \in r_M(N)$. Conversely, if $a \in r_M(N)$ then a belongs to $r_M(N_i)$ for $i = 1, 2$, and therefore $a \in P$. ♣

We now prepare to prove that every submodule of a Noetherian module has a primary decomposition.

1.2.3 Definition

The proper submodule N of M is *irreducible* if N cannot be expressed as $N_1 \cap N_2$ with N properly contained in the submodules $N_i, i = 1, 2$.

1.2.4 Proposition

If N is an irreducible submodule of the Noetherian module M, then N is primary.

Proof. If not, then for some $a \in R$, $\lambda_a : M/N \to M/N$ is neither injective nor nilpotent. The chain $\ker \lambda_a \subseteq \ker \lambda_a^2 \subseteq \ker \lambda_a^3 \subseteq \cdots$ terminates by the ascending chain condition, say at $\ker \lambda_a^i$. Let $\varphi = \lambda_a^i$; then $\ker \varphi = \ker \varphi^2$ and we claim that $\ker \varphi \cap \operatorname{im} \varphi = 0$. Suppose $x \in \ker \varphi \cap \operatorname{im} \varphi$, and let $x = \varphi(y)$. Then $0 = \varphi(x) = \varphi^2(y)$, so $y \in \ker \varphi^2 = \ker \varphi$, so $x = \varphi(y) = 0$.

Now λ_a is not injective, so $\ker \varphi \neq 0$, and λ_a is not nilpotent, so λ_a^i can't be 0 (because $a^i M \not\subseteq N$). Consequently, $\operatorname{im} \varphi \neq 0$.

Let $p : M \to M/N$ be the canonical epimorphism, and set $N_1 = p^{-1}(\ker \varphi)$, $N_2 = p^{-1}(\operatorname{im} \varphi)$. We will prove that $N = N_1 \cap N_2$. If $x \in N_1 \cap N_2$, then $p(x)$ belongs to both $\ker \varphi$ and $\operatorname{im} \varphi$, so $p(x) = 0$, in other words, $x \in N$. Conversely, if $x \in N$, then $p(x) = 0 \in \ker \varphi \cap \operatorname{im} \varphi$, so $x \in N_1 \cap N_2$.

Finally, we will show that N is properly contained in both N_1 and N_2, so N is reducible, a contradiction. Choose a nonzero element $y \in \ker \varphi$. Since p is surjective, there exists $x \in M$ such that $p(x) = y$. Thus $x \in p^{-1}(\ker \varphi) = N_1$ (because $y = p(x) \in \ker \varphi$), but $x \notin N$ (because $p(x) = y \neq 0$). Similarly, $N \subset N_2$ (with $0 \neq y \in \operatorname{im} \varphi$), and the result follows. ♣

1.2.5 Theorem

If N is a proper submodule of the Noetherian module M, then N has a primary decomposition, hence a reduced primary decomposition.

Proof. We will show that N can be expressed as a finite intersection of irreducible submodules of M, so that (1.2.4) applies. Let $\mathcal{S}$ be the collection of all submodules of M that cannot be expressed in this form. If $\mathcal{S}$ is nonempty, then $\mathcal{S}$ has a maximal element N (because M is Noetherian). By definition of $\mathcal{S}$, N must be reducible, so we can write $N = N_1 \cap N_2$, $N \subset N_1$, $N \subset N_2$. By maximality of N, N_1 and N_2 can be expressed as finite intersections of irreducible submodules, hence so can N, contradicting $N \in \mathcal{S}$. Thus $\mathcal{S}$ is empty. ♣

1.3 Associated Primes

1.3.1 Definitions and Comments

Let M be an R-module, and P a prime ideal of R. We say that P is an *associated prime* of M (or that P is *associated* to M) if P is the annihilator of some $x \in M$. (Note that $x \neq 0$ because P is proper.) The set of associated primes of M is denoted by $\mathrm{AP}(M)$. (The standard notation is Ass(M). Please do not use this regrettable terminology.)

Here is a useful characterization of associated primes.

1.3.2 Proposition

The prime ideal P is associated to M if and only if there is an injective R-module homomorphism from R/P to M. Therefore if N is a submodule of M, then $\mathrm{AP}(N) \subseteq \mathrm{AP}(M)$.

Proof. If P is the annihilator of $x \neq 0$, the desired homomorphism is given by $r+P \to rx$. Conversely, if an injective R-homomorphism from R/P to M exists, let x be the image of $1 + P$. We will show that $P = \operatorname{ann}_R(x)$, the set of elements $r \in R$ such that $rx = 0$. If $r \in P$, then $r + P = 0$, so $rx = 0$, and therefore $r \in \operatorname{ann}_R(x)$. If $rx = 0$, then by injectivity, $r + P = 0$, so $r \in P$. ♣

Associated primes exist under wide conditions, and are sometimes unique.

1.3.3 Proposition

If $M = 0$, then $\mathrm{AP}(M)$ is empty. The converse holds if R is a Noetherian ring.

Proof. There are no nonzero elements in the zero module, hence no associated primes. Assuming that $M \neq 0$ and R is Noetherian, there is a maximal element $I = \operatorname{ann}_R x$ in the collection of all annihilators of nonzero elements of M. The ideal I must be proper, for if $I = R$, then $x = 1x = 0$, a contradiction. If we can show that I is prime, we have $I \in \mathrm{AP}(M)$, as desired. Let $ab \in I$ with $a \notin I$. Then $abx = 0$ but $ax \neq 0$, so $b \in \operatorname{ann}(ax)$. But $I = \operatorname{ann} x \subseteq \operatorname{ann}(ax)$, and the maximality of I gives $I = \operatorname{ann}(ax)$. Consequently, $b \in I$. ♣

1.3.4 Proposition

For any prime ideal P, $\mathrm{AP}(R/P) = \{P\}$.

Proof. By (1.3.2), P is an associated prime of R/P because there certainly is an R-monomorphism from R/P to itself. If $Q \in \mathrm{AP}(R/P)$, we must show that $Q = P$. Suppose that $Q = \operatorname{ann}(r + P)$. Then $s \in Q$ iff $sr \in P$ iff $s \in P$ (because P is prime). ♣

1.3.5 Remark

Proposition 1.3.4 shows that the annihilator of any nonzero element of R/P is P.

The next result gives us considerable information about the elements that belong to associated primes.

1.3.6 Theorem

Let $z(M)$ be the set of zero-divisors of M, that is, the set of all $r \in R$ such that $rx = 0$ for some nonzero $x \in M$. Then $\cup\{P : P \in \mathrm{AP}(M)\} \subseteq z(M)$, with equality if R is Noetherian.

Proof. The inclusion follows from the definition of associated prime; see (1.3.1). Thus assume $a \in z(M)$, with $ax = 0$, $x \in M$, $x \neq 0$. Then $Rx \neq 0$, so by (1.3.3) [assuming R Noetherian], Rx has an associated prime $P = \operatorname{ann}(bx)$. Since $ax = 0$ we have $abx = 0$, so $a \in P$. But $P \in \mathrm{AP}(Rx) \subseteq \mathrm{AP}(M)$ by (1.3.2). Therefore $a \in \cup\{P : P \in \mathrm{AP}(M)\}$. ♣

Now we prove a companion result to (1.3.2).

1.3.7 Proposition

If N is a submodule of M, then $\mathrm{AP}(M) \subseteq \mathrm{AP}(N) \cup \mathrm{AP}(M/N)$.

Proof. Let $P \in \mathrm{AP}(M)$, and let $h : R/P \to M$ be a monomorphism. Set $H = h(R/P)$ and $L = H \cap N$.
Case 1: $L = 0$. Then the map from H to M/N given by $h(r + P) \to h(r + P) + N$ is a monomorphism. (If $h(r + P)$ belongs to N, it must belong to $H \cap N = 0$.) Thus H is isomorphic to a submodule of M/N, so by definition of H, there is a monomorphism from R/P to M/N. Thus $P \in \mathrm{AP}(M/N)$.
Case 2: $L \neq 0$. If L has a nonzero element x, then x must belong to both H and N, and H is isomorphic to R/P via h. Thus $x \in N$ and the annihilator of x coincides with the annihilator of some nonzero element of R/P. By (1.3.5), $\operatorname{ann} x = P$, so $P \in \mathrm{AP}(N)$. ♣

1.3.8 Corollary

$$\mathrm{AP}(\bigoplus_{j \in J} M_j) = \bigcup_{j \in J} \mathrm{AP}(M_j).$$

Proof. By (1.3.2), the right side is contained in the left side. The result follows from (1.3.7) when the index set is finite. For example,

$$\begin{aligned}\mathrm{AP}(M_1 \oplus M_2 \oplus M_3) &\subseteq \mathrm{AP}(M_1) \cup \mathrm{AP}(M/M_1)\\ &= \mathrm{AP}(M_1) \cup \mathrm{AP}(M_2 \oplus M_3)\\ &\subseteq \mathrm{AP}(M_1) \cup \mathrm{AP}(M_2) \cup \mathrm{AP}(M_3).\end{aligned}$$

In general, if P is an associated prime of the direct sum, then there is a monomorphism from R/P to $\oplus M_j$. The image of the monomorphism is contained in the direct sum of finitely many components, as R/P is generated as an R-module by the single element $1 + P$. This takes us back to the finite case. ♣

We now establish the connection between associated primes and primary decomposition, and show that under wide conditions, there are only finitely many associated primes.

1.3.9 Theorem

Let M be a nonzero finitely generated module over the Noetherian ring R, so that by (1.2.5), every proper submodule of M has a reduced primary decomposition. In particular, the zero module can be expressed as $\cap_{i=1}^{r} N_i$, where N_i is P_i-primary. Then $\mathrm{AP}(M) = \{P_1, \dots, P_r\}$, a finite set.

Proof. Let P be an associated prime of M, so that $P = \operatorname{ann}(x)$, $x \neq 0$, $x \in M$. Renumber the N_i so that $x \notin N_i$ for $1 \leq i \leq j$ and $x \in N_i$ for $j + 1 \leq i \leq r$. Since N_i is P_i-primary, we have $P_i = r_M(N_i)$ (see (1.1.1)). Since P_i is finitely generated, $P_i^{n_i} M \subseteq N_i$ for some $n_i \geq 1$. Therefore

$$(\bigcap_{i=1}^{j} P_i^{n_i})x \ \subseteq \ \bigcap_{i=1}^{r} N_i = (0)$$

so $\cap_{i=1}^{j} P_i^{n_i} \subseteq \operatorname{ann}(x) = P$. (By our renumbering, there is a j rather than an r on the left side of the inclusion.) Since P is prime, $P_i \subseteq P$ for some $i \leq j$. We claim that $P_i = P$, so that every associated prime must be one of the P_i. To verify this, let $a \in P$. Then $ax = 0$ and $x \notin N_i$, so λ_a is not injective and therefore must be nilpotent. Consequently, $a \in r_M(N_i) = P_i$, as claimed.

Conversely, we show that each P_i is an associated prime. Without loss of generality, we may take $i = 1$. Since the decomposition is reduced, N_1 does not contain the intersection of the other N_i's, so we can choose $x \in N_2 \cap \cdots \cap N_r$ with $x \notin N_1$. Now N_1 is P_1-primary, so as in the preceding paragraph, for some $n \geq 1$ we have $P_1^n x \subseteq N_1$ but $P_1^{n-1} x \not\subseteq N_1$. (Take $P_1^0 x = Rx$ and recall that $x \notin N_1$.) If we choose $y \in P_1^{n-1} x \setminus N_1$ (hence $y \neq 0$), the proof will be complete upon showing that P_1 is the annihilator of y. We have $P_1 y \subseteq P_1^n x \subseteq N_1$ and $x \in \cap_{i=2}^r N_i$, so $P_1^n x \subseteq \cap_{i=2}^r N_i$. Thus $P_1 y \subseteq \cap_{i=1}^r N_i = (0)$, so $P_1 \subseteq \operatorname{ann} y$. On the other hand, if $a \in R$ and $ay = 0$, then $ay \in N_1$ but $y \notin N_1$, so $\lambda_a : M/N_1 \to M/N_1$ is not injective and is therefore nilpotent. Thus $a \in r_M(N_1) = P_1$. ♣

We can now say something about uniqueness in primary decompositions.

1.3.10 First Uniqueness Theorem

Let M be a finitely generated module over the Noetherian ring R. If $N = \cap_{i=1}^r N_i$ is a reduced primary decomposition of the submodule N, and N_i is P_i-primary, $i = 1, \dots, r$, then (regarding M and R as fixed) the P_i are uniquely determined by N.

Proof. By the correspondence theorem, a reduced primary decomposition of (0) in M/N is given by $(0) = \cap_{i=1}^r N_i/N$, and N_i/N is P_i-primary, $1 \leq i \leq r$. By (1.3.9),

$$\operatorname{AP}(M/N) = \{P_1, \dots, P_r\}.$$

But [see (1.3.1)] the associated primes of M/N are determined by N. ♣

1.3.11 Corollary

Let N be a submodule of M (finitely generated over the Noetherian ring R). Then N is P-primary iff $\operatorname{AP}(M/N) = \{P\}$.

Proof. The "only if" part follows from the displayed equation above. Conversely, if P is the only associated prime of M/N, then N coincides with a P-primary submodule N', and hence $N(= N')$ is P-primary. ♣

1.3.12 Definitions and Comments

Let $N = \cap_{i=1}^r N_i$ be a reduced primary decomposition, with associated primes $P_1, \dots, P_r$. We say that N_i is an *isolated* (or *minimal*) component if P_i is minimal, that is P_i does not properly contain any P_j, $j \neq i$. Otherwise, N_i is an *embedded* component (see Exercise 5 for an example). Embedded components arise in algebraic geometry in situations where one irreducible algebraic set is properly contained in another.

1.4 Associated Primes and Localization

To get more information about uniqueness in primary decompositions, we need to look at associated primes in localized rings and modules. In this section, S will be a multiplicative subset of the Noetherian ring R, R_S the localization of R by S, and M_S the localization of the R-module M by S. Recall that $P \to P_S = PR_S$ is a bijection of C, the set of prime ideals of R not meeting S, and the set of all prime ideals of R_S.

The set of associated primes of the R-module M will be denoted by $\mathrm{AP}_R(M)$. We need a subscript to distinguish this set from $\mathrm{AP}_{R_S}(M_S)$, the set of associated primes of the R_S-module M_S.

1.4.1 Lemma

Let P be a prime ideal not meeting S. If $P \in \mathrm{AP}_R(M)$, then $P_S = PR_S \in \mathrm{AP}_{R_S}(M_S)$. (By the above discussion, the map $P \to P_S$ is the restriction of a bijection and therefore must be injective.)

Proof. If P is the annihilator of the element $x \in M$, then P_S is the annihilator of the element $x/1 \in M_S$. For if $a \in P$ and $a/s \in P_S$, then $(a/s)(x/1) = ax/s = 0$. Conversely, if $(a/s)(x/1) = 0$, then there exists $t \in S$ such that $tax = 0$, and it follows that $a/s = at/st \in P_S$. ♣

1.4.2 Lemma

The map of (1.4.1) is surjective, hence is a bijection of $\mathrm{AP}_R(M) \cap C$ and $\mathrm{AP}_{R_S}(M_S)$.

Proof. Let P be generated by $a_1, \dots, a_n$. Suppose that P_S is the annihilator of the element $x/t \in M_S$. Then $(a_i/1)(x/t) = 0$, $1 \le i \le n$. For each i there exists $s_i \in S$ such that $s_i a_i x = 0$. If s is the product of the s_i, then $sa_i x = 0$ for all i, hence $sax = 0$ for all $a \in P$. Thus $P \subseteq \mathrm{ann}(sx)$. On the other hand, suppose b annihilates sx. Then $(b/1)(x/t) = bsx/st = 0$, so $b/1 \in P_S$, and consequently $b/1 = b'/s'$ for some $b' \in P$ and $s' \in S$. This means that for some $u \in S$ we have $u(bs' - b') = 0$. Now b', hence ub', belongs to P, and therefore so does ubs'. But $us' \notin P$ (because $S \cap P = \emptyset$). We conclude that $b \in P$, so $P = \mathrm{ann}(sx)$. ♣

1.4.3 Lemma

Let M be a finitely generated module over the Noetherian ring R, and N a P-primary submodule of M. Let P' be any prime ideal of R, and set $M' = M_{P'}$, $N' = N_{P'}$. If $P \not\subseteq P'$, then $N' = M'$.

Proof. By (1.4.1) and (1.4.2), there is a bijection between $\mathrm{AP}_{R_{P'}}(M/N)_{P'}$ (which coincides with $\mathrm{AP}_{R_{P'}}(M'/N')$) and the intersection $\mathrm{AP}_R(M/N) \cap C$, where C is the set of prime ideals contained in P' (in other words, not meeting $S = R \setminus P'$). By (1.3.11), there is only one associated prime of M/N over R, namely P, which is not contained in P' by hypothesis. Thus $\mathrm{AP}_R(M/N) \cap C$ is empty, so by (1.3.3), $M'/N' = 0$, and the result follows. ♣

At the beginning of the proof of (1.4.3), we have taken advantage of the isomorphism between $(M/N)_{P'}$ and M'/N'. The result comes from the exactness of the localization functor. If this is unfamiliar, look ahead to the proof of (1.5.3), where the technique is spelled out. See also TBGY, Section 8.5, Problem 5.

1.4.4 Lemma

In (1.4.3), if $P \subseteq P'$, then $N = f^{-1}(N')$, where f is the natural map from M to M'.

Proof. As in (1.4.3), $\mathrm{AP}_R(M/N) = \{P\}$. Since $P \subseteq P'$, we have $R \setminus P' \subseteq R \setminus P$. By (1.3.6), $R \setminus P'$ contains no zero-divisors of M/N, because all such zero-divisors belong to P. Thus the natural map $g : x \to x/1$ of M/N to $(M/N)_{P'} \cong M'/N'$ is injective. (If $x/1 = 0$, then $sx = 0$ for some $s \in S = R \setminus P'$, and since s is not a zero-divisor, we have $x = 0$.)

If $x \in N$, then $f(x) \in N'$ by definition of f, so assume $x \in f^{-1}(N')$. Then $f(x) \in N'$, so $f(x) + N'$ is 0 in M'/N'. By injectivity of the natural map $M/N \to (M/N)_{P'}$, $x + N$ is 0 in M/N, in other words, $x \in N$, and the result follows. ♣

1.4.5 Second Uniqueness Theorem

Let M be a finitely generated module over the Noetherian ring R. Suppose that $N = \cap_{i=1}^{r} N_i$ is a reduced primary decomposition of the submodule N, and N_i is P_i-primary, $i = 1, \dots, r$. If P_i is minimal, then (regarding M and R as fixed) N_i is uniquely determined by N.

Proof. Suppose that P_1 is minimal, so that $P_1 \not\supseteq P_i$, $i > 1$. By (1.4.3) with $P = P_i$, $P' = P_1$, we have $(N_i)_{P_1} = M_{P_1}$ for $i > 1$. By (1.4.4) with $P = P' = P_1$, we have $N_1 = f^{-1}[(N_1)_{P_1}]$, where f is the natural map from M to M_{P_1}. Now

$$N_{P_1} = (N_1)_{P_1} \cap \cap_{i=2}^{r} (N_i)_{P_1} = (N_1)_{P_1} \cap M_{P_1} = (N_1)_{P_1}.$$

Thus $N_1 = f^{-1}[(N_1)_{P_1}] = f^{-1}(N_{P_1})$ depends only on N and P_1, and since P_1 depends on the fixed ring R, it follows that N_1 depends only on N. ♣

1.5 The Support of a Module

The support of a module M is closely related to the set of associated primes of M. We will need the following result in order to proceed.

1.5.1 Proposition

M is the zero module if and only if $M_P = 0$ for every prime ideal P, if and only if $M_{\mathcal{M}} = 0$ for every maximal ideal $\mathcal{M}$.

Proof. It suffices to show that if $M_{\mathcal{M}} = 0$ for all maximal ideals $\mathcal{M}$, then $M = 0$. Choose a nonzero element $x \in M$, and let I be the annihilator of x. Then $1 \notin I$ (because $1x = x \neq 0$), so I is a proper ideal and is therefore contained in a maximal ideal $\mathcal{M}$. By hypothesis, $x/1$ is 0 in $M_{\mathcal{M}}$, hence there exists $a \notin \mathcal{M}$ (so $a \notin I$) such that $ax = 0$. But then by definition of I we have $a \in I$, a contradiction. ♣

1.5.2 Definitions and Comments

The *support* of an R-module M (notation $\operatorname{Supp} M$) is the set of prime ideals P of R such that $M_P \neq 0$. Thus $\operatorname{Supp} M = \emptyset$ iff $M_P = 0$ for all prime ideals P. By (1.5.1), this is equivalent to $M = 0$.

If I is any ideal of R, we define $V(I)$ as the set of prime ideals containing I. In algebraic geometry, the *Zariski topology* on $\operatorname{Spec} R$ has the sets $V(I)$ as its closed sets.

1.5.3 Proposition

$\operatorname{Supp} R/I = V(I)$.

Proof. We apply the localization functor to the exact sequence $0 \to I \to R \to R/I \to 0$ to get the exact sequence $0 \to I_P \to R_P \to (R/I)_P \to 0$. Consequently, $(R/I)_P \cong R_P/I_P$. Thus $P \in \operatorname{Supp} R/I$ iff $R_P \supset I_P$ iff I_P is contained in a maximal ideal, necessarily PR_P. But this is equivalent to $I \subseteq P$. To see this, suppose $a \in I$, with $a/1 \in I_P \subseteq PR_P$. Then $a/1 = b/s$ for some $b \in P,\ s \notin P$. There exists $c \notin P$ such that $c(as - b) = 0$. We have $cas = cb \in P$, a prime ideal, and $cs \notin P$. We conclude that $a \in P$. ♣

1.5.4 Proposition

Let $0 \to M' \to M \to M'' \to 0$ be exact, hence $0 \to M'_P \to M_P \to M''_P \to 0$ is exact. Then

$$\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''.$$

Proof. Let P belong to $\operatorname{Supp} M \setminus \operatorname{Supp} M'$. Then $M'_P = 0$, so the map $M_P \to M''_P$ is injective as well as surjective, hence is an isomorphism. But $M_P \neq 0$ by assumption, so $M''_P \neq 0$, and therefore $P \in \operatorname{Supp} M''$. On the other hand, since M'_P is isomorphic to a submodule of M_P, it follows that $\operatorname{Supp} M' \subseteq \operatorname{Supp} M$. If $M_P = 0$, then $M''_P = 0$ (because $M_P \to M''_P$ is surjective). Thus $\operatorname{Supp} M'' \subseteq \operatorname{Supp} M$. ♣

Supports and annihilators are connected by the following basic result.

1.5.5 Theorem

If M is a finitely generated R-module, then $\operatorname{Supp} M = V(\operatorname{ann} M)$.

Proof. Let $M = Rx_1 + \cdots + Rx_n$, so that $M_P = (Rx_1)_P + \cdots + (Rx_n)_P$. Then $\operatorname{Supp} M = \cup_{i=1}^n \operatorname{Supp} Rx_i$, and by the first isomorphism theorem, $Rx_i \cong R/\operatorname{ann} x_i$. By (1.5.3), $\operatorname{Supp} Rx_i = V(\operatorname{ann} x_i)$. Therefore $\operatorname{Supp} M = \cup_{i=1}^n V(\operatorname{ann} x_i) = V(\operatorname{ann} M)$. To justify the last equality, note that if $P \in V(\operatorname{ann} x_i)$, then $P \supseteq \operatorname{ann} x_i \supseteq \operatorname{ann} M$. Conversely, if $P \supseteq \operatorname{ann} M = \cap_{i=1}^n \operatorname{ann} x_i$, then $P \supseteq \operatorname{ann} x_i$ for some i. ♣

And now we connect associated primes and annihilators.

1.5.6 Proposition

If M is a finitely generated module over the Noetherian ring R, then

$$\bigcap_{P \in \mathrm{AP}(M)} P = \sqrt{\operatorname{ann} M}.$$

Proof. If $M = 0$, then by (1.3.3), $\mathrm{AP}(M) = \emptyset$, and the result to be proved is $R = R$. Thus assume $M \neq 0$, so that (0) is a proper submodule. By (1.2.5) and (1.3.9), there is a reduced primary decomposition $(0) = \cap_{i=1}^r N_i$, where for each i, N_i is P_i-primary and $\mathrm{AP}(M) = \{P_1, \dots, P_r\}$.

If $a \in \sqrt{\operatorname{ann} M}$, then for some $n \geq 1$ we have $a^n M = 0$. Thus for each i, $\lambda_a : M/N_i \to M/N_i$ is nilpotent [see (1.1.1)]. Consequently, $a \in \cap_{i=1}^r r_M(N_i) = \cap_{i=1}^r P_i$. Conversely, if a belongs to this intersection, then for all i there exists $n_i \geq 1$ such that $a^{n_i} M \subseteq N_i$. If $n = \max n_i$, then $a^n M = 0$, so $a \in \sqrt{\operatorname{ann} M}$. ♣

1.5.7 Corollary

If R is a Noetherian ring, then the nilradical of R is the intersection of all associated primes of R.

Proof. Take $M = R$ in (1.5.6). Since $\operatorname{ann} R = 0$, $\sqrt{\operatorname{ann} R}$ is the nilradical. ♣

And now, a connection between supports, associated primes and annihilators.

1.5.8 Proposition

Let M be a finitely generated module over the Noetherian ring R, and let P be any prime ideal of R. The following conditions are equivalent:

(1) $P \in \operatorname{Supp} M$;
(2) $P \supseteq P'$ for some $P' \in \mathrm{AP}(M)$;
(3) $P \supseteq \operatorname{ann} M$.

Proof. Conditions (1) and (3) are equivalent by (1.5.5). To prove that (1) implies (2), let $P \in \operatorname{Supp} M$. If P does not contain any associated prime of M, then P does not contain the intersection of all associated primes (because P is prime). By (1.5.6), P does not contain $\sqrt{\operatorname{ann} M}$, hence P cannot contain $\operatorname{ann} M$. (See TBGY, Section 8.3, Problem 2.) This contradicts (1.5.5). To prove that (2) implies (3), let Q be the intersection of all associated primes. Then $P \supseteq P' \supseteq Q =$ [by (1.5.6)] $\sqrt{\operatorname{ann} M} \supseteq \operatorname{ann} M$. ♣

Here is the most important connection between supports and associated primes.

1.5.9 Theorem

Let M be a finitely generated module over the Noetherian ring R. Then $\mathrm{AP}(M) \subseteq \operatorname{Supp} M$, and the minimal elements of $\mathrm{AP}(M)$ and $\operatorname{Supp} M$ are the same.

Proof. We have $\mathrm{AP}(M) \subseteq \operatorname{Supp} M$ by (2) implies (1) in (1.5.8), with $P = P'$. If P is minimal in $\operatorname{Supp} M$, then by (1) implies (2) in (1.5.8), P contains some $P' \in \mathrm{AP}(M) \subseteq$

$\operatorname{Supp} M$. By minimality, $P = P'$. Thus $P \in \mathrm{AP}(M)$, and in fact, P must be a minimal associated prime. Otherwise, $P \supset Q \in \mathrm{AP}(M) \subseteq \operatorname{Supp} M$, so that P is not minimal in $\operatorname{Supp} M$, a contradiction. Finally, let P be minimal among associated primes but not minimal in $\operatorname{Supp} M$. If $P \supset Q \in \operatorname{Supp} M$, then by (1) implies (2) in (1.5.8), $Q \supseteq P' \in \mathrm{AP}(M)$. By minimality, $P = P'$, contradicting $P \supset Q \supseteq P'$. ♣

Here is another way to show that there are only finitely many associated primes.

1.5.10 Theorem

Let M be a nonzero finitely generated module over the Noetherian ring R. Then there is a chain of submodules $0 = M_0 < M_1 < \cdots < M_n = M$ such that for each $j = 1, \dots, n$, $M_j/M_{j-1} \cong R/P_j$, where the P_j are prime ideals of R. For any such chain, $\mathrm{AP}(M) \subseteq \{P_1, \dots, P_n\}$.

Proof. By (1.3.3), M has an associated prime $P_1 = \operatorname{ann} x_1$, with x_1 a nonzero element of M. Take $M_1 = Rx_1 \cong R/P_1$ (apply the first isomorphism theorem). If $M \neq M_1$, then the quotient module M/M_1 is nonzero, hence [again by (1.3.3)] has an associated prime $P_2 = \operatorname{ann}(x_2 + M_1)$, $x_2 \notin M_1$. Let $M_2 = M_1 + Rx_2$. Now map R onto M_2/M_1 by $r \to rx_2 + M_1$. By the first isomorphism theorem, $M_2/M_1 \cong R/P_2$. Continue inductively to produce the desired chain. (Since M is Noetherian, the process terminates in a finite number of steps.) For each $j = 1, \dots, n$, we have $\mathrm{AP}(M_j) \subseteq \mathrm{AP}(M_{j-1}) \cup \{P_j\}$ by (1.3.4) and (1.3.7). Another inductive argument shows that $\mathrm{AP}(M) \subseteq \{P_1, \dots, P_n\}$. ♣

1.5.11 Proposition

In (1.5.10), each P_j belongs to $\operatorname{Supp} M$. Thus (replacing $\mathrm{AP}(M)$ by $\{P_1, \dots, P_n\}$ in the proof of (1.5.9)), the minimal elements of all three sets $\mathrm{AP}(M)$, $\{P_1, \dots, P_n\}$ and $\operatorname{Supp} M$ are the same.

Proof. By (1.3.4) and (1.5.9), $P_j \in \operatorname{Supp} R/P_j$, so by (1.5.10), $P_j \in \operatorname{Supp} M_j/M_{j-1}$. By (1.5.4), $\operatorname{Supp} M_j/M_{j-1} \subseteq \operatorname{Supp} M_j$, and finally $\operatorname{Supp} M_j \subseteq \operatorname{Supp} M$ because $M_j \subseteq M$. ♣

The following observation clarifies the relation between associated primes and supports. A prime ideal P belongs to $\mathrm{AP}(M)$ iff $\operatorname{ann}(x) = P$ and belongs to $\operatorname{Supp} M$ iff $\operatorname{ann}(x) \subseteq P$ for some $x \in M$.

1.6 Artinian Rings

1.6.1 Definitions and Comments

Recall that an R-module is Artinian if it satisfies the descending chain condition on submodules. If the ring R is Artinian as a module over itself, in other words, R satisfies the dcc on ideals, then R is said to be an *Artinian ring*. Note that $\mathbb{Z}$ is a Noetherian ring that is not Artinian. Any finite ring, for example $\mathbb{Z}_n$, is both Noetherian and Artinian, and in fact we will prove later in the section that an Artinian ring must be Noetherian. The theory of associated primes and supports will help us to analyze Artinian rings.

1.6.2 Lemma

If I is an ideal in the Artinian ring R, then R/I is an Artinian ring.

Proof. Since R/I is a quotient of an Artinian R-module, it is also an Artinian R-module. In fact it is an R/I module via $(r+I)(x+I) = rx + I$, and the R-submodules are identical to the R/I-submodules. Thus R/I is an Artinian R/I-module, in other words, an Artinian ring. ♣

1.6.3 Lemma

An Artinian integral domain is a field.

Proof. Let a be a nonzero element of the Artinian domain R. We must produce a multiplicative inverse of a. The chain of ideals $(a) \supseteq (a^2) \supseteq (a^3) \supseteq \cdots$ stabilizes, so for some t we have $(a^t) = (a^{t+1})$. If $a^t = ba^{t+1}$, then since R is a domain, $ba = 1$. ♣

1.6.4 Proposition

If R is an Artinian ring, then every prime ideal of R is maximal. Therefore, the nilradical $N(R)$ coincides with the Jacobson radical $J(R)$.

Proof. Let P be a prime ideal of R, so that R/P is an integral domain, Artinian by (1.6.2). By (1.6.3), R/P is a field, hence P is maximal. ♣

One gets the impression that the Artinian property puts strong constraints on a ring. The following two results reinforce this conclusion.

1.6.5 Proposition

An Artinian ring has only finitely many maximal ideals.

Proof. Let Σ be the collection of all finite intersections of maximal ideals. Then Σ is nonempty and has a minimal element $I = \mathcal{M}_1 \cap \cdots \cap \mathcal{M}_r$ (by the Artinian property). If $\mathcal{M}$ is any maximal ideal, then $\mathcal{M} \supseteq \mathcal{M} \cap I \in \Sigma$, so by minimality of I we have $\mathcal{M} \cap I = I$. But then $\mathcal{M}$ must contain one of the $\mathcal{M}_i$ (because $\mathcal{M}$ is prime), hence $\mathcal{M} = \mathcal{M}_i$ (because $\mathcal{M}$ and $\mathcal{M}_i$ are maximal). ♣

1.6.6 Proposition

If R is Artinian, then the nilradical $N(R)$ is nilpotent, hence by (1.6.4), the Jacobson radical $J(R)$ is nilpotent.

Proof. Let $I = N(R)$. The chain $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ stabilizes, so for some i we have $I^i = I^{i+1} = \cdots = L$. If $L = 0$ we are finished, so assume $L \neq 0$. Let Σ be the collection of all ideals K of R such that $KL \neq 0$. Then Σ is nonempty, since L (as well as R) belongs to Σ. Let K_0 be a minimal element of Σ, and choose $a \in K_0$ such that $aL \neq 0$. Then $Ra \subseteq K_0$ (because K_0 is an ideal), and $RaL = aL \neq 0$, hence $Ra \in \Sigma$. By minimality of K_0 we have $Ra = K_0$.

We will show that the principal ideal $(a) = Ra$ coincides with aL. We have $aL \subseteq Ra = K_0$, and $(aL)L = aL^2 = aL \neq 0$, so $aL \in \Sigma$. By minimality of K_0 we have $aL = K_0 = Ra$.

From $(a) = aL$ we get $a = ab$ for some $b \in L \subseteq N(R)$, so $b^n = 0$ for some $n \geq 1$. Therefore $a = ab = (ab)b = ab^2 = \cdots = ab^n = 0$, contradicting our choice of a. Since the assumption $L \neq 0$ has led to a contradiction, we must have $L = 0$. But L is a power of the nilradical I, and the result follows. ♣

We now prove a fundamental structure theorem for Artinian rings.

1.6.7 Theorem

Every Artinian ring R is isomorphic to a finite direct product of Artinian local rings R_i.

Proof. By (1.6.5), R has only finitely many maximal ideals $\mathcal{M}_1, \dots, \mathcal{M}_r$. The intersection of the $\mathcal{M}_i$ is the Jacobson radical $J(R)$, which is nilpotent by (1.6.6). By the Chinese remainder theorem, the intersection of the $\mathcal{M}_i$ coincides with their product. Thus for some $k \geq 1$ we have $(\prod_1^r \mathcal{M}_i)^k = \prod_1^r \mathcal{M}_i^k = 0$. Powers of the $\mathcal{M}_i$ still satisfy the hypothesis of the Chinese remainder theorem, so the natural map from R to $\prod_1^r R/\mathcal{M}_i^k$ is an isomorphism. By (1.6.2), $R/\mathcal{M}_i^k$ is Artinian, and we must show that it is local. A maximal ideal of $R/\mathcal{M}_i^k$ corresponds to a maximal ideal $\mathcal{M}$ of R with $\mathcal{M} \supseteq \mathcal{M}_i^k$, hence $\mathcal{M} \supseteq \mathcal{M}_i$ (because $\mathcal{M}$ is prime). By maximality, $\mathcal{M} = \mathcal{M}_i$. Thus the unique maximal ideal of $R/\mathcal{M}_i^k$ is $\mathcal{M}_i/\mathcal{M}_i^k$. ♣

1.6.8 Remarks

A finite direct product of Artinian rings, in particular, a finite direct product of fields, is Artinian. To see this, project a descending chain of ideals onto one of the coordinate rings. At some point, all projections will stabilize, so the original chain will stabilize. A sequence of exercises will establish the uniqueness of the Artinian local rings in the decomposition (1.6.7).

It is a standard result that an R-module M has finite length $l_R(M)$ if and only if M is both Artinian and Noetherian. We can relate this condition to associated primes and supports.

1.6.9 Proposition

Let M be a finitely generated module over the Noetherian ring R. The following conditions are equivalent:
(1) $l_R(M) < \infty$;
(2) Every associated prime ideal of M is maximal;
(3) Every prime ideal in the support of M is maximal.

Proof.
(1) $\Rightarrow$ (2): As in (1.5.10), there is a chain of submodules $0 = M_0 < \cdots < M_n = M$, with $M_i/M_{i-1} \cong R/P_i$. Since M_i/M_{i-1} is a submodule of a quotient M/M_{i-1} of M, the hypothesis (1) implies that R/P_i has finite length for all i. Thus R/P_i is an Artinian R-module, hence an Artinian R/P_i-module (note that P_i annihilates R/P_i). In other words, R/P_i is an Artinian ring. But P_i is prime, so R/P_i is an integral domain, hence

a field by (1.6.3). Therefore each P_i is a maximal ideal. Since every associated prime is one of the P_i's [see (1.5.10)], the result follows.

(2) $\Rightarrow$ (3): If $P \in \operatorname{Supp} M$, then by (1.5.8), P contains some associated prime Q. By hypothesis, Q is maximal, hence so is P.

(3) $\Rightarrow$ (1): By (1.5.11) and the hypothesis (3), every P_i is maximal, so R/P_i is a field. Consequently, $l_R(M_i/M_{i-1}) = l_R(R/P_i) = 1$ for all i. But length is additive, that is, if N is a submodule of M, then $l(M) = l(N) + l(M/N)$. Summing on i from 1 to n, we get $l_R(M) = n < \infty$. ♣

1.6.10 Corollary

Let M be finitely generated over the Noetherian ring R. If $l_R(M) < \infty$, then $\mathrm{AP}(M) = \operatorname{Supp} M$.

Proof. By (1.5.9), $\mathrm{AP}(M) \subseteq \operatorname{Supp} M$, so let $P \in \operatorname{Supp} M$. By (1.5.8), $P \supseteq P'$ for some $P' \in \mathrm{AP}(M)$. By (1.6.9), P and P' are both maximal, so $P = P' \in \mathrm{AP}(M)$. ♣

We can now characterize Artinian rings in several ways.

1.6.11 Theorem

Let R be a Noetherian ring. The following conditions are equivalent:
(1) R is Artinian;
(2) Every prime ideal of R is maximal;
(3) Every associated prime ideal of R is maximal.

Proof. (1) implies (2) by (1.6.4), and (2) implies (3) is immediate. To prove that (3) implies (1), note that by (1.6.9), $l_R(R) < \infty$, hence R is Artinian. ♣

1.6.12 Theorem

The ring R is Artinian if and only if $l_R(R) < \infty$.

Proof. The "if" part follows because any module of finite length is Artinian and Noetherian. Thus assume R Artinian. As in (1.6.7), the zero ideal is a finite product $\mathcal{M}_1 \cdots \mathcal{M}_k$ of not necessarily distinct maximal ideals. Now consider the chain

$$R = \mathcal{M}_0 \supseteq \mathcal{M}_1 \supseteq \mathcal{M}_1\mathcal{M}_2 \supseteq \cdots \supseteq \mathcal{M}_1 \cdots \mathcal{M}_{k-1} \supseteq \mathcal{M}_1 \cdots \mathcal{M}_k = 0.$$

Since any submodule or quotient module of an Artinian module is Artinian, it follows that $T_i = \mathcal{M}_1 \cdots \mathcal{M}_{i-1}/\mathcal{M}_1 \cdots \mathcal{M}_i$ is an Artinian R-module, hence an Artinian $R/\mathcal{M}_i$-module. (Note that $\mathcal{M}_i$ annihilates $\mathcal{M}_1 \cdots \mathcal{M}_{i-1}/\mathcal{M}_1 \cdots \mathcal{M}_i$.) Thus T_i is a vector space over the field $R/\mathcal{M}_i$, and this vector space is finite-dimensional by the descending chain condition. Thus T_i has finite length as an $R/\mathcal{M}_i$-module, hence as an R-module. By additivity of length [as in (3) implies (1) in (1.6.9)], we conclude that $l_R(R) < \infty$. ♣

1.6.13 Theorem

The ring R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal.

Proof. The "if" part follows from (1.6.11). If R is Artinian, then $l_R(R) < \infty$ by (1.6.12), hence R is Noetherian. By (1.6.4) or (1.6.11), every prime ideal of R is maximal. ♣

1.6.14 Corollary

Let M be finitely generated over the Artinian ring R. Then $l_R(M) < \infty$.

Proof. By (1.6.13), R is Noetherian, hence the module M is both Artinian and Noetherian. Consequently, M has finite length. ♣