Chapter 6

Factorization of Analytic Functions

In this chapter we will consider the problems of factoring out the zeros of an analytic function f on a region Ω (à la polynomials), and of decomposing a meromorphic function (à la partial fractions for rational functions). Suppose f is analytic on a region Ω and $f \not\equiv 0$. What can be said about $Z(f)$? Theorem 2.4.8, the identity theorem, asserts that $Z(f)$ has no limit point in Ω. It turns out that no more can be said in general. That is, if A is any subset of Ω with no limit point in Ω, then there exists $f \in A(\Omega)$ whose set of zeros is precisely A. Furthermore, we can prescribe the order of the zero which f shall have at each point of A. Now if A is a finite subset of Ω, say $\{z_1, \dots, z_n\}$, and $m_1, \dots, m_n$ are the corresponding desired multiplicities, then the finite product

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n}$$

would be such a function. However, in general the construction of such an f is accomplished using *infinite products*, which we now study in detail.

6.1 Infinite Products

Let $\{z_n\}$ be a sequence of complex numbers and put $P_n = \prod_{k=1}^{n} z_k$, the n-th *partial product*. We say that the infinite product $\prod_{n=1}^{\infty} z_n$ *converges* if the sequence $\{P_n\}$ is convergent to a complex number P, and in this case we write $P = \prod_{n=1}^{\infty} z_n$.

This particular definition of convergence of infinite products is a natural one if the usual definition of convergence of infinite series is extended directly to products. Many textbook authors, however, find this approach objectionable, primarily for the following two reasons.

(a) If one of the factors is zero, then the product converges to zero, no matter what the other factors are, and a "correct" notion of convergence should presumably depend on *all* (but possibly finitely many) of the factors.

(b) It is possible for a product to converge to zero without *any* of the factors being zero, unlike the situation for a finite product.

Nevertheless, we have chosen to take the *naive* approach, and will deal with the above if and when they are relevant.

Note that if $P_n \to P \neq 0$, then $z_n = P_n/P_{n-1} \to P/P = 1$ as $n \to \infty$. Thus a necessary (but not sufficient) condition for convergence of the infinite product to a nonzero limit is that $z_n \to 1$.

A natural approach to the study of an infinite product is to formally convert the product into a sum by taking logarithms. In fact this approach is quite fruitful, as the next result shows.

6.1.1 Lemma

Suppose that $z_n \neq 0, n = 1, 2, \dots$. Then $\prod_{n=1}^{\infty} z_n$ converges to a nonzero limit iff the series $\sum_{n=1}^{\infty} \operatorname{Log} z_n$ converges. (Recall that Log denotes the particular branch of the logarithm such that $-\pi \leq \operatorname{Im}(\operatorname{Log} z) < \pi$.)

Proof. Let $P_n = \prod_{k=1}^{n} z_k$ and $S_n = \sum_{k=1}^{n} \operatorname{Log} z_k$. If $S_n \to S$, then $P_n = e^{S_n} \to e^S \neq 0$. Conversely, suppose that $P_n \to P \neq 0$. Choose any θ such that $\arg_\theta$ is continuous at P (see Theorem 3.1.2). Then $\log_\theta P_n = \ln|P_n| + i\arg_\theta(P_n) \to \ln|P| + i\arg_\theta(P) = \log_\theta P$. Since $e^{S_n} = P_n$, we have $S_n = \log_\theta P_n + 2\pi i l_n$ for some integer l_n. But $S_n - S_{n-1} = \operatorname{Log} z_n \to \operatorname{Log} 1 = 0$. Consequently, $\log_\theta P_n - \log_\theta P_{n-1} + 2\pi i(l_n - l_{n-1}) \to 0$. Since $\log_\theta P_n - \log_\theta P_{n-1} \to \log_\theta P - \log_\theta P = 0$ and $l_n - l_{n-1}$ is an integer, it follows that $l_n - l_{n-1}$ is eventually zero. Therefore l_n is eventually a constant l. Thus $S_n \to \log_\theta P + 2\pi i l$. ♣

6.1.2 Lemma

If $a_n \geq 0$ for all n, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges iff $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $1 + x \leq e^x$, we have, for every $n = 1, 2, \dots$,

$$a_1 + \cdots + a_n \leq (1 + a_1) \cdots (1 + a_n) \leq e^{a_1 + \cdots + a_n}. \quad ♣$$

Lemma 6.1.2 suggests the following useful notion of absolute convergence for infinite products.

6.1.3 Definition

The infinite product $\prod_{n=1}^{\infty}(1 + z_n)$ is said to converge *absolutely* if $\prod_{n=1}^{\infty}(1 + |z_n|)$ converges. Thus by (6.1.2), absolute convergence of $\prod_{n=1}^{\infty}(1 + z_n)$ is equivalent to absolute convergence of the series $\sum_{n=1}^{\infty} z_n$.

With this definition of absolute convergence, we can state and prove a result analogous to a well known property of infinite series.

6.1.4 Lemma

If the infinite product $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely, then it converges.

Proof. By Lemma 6.1.2, convergence of $\prod_{n=1}^{\infty}(1+|z_n|)$ implies that of $\sum_{n=1}^{\infty}|z_n|$, hence $|z_n| \to 0$ in particular. So we can assume that $|z_n| < 1$ for all n. Now for $|z| < 1$, we have

$$\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = zh(z)$$

where $h(z) = 1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4} + \cdots \to 1$ as $z \to 0$. Consequently, for $m \leq p$,

$$|\sum_{n=m}^{p} \text{Log}(1+z_n)| \leq \sum_{n=m}^{p} |z_n||h(z_n)|.$$

Since $\{h(z_n) : n = 1, 2, \dots\}$ is a bounded set and $\sum_{n=1}^{\infty}|z_n|$ converges, it follows from the preceding inequality that $|\sum_{n=m}^{p}\text{Log}(1+z_n)| \to 0$ as $m, p \to \infty$. Thus $\sum_{n=1}^{\infty}\text{Log}(1+z_n)$ is convergent, which by (6.1.1) implies that $\prod_{n=1}^{\infty}(1+z_n)$ converges.

The preceding result may be combined with (6.1.2) to obtain a rearrangement theorem for absolutely convergent products.

6.1.5 Theorem

If $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely, then so does every rearrangement, and to the same limit. That is, if $\prod_{n=1}^{\infty}(1+|z_n|)$ converges and $P = \prod_{n=1}^{\infty}(1+z_n)$, then for every permutation $k \to n_k$ of the positive integers, $\prod_{k=1}^{\infty}(1+z_{n_k})$ also converges to P.

Proof. Since $\prod_{n=1}^{\infty}(1+|z_n|)$ converges, so does $\sum_{n=1}^{\infty}|z_n|$ by (6.1.2). But then every rearrangement of this series converges, so by (6.1.2) again, $\prod_{k=1}^{\infty}(1+|z_{n_k}|)$ converges. Thus it remains to show that $\prod_{k=1}^{\infty}(1+z_{n_k})$ converges to the same limit as does $\prod_{n=1}^{\infty}(1+z_n)$. To this end let $\epsilon > 0$ and for $j = 1, 2, \dots$, let Q_j be the j-th partial product of $\prod_{k=1}^{\infty}(1+z_{n_k})$. Choose N so large that $\sum_{n=N+1}^{\infty}|z_n| < \epsilon$ and J so large that $j \geq J$ implies that $\{1, 2, \dots, N\} \subseteq \{n_1, n_2, \dots, n_j\}$. (The latter is possible because $j \to n_j$ is a permutation of the positive integers.) Then for $j \geq J$ we have

$$\begin{aligned} |Q_j - P| &\leq |Q_j - P_N| + |P_N - P| \\ &= |P_N|| \prod_k (1+z_{n_k}) - 1| + |P_N - P| \end{aligned} \tag{1}$$

where the product is taken over those $k \leq j$ such that $n_k > N$. Now for any complex numbers $w_1, \dots, w_n$ we have (by induction) $|\prod_{k=1}^{n}(1+w_k) - 1| \leq \prod_{k=1}^{n}(1+|w_k|) - 1$. Using this, we get from (1) that

$$\begin{aligned} |Q_j - P| &\leq |P_N|(\prod_k (1+|z_{n_k}|) - 1) + |P_N - P| \\ &\leq |P_N|(e^{\epsilon} - 1) + |P_N - P|. \end{aligned}$$

But the right side of the above inequality can be made as small as we wish by choosing ϵ sufficiently small and N sufficiently large. Therefore $Q_j \to P$ also, and the proof is complete. ♣

6.1.6 Proposition

Let $g_1, g_2, \dots$ be a sequence of bounded complex-valued functions, each defined on a set S. If the series $\sum_{n=1}^{\infty} |g_n|$ converges uniformly on S, then the product $\prod_{n=1}^{\infty}(1+g_n)$ converges absolutely and uniformly on S. Furthermore, if $f(z)=\prod_{n=1}^{\infty}(1+g_n(z)), z \in S$, then $f(z)=0$ for some $z \in S$ iff $1+g_n(z)=0$ for some n.

Proof. Absolute convergence of the product follows from (6.1.2). If $\sum |g_n|$ converges uniformly on S, there exists N such that $n \geq N$ implies $|g_n(z)|<1$ for all $z \in S$. Now for any $r \geq N$,

$$\prod_{n=1}^{r}(1+g_n(z)) = \prod_{n=1}^{N-1}(1+g_n(z)) \prod_{n=N}^{r}(1+g_n(z)).$$

As in the proof of (6.1.4), with the same h and with $m, p \geq N$,

$$|\sum_{n=m}^{p} \operatorname{Log}(1+g_n(z))| \leq \sum_{n=m}^{p} |g_n(z)||h(g_n(z))| \to 0$$

uniformly on S as $m, p \to \infty$. Therefore $\sum_{n=N}^{\infty} \operatorname{Log}(1+g_n(z))$ converges uniformly on S. Since the functions $g_N, g_{N+1}, \dots$ are bounded on S, it follows that the series $\sum_{n=N}^{\infty} |g_n(z)||h(g_n(z))|$ is bounded on S and thus by the above inequality, the same is true of $\sum_{n=N}^{\infty} \operatorname{Log}(1+g_n(z))$. However, the exponential function is uniformly continuous on bounded subsets of $\mathbb{C}$, so we may infer that

$$\exp\left\{\sum_{n=N}^{r} \operatorname{Log}(1+g_n(z))\right\} \to \exp\left\{\sum_{n=N}^{\infty} \operatorname{Log}(1+g_n(z))\right\} \neq 0$$

uniformly on S as $r \to \infty$. This proves uniform convergence on S of $\prod_{n=N}^{\infty}(1+g_n(z))$. Now $1+g_n(z)$ is never 0 on S for $n \geq N$, so if $f(z)=\prod_{n=1}^{\infty}(1+g_n(z))$, then $f(z)=0$ for some $z \in S$ iff $1+g_n(z)=0$ for some $n<N$. ♣

Remark

The product $\prod_{n=1}^{\infty}(1+|g_n|)$ also converges uniformly on S, as follows from the inequality

$$\prod_{n=m}^{p}(1+|g_n|) \leq \exp\left\{\sum_{n=m}^{p} |g_n|\right\}$$

or by applying (6.1.6) to $|g_1|, |g_2|, \dots$.

Proposition (6.1.6) supplies the essential ingredients for an important theorem on products of analytic functions.

6.1.7 Theorem

Let $f_1, f_2, \dots$ be analytic on Ω. If $\sum_{n=1}^{\infty}|f_n-1|$ converges uniformly on compact subsets of Ω, then $f(z)=\prod_{n=1}^{\infty} f_n(z)$ defines a function f that is analytic on Ω. Furthermore, for any $z \in \Omega$ we have $f(z)=0$ iff $f_n=0$ for some n.

Proof. By (6.1.6) with $g_n = f_n - 1$, the product $\prod_{n=1}^{\infty} f_n(z)$ converges uniformly on compact subsets of Ω, hence f is analytic on Ω. The last statement of the theorem is also a direct consequence of (6.1.6). ♣

Problems

1. Let $f_1, f_2, \dots$ and f be as in Theorem 6.1.7. Assume in addition that no f_n is identically zero on any component of Ω. Prove that for each $z \in \Omega, m(f,z) = \sum_{n=1}^{\infty} m(f_n, z)$. (Recall that $m(f,z)$ is the order of the zero of f at z; $m(f,z) = 0$ if $f(z) \neq 0$.)

2. Show that $-\ln(1-x) = x + g(x)x^2, |x| < 1$, where $g(x) \to 1/2$ as $x \to 0$. Conclude that if $a_1, a_2, \dots$ are real numbers and $\sum_{n=1}^{\infty} a_n$ converges, then the infinite product $\prod_n (1 - a_n)$ converges to a nonzero limit iff $\sum_{n=1}^{\infty} a_n^2 < \infty$. Also, if $\sum_{n=1}^{\infty} a_n^2 < \infty$, then $\prod_n (1 - a_n)$ converges to a nonzero limit iff $\sum_{n=1}^{\infty} a_n$ converges.

3. Determine whether or not the following infinite products are convergent.
(a) $\prod_n (1 - 2^{-n})$, (b) $\prod_n (1 - \frac{1}{n+1})$, (c) $\prod_n (1 + \frac{(-1)^n}{\sqrt{n}})$, $\prod_n (1 - \frac{1}{n^2})$.

4. (a) Give an example of an infinite product $\prod_n (1 + a_n)$ such that $\sum a_n$ converges but $\prod_n (1 + a_n)$ diverges.
(b) Give an example of an infinite product $\prod_n (1 + a_n)$ such that $\sum a_n$ diverges but $\prod_n (1 + a_n)$ converges to a nonzero limit.

5. Show that the following infinite products define entire functions.
(a) $\prod_{n=1}^{\infty} (1 + a^n z)$, $|a| < 1$, (b) $\prod_{n \in \mathbb{Z}, n \neq 0} (1 - z/n) e^{z/n}$,
(c) $\prod_{n=2}^{\infty} [1 + \frac{z}{n(\ln n)^2}]$.

6. Criticize the following argument. We know that $\prod_n (1+z_n)$ converges to a nonzero limit iff $\sum_n \operatorname{Log}(1 + z_n)$ converges. The Taylor expansion of $\operatorname{Log}(1 + z)$ yields $\operatorname{Log}(1 + z) = zg(z)$, where $g(z) \to 1$ as $z \to 0$. If $z_n \to 0$, then $g(z_n)$ will be arbitrarily close to 1 for large n, and thus $\sum_n z_n g(z_n)$ will converge iff $\sum_n z_n$ converges. Consequently, $\prod_n (1 + z_n)$ converges to a nonzero limit iff $\sum_n z_n$ converges.

6.2 Weierstrass Products

In this section we will consider the problem of constructing an analytic function f with a prescribed sequence of complex numbers as its set of zeros, as was discussed at the beginning of the chapter. A naive approach is simply to write $\prod_n (z - a_n)^{m_n}$ where $a_1, a_2, \dots$ is the sequence of (distinct) desired zeros and m_n is the specified multiplicity of the zero, that is, $m(f, a_n) = m_n$. But if $a_1, a_2, \dots$ is an infinite sequence, then the infinite product $\prod_n (z - a_n)^{m_n}$ need not converge. A more subtle approach is required, one that achieves convergence by using factors more elaborate than $(z - a_n)$. These "primary factors" were introduced by Weierstrass.

6.2.1 Definition

Define $E_0(z) = 1 - z$ and for $m = 1, 2, \dots,$

$$E_m(z) = (1-z)\exp\left[z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right].$$

Note that if $|z| < 1$, then as $m \to \infty$, $E_m(z) \to (1-z)\exp[-\operatorname{Log}(1-z)] = 1$. Indeed, $E_m(z) \to 1$ uniformly on compact subsets of the unit disk D. Also, the E_m are entire functions, and E_m has a zero of order 1 at $z = 1$, and no other zeros.

6.2.2 Lemma

$|1 - E_m(z)| \le |z|^{m+1}$ for $|z| \le 1$.

Proof. If $m = 0$, equality holds, so assume $m \ge 1$. Then a calculation shows that

$$E_m'(z) = -z^m \exp\left[z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right]$$

so that

$$(1 - E_m(z))' = z^m \exp\left[z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right]. \tag{1}$$

This shows that the derivative of $1 - E_m$ has a zero of order m at 0. Since $1 - E_m(0) = 0$, it follows that $1 - E_m$ has a zero of order $m + 1$ at $z = 0$. Thus $(1 - E_m(z))/z^{m+1}$ has a removable singularity at 0 and so has a Taylor expansion $\sum_{n=0}^{\infty} a_n z^n$ valid everywhere on $\mathbb{C}$. Equation (1) shows also that the derivative of $1 - E_m$ has nonnegative Taylor coefficients and hence the same must be true of $(1 - E_m(z))/z^{m+1}$. Thus $a_n \ge 0$ for all n. Consequently,

$$\left|\frac{1 - E_m(z)}{z^{m+1}}\right| \le \sum_{n=0}^{\infty} |a_n||z|^n \le \sum_{n=0}^{\infty} a_n \text{ if } |z| \le 1.$$

But $\sum_{n=0}^{\infty} a_n = [(1 - E_m(1)]/1^{m+1} = 1$, and the result follows. ♣

Weierstrass' primary factors E_m will now be used to construct functions with prescribed zeros. We begin by constructing *entire* functions with given zeros.

6.2.3 Theorem

Let $\{z_n\}$ be a sequence of nonzero complex numbers such that $|z_n| \to \infty$. Then there is a sequence $\{m_n\}$ of nonnegative integers such that the infinite product $\prod_{n=1}^{\infty} E_{m_n}(z/z_n)$ defines an entire function f. Furthermore, $f(z) = 0$ iff $z = z_n$ for some n. Thus it is possible to construct an entire function having zeros precisely at the z_n, with prescribed multiplicities. (If a appears k times in the sequence $\{z_n\}$, then f has a zero of order k at a. Also, a zero at the origin is handled by multiplying the product by z^m.)

Proof. Let $\{m_n\}$ be a sequence of nonnegative integers with the property that

$$\sum_{n=1}^{\infty}\left(\frac{r}{|z_n|}\right)^{m_n+1}<\infty$$

for every $r>0$. (One such sequence is $m_n=n-1$ since for any $r>0$, $r/|z_n|$) is eventually less than 1/2.) For fixed $r>0$, (6.2.2) implies that

$$|1-E_{m_n}(z/z_n)|\le|z/z_n|^{m_n+1}\le(r/z_n)^{m_n+1}$$

for all $z\in D(0,r)$. Thus the series $\sum|1-E_{m_n}(z/z_n)|$ converges uniformly on $D(0,r)$. Since r is arbitrary, the series converges uniformly on compact subsets of $\mathbb{C}$. The result follows from (6.1.7). ♣

6.2.4 Remark

Let $\{z_n\}$ be as in (6.2.3). If $|z_n|$ grows sufficiently rapidly, it may be possible to take $\{m_n\}$ to be a constant sequence. For example, if $|z_n|=n$, then we may choose $m_n\equiv 1$. The corresponding product is $\prod_{n=1}^{\infty}E_1(z/z_n)=\prod_{n=1}^{\infty}(1-z/z_n)e^{z/z_n}$. In this case, $m=1$ is the *smallest* nonnegative integer for which $\sum_{n=1}^{\infty}(r/|z_n|)^{m+1}<\infty$ for all $r>0$, and $\prod_{n=1}^{\infty}E_m(z/z_n)$ can be viewed as the *canonical product* associated with the sequence $\{z_n\}$. On the other hand, if $|z_n|=\ln n$, then $\sum_{n=1}^{\infty}(1/|z_n|)^m=+\infty$ for every nonnegative integer m, so no constant sequence suffices. These concepts arise in the study of the order of growth of entire functions, but we will not pursue this area further.

Theorem 6.2.3 allows us to factor out the zeros of an entire function.Specifically, we have a representation of an entire function as a product involving the primary factors E_m.

6.2.5 Weierstrass Factorization Theorem

Let f be an entire function, $f\not\equiv 0$, and let $k\ge 0$ be the order of the zero of f at 0. Let the remaining zeros of f be at $z_1,z_2,\dots$, where each z_n is repeated as often as its multiplicity. Then

$$f(z)=e^{g(z)}z^k\prod_n E_{m_n}(z/z_n)$$

for some entire function g and nonnegative integers m_n.

Proof. If f has finitely many zeros, the result is immediate, so assume that there are infinitely many z_n. Since $f\not\equiv 0, |z_n|\to\infty$. By (6.2.3) there is a sequence $\{m_n\}$ such that

$$h(z)=f(z)/[z^k\prod_{n=1}^{\infty}E_{m_n}(z/z_n)]$$

has a zero-free extension to an entire function, which we will persist in calling h. But now h has an analytic logarithm g on $\mathbb{C}$, hence $h(z)=e^{g(z)}$ and we have the desired representation. ♣

More generally, versions of (6.2.3) and its consequence (6.2.5) are available for any *proper* open subset of $\hat{\mathbb{C}}$. We begin with the generalization of (6.2.3).

6.2.6 Theorem

Let Ω be a proper open subset of $\hat{\mathbb{C}}$, $A = \{a_n : n = 1, 2, \dots\}$ a set of distinct points in Ω with no limit point in Ω, and $\{m_n\}$ a sequence of positive integers. Then there exists $f \in A(\Omega)$ such that $Z(f) = A$ and such that for each n we have $m(f, a_n) = m_n$.

Proof. We first show that it is sufficient to prove the theorem in the special case where Ω is a deleted neighborhood of ∞ in $\hat{\mathbb{C}}$ and $\infty \notin A$. For suppose that the theorem has been established in this special case. Then let Ω_1 and A_1 be arbitrary but as in the hypothesis of the the theorem. Choose a point $a \neq \infty$ in $\Omega_1 \setminus A_1$ and define $T(z) = 1/(z-a)$, $z \in \hat{\mathbb{C}}$. Then T is a linear fractional transformation of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$ and thus is a one-to-one continuous map of the open set Ω_1 in $\hat{\mathbb{C}}$ onto an open set Ω. Further, if $A = \{T(a_n) : n = 1, 2 \dots\}$ then Ω and A satisfy the hypotheses of the special case. Having assumed the special case, there exists f analytic on Ω such that $Z(f) = A$ and $m(f, T(a_n)) = m_n$. Now consider the function $f_1 = f \circ T$. Since T is analytic on $\Omega_1 \setminus \{a\}$, so is f_1. But as $z \to a$, $T(z) \to \infty$, and since f is analytic at ∞, $f(T(z))$ approaches a nonzero limit as $z \to a$. Thus f_1 has a removable singularity at a with $f_1(a) \neq 0$. The statement regarding the zeros of f_1 and their multiplicities follows from the fact that T is one-to-one.

Now we must establish the special case. First, if A is a finite set $\{a_1, \dots, a_n\}$, then we can simply take

$$f(z) = \frac{(z-a_1)^{m_1} \cdots (z-a_n)^{m_n}}{(z-b)^{m_1 + \cdots + m_n}}$$

where $b \in \mathbb{C} \setminus \Omega$. The purpose of the denominator is to assure that f is analytic and nonzero at ∞.

Now suppose that $A = \{a_1, a_2, \dots\}$ is an infinite set. Let $\{z_n\}$ be a sequence whose range is A but such that for each j, we have $z_n = a_j$ for exactly m_j values of n. Since $\mathbb{C} \setminus \Omega$ is a *nonempty* compact subset of $\mathbb{C}$, for each $n \geq 1$ there exists a point w_n in $\mathbb{C} \setminus \Omega$ such that $|w_n - z_n| = \operatorname{dist}(z_n, \mathbb{C} \setminus \Omega)$. Note that $|w_n - z_n| \to 0$ as $n \to \infty$ because the sequence $\{z_n\}$ has no limit point in Ω. Let $\{f_n\}$ be the sequence of functions on Ω defined by

$$f_n(z) = E_n\left(\frac{z_n - w_n}{z - w_n}\right),$$

where $f_n(\infty) = E_n(0) = 1$. Then f_n has a simple zero at z_n and no other zeros. Furthermore, $\sum |f_n - 1|$ converges uniformly on compact subsets of Ω. For if $K \subseteq \Omega$, K compact, then eventually $|z_n - w_n|/|z - w_n|$ is uniformly bounded by $1/2$ on K. Thus by Lemma 6.2.2,

$$|f_n(z) - 1| = \left|1 - E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| \leq \left|\frac{z_n - w_n}{z - w_n}\right|^{n+1} \leq (1/2)^{n+1}$$

for each $z \in K$. The statement of the theorem then follows from (6.1.7) by setting $f(z) = \prod_{n=1}^{\infty} f_n(z)$. ♣

It is interesting to see what the preceding argument yields in the special case $\Omega = \mathbb{C}$, a case which was established directly in (6.2.3). Specifically, suppose that $A = \{a_1, a_2, \dots\}$ is an infinite set of distinct points in $\mathbb{C}$ (with no limit point in $\mathbb{C}$), and assume that $0 \notin A$. Let $\{m_j\}$ and $\{z_n\}$ be as in the preceding proof. We are going to reconstruct the proof in the case where $\infty \in \Omega \setminus A$. In order to do this, consider the transformation $T(z) = 1/z$. This maps $\mathbb{C}$ onto $\hat{\mathbb{C}} \setminus \{0\}$ and the sequence $\{z_n\}$ in $\mathbb{C} \setminus \{0\}$ onto the sequence $\{1/z_n\}$ in $T(\mathbb{C})$. The points w_n obtained in the proof of (6.2.6) are all 0, and the corresponding functions f_n would be given by

$$f_n(z) = E_n(1/z_n z), \quad z \in \mathbb{C} \setminus \{0\}.$$

Thus $f(z) = \prod_{n=1}^{\infty} f_n(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ and f has a zero of order m_j at $1/a_j$. Transforming $\hat{\mathbb{C}} \setminus \{0\}$ back to $\mathbb{C}$, it follows that

$$F(z) = f(1/z) = \prod_{n=1}^{\infty} E_n(z/z_n)$$

is an entire function with zeros of order m_j at a_j and no other zeros. That is, we obtain (6.2.3) with $m_n = n$. (Note that this m_n from (6.2.3) is unrelated to the sequence $\{m_j\}$ above.)

The fact that we can construct analytic functions with prescribed zeros has an interesting consequence, which was referred to earlier in (4.2.5).

6.2.7 Theorem

Let h be meromorphic on the open set $\Omega \subseteq \mathbb{C}$. Then $h = f/g$ where f and g are analytic on Ω.

Proof. Let A be the set of poles of h in Ω. Then A satisfies the hypothesis in (6.2.6). Let g be an analytic function on Ω with zeros precisely at the points in A and such that for each $a \in A$, the order of the zero of g at a equals the order of the pole of h at a. Then gh has only removable singularities in Ω and thus can be extended to an analytic function $f \in A(\Omega)$. ♣

Problems

1. Determine the canonical products associated with each of the following sequences. [See the discussion in (6.2.4).]
   (a) $z_n = 2^n$, (b) $z_n = n^b, b > 0$, (c) $z_n = n(\ln n)^2$.
2. Apply Theorem 6.2.6 to construct an analytic function f on the unit disk D such that f has no proper analytic extension to a region $\Omega \supset D$. (Hint: Construct a countable set $A = \{a_n : n = 1, 2, \dots\}$ in D such that every point in ∂D is an accumulation point of A.) Compare this approach to that in Theorem 4.9.5, where essentially the same result is obtained by quite different means.

6.3 Mittag-Leffler's Theorem and Applications

Let Ω be an open subset of $\mathbb{C}$ and let $A = \{a_n : n = 1, 2, \dots\}$ be a set of distinct points in Ω with no limit point in Ω. If $\{m_n\}$ is a sequence of positive integers, then Theorem 6.2.6 implies (by using $1/f$) that there is a meromorphic function f on Ω such that f has poles of order precisely m_n at precisely the points a_n. The theorem of Mittag-Leffler, which we will prove next, states that we can actually specify the coefficients of the principal part at each pole a_n. The exact statement follows; the proof requires Runge's theorem.

6.3.1 Mittag-Leffler's Theorem

Let Ω be an open subset of $\mathbb{C}$ and B a subset of Ω with no limit point in Ω. Thus $B = \{b_j : j \in J\}$ where J is some finite or countably infinite index set. Suppose that to each $j \in J$ there corresponds a rational function of the form

$$S_j(z) = \frac{a_{j1}}{z - b_j} + \frac{a_{j2}}{(z - b_j)^2} + \cdots + \frac{a_{jn_j}}{(z - b_j)^{n_j}}.$$

Then there is a meromorphic function f on Ω such that f has poles at precisely the points b_j and such that the principal part of the Laurent expansion of f at b_j is exactly S_j.

Proof. Let $\{K_n\}$ be the sequence of compact sets defined in (5.1.1). Recall that $\{K_n\}$ has the properties that $K_n \subseteq K_{n+1}^o$ and $\cup K_n = \Omega$. Furthermore, by Problem 5.2.5, each component of $\mathbb{C} \setminus K_n$ contains a component of $\mathbb{C} \setminus \Omega$, in particular, $\mathbb{C} \setminus \Omega$ *meets* each component of $\mathbb{C} \setminus K_n$. Put $K_0 = \emptyset$ and for $n = 1, 2, \dots,$ define

$$J_n = \{j \in J : b_j \in K_n \setminus K_{n-1}\}.$$

The sets J_n are pairwise disjoint (possibly empty), each J_n is finite (since B has no limit point in Ω), and $\cup J_n = J$. For each n, define Q_n by

$$Q_n(z) = \sum_{j \in J_n} S_j(z)$$

where $Q_n \equiv 0$ if J_n is empty. Then Q_n is a rational function whose poles lie in $K_n \setminus K_{n-1}$. In particular, Q_n is analytic on a neighborhood of K_{n-1}. Hence by Runge's theorem (5.2.8) with $S = \mathbb{C} \setminus \Omega$, there is a rational function R_n whose poles lie in $\mathbb{C} \setminus \Omega$ such that

$$|Q_n(z) - R_n(z)| \le (1/2)^n, \; z \in K_{n-1}.$$

It follows that for any fixed $m \ge 1$, the series $\sum_{n=m+1}^{\infty}(Q_n - R_n)$ converges uniformly on K_m to a function which is analytic on $K_m^o \supseteq K_{m-1}$. Thus it is meaningful to define a function $f : \Omega \to \mathbb{C}$ by

$$f(z) = Q_1(z) + \sum_{n=2}^{\infty}(Q_n(z) - R_n(z)), \; z \in \Omega.$$

Indeed, note that for any fixed m, f is the sum of the rational function $Q_1 + \sum_{n=2}^{m}(Q_n - R_n)$ and the series $\sum_{n=m+1}^{\infty}(Q_n - R_n)$, which is analytic on K_m^o. Therefore f is meromorphic on Ω, as well as analytic on $\Omega \setminus B$. It remains to show that f has the required principal part at each point $b \in B$. But for any $b_j \in B$, we have $f(z) = S_j(z)$ plus a function that is analytic on a neighborhood of b_j. Thus f has a pole at b_j with the required principal part S_j. ♣

6.3.2 Remark

Suppose g is analytic at the complex number b and g has a zero of order $m \geq 1$ at b. Let $c_1, c_2, \ldots, c_m$ be given complex numbers, and let R be the rational function given by

$$R(z) = \frac{c_1}{z-b} + \cdots + \frac{c_m}{(z-b)^m}.$$

Then gR has a removable singularity at b, so there exist complex numbers $a_0, a_1, a_2, \ldots$ such that for z in a neighborhood of b,

$$g(z)R(z) = a_0 + a_1(z-b) + \cdots + a_{m-1}(z-b)^{m-1} + \cdots .$$

Furthermore, if we write the Taylor series expansion

$$g(z) = b_0(z-b)^m + b_1(z-b)^{m+1} + \cdots + b_{m-1}(z-b)^{2m-1} + \cdots ,$$

then the coefficients $a_0, a_1, \ldots$ for gR must satisfy

$$\begin{aligned} a_0 &= b_0 c_m \\ a_1 &= b_0 c_{m-1} + b_1 c_m \\ &\vdots \\ a_{m-1} &= b_0 c_1 + b_1 c_2 + \cdots + b_{m-1} c_m \end{aligned}$$

That is, if $c_1, c_2, \ldots, c_m$ are *given*, then $a_0, a_1, \ldots, a_{m-1}$ are *determined* by the above equations. Conversely, if g is given as above, and $a_0, a_1, \ldots, a_{m-1}$ are *given* complex numbers, then since $b_0 \neq 0$, one can sequentially solve the equations to obtain, in order, $c_m, c_{m-1}, \ldots, c_1$. This observation plays a key role in the next result, where it is shown that not only is it possible to construct analytic functions with prescribed zeros and with prescribed orders at these zeros, as in (6.2.3) and (6.2.6), but we can specify the values of f and finitely many of its derivatives in an arbitrary way. To be precise, we have the following extension of (6.2.6).

6.3.3 Theorem

Let Ω be an open subset of $\mathbb{C}$ and B a subset of Ω with no limit point in Ω. Index B by J, as in Mittag-Leffler's theorem, so $B = \{b_j : j \in J\}$. Suppose that corresponding to each $j \in J$, there is a nonnegative integer n_j and complex numbers $a_{0j}, a_{1j}, \ldots, a_{n_j,j}$. Then there exists $f \in A(\Omega)$ such that for each $j \in J$,

$$\frac{f^{(k)}(b_j)}{k!} = a_{kj},\ 0 \leq k \leq n_j.$$

Proof. First apply (6.2.6) to produce a function $g \in A(\Omega)$ such that $Z(g) = B$ and for each j, $m(g, b_j) = n_j + 1 = m_j$, say. Next apply the observations made above in (6.3.2) to obtain, for each $b_j \in B$, complex numbers $c_{1j}, c_{2j}, \dots, c_{m_j,j}$ such that

$$g(z)\sum_{k=1}^{m_j} \frac{c_{kj}}{(z-b_j)^k} = a_{0j} + a_{1j}(z-b_j) + \cdots + a_{n_j,j}(z-b_j)^{n_j} + \cdots$$

for z near b_j. Finally, apply Mittag-Leffler's theorem to obtain h, meromorphic on Ω, such that for each j,

$$h - \sum_{k=1}^{m_j} \frac{c_{kj}}{(z-b_j)^k}$$

has a removable singularity at b_j. It follows that the analytic extension of gh to Ω is the required function f. (To see this, note that

$$gh = g\left(h - \sum_{k=1}^{m_j} \frac{c_{kj}}{(z-b_j)^k}\right) + g\sum_{k=1}^{m_j} \frac{c_{kj}}{(z-b_j)^k}$$

and $m(g, b_j) > n_j$.) ♣

6.3.4 Remark

Theorem 6.3.3 will be used to obtain a number of *algebraic* properties of the ring $A(\Omega)$. This theorem, together with most of results to follow, were obtained (in the case $\Omega = \mathbb{C}$) by Olaf Helmer, Duke Mathematical Journal, volume 6, 1940, pp.345-356.

Assume in what follows that Ω is connected. Thus by Problem 2.4.11, $A(\Omega)$ is an integral domain. Recall that in a ring, such as $A(\Omega)$, g *divides* f if $f = gq$ for some $q \in A(\Omega)$. Also, g is a *greatest common divisor* of a *set* $\mathcal{F}$ if g is a divisor of each $f \in \mathcal{F}$ and if h divides each $f \in \mathcal{F}$, then h divides g.

6.3.5 Proposition

Each nonempty subfamily $\mathcal{F} \subseteq A(\Omega)$ has a greatest common divisor, provided $\mathcal{F} \neq \{0\}$.

Proof. Put $B = \cap\{Z(f) : f \in \mathcal{F}\}$. Apply Theorem 6.2.6 to obtain $g \in A(\Omega)$ such that $Z(g) = B$ and for each $b \in B, m(g, b) = \min\{m(f, b) : f \in \mathcal{F}\}$. Then $f \in \mathcal{F}$ implies that $g|f$ (g divides f). Furthermore, if $h \in A(\Omega)$ and $h|f$ for each $f \in \mathcal{F}$, then $Z(h) \subseteq B$ and for each $b \in B, m(h, b) \leq \min\{m(f, b) : f \in \mathcal{F}\} = m(g, b)$. Thus $h|g$, and consequently g is a greatest common divisor of $\mathcal{F}$. ♣

6.3.6 Definitions

A *unit* in $A(\Omega)$ is a function $f \in A(\Omega)$ such that $1/f \in A(\Omega)$. Thus f is a unit iff f has no zeros in Ω. If $f, g \in A(\Omega)$, we say that f and g are *relatively prime* if each greatest common divisor of f and g is a unit. It follows that f and g are relatively prime iff $Z(f) \cap Z(g) = \emptyset$. (Note that f and g have a common zero iff they have a nonunit common factor.)

6.3.7 Proposition

If the functions $f_1, f_2 \in A(\Omega)$ are relatively prime, then there exist $g_1, g_2 \in A(\Omega)$ such that $f_1g_1 + f_2g_2 \equiv 1$.

Proof. By the remarks above, $Z(f_1) \cap Z(f_2) = \emptyset$. By working backwards, i.e., solving $f_1g_1 + f_2g_2 = 1$ for g_1, we see that it suffices to obtain g_2 such that $(1 - f_2g_2)/f_1$ has only removable singularities. But this entails obtaining g_2 such that $Z(f_1) \subseteq Z(1 - f_2g_2)$ and such that for each $a \in Z(f_1), m(f_1, a) \leq m(1 - f_2g_2, a)$. However, the latter condition may be satisfied by invoking (6.3.3) to obtain $g_2 \in A(\Omega)$ such that for each $a \in Z(f_1)$ (recalling that $f_2(a) \neq 0$),

$$\begin{aligned}
0 &= 1 - f_2(a)g_2(a) = (1 - f_2g_2)(a) \\
0 &= f_2(a)g_2'(a) + f_2'(a)g_2(a) = (1 - f_2g_2)'(a) \\
0 &= f_2(a)g_2''(a) + 2f_2'(a)g_2'(a) + f_2''(a)g_2(a) = (1 - f_2g_2)''(a) \\
&\vdots \\
0 &= f_2(a)g_2^{(m-1)}(a) + \cdots + f_2^{(m-1)}(a)g_2(a) = (1 - f_2g_2)^{(m-1)}(a)
\end{aligned}$$

where $m = m(f_1, a)$. [Note that these equations successively determine $g_2(a), g_2'(a), \dots, g_2^{(m-1)}(a)$.] This completes the proof of the proposition. ♣

The preceding result can be generalized to an arbitrary finite collection of functions.

6.3.8 Proposition

If $\{f_1, f_2, \dots, f_n\} \subseteq A(\Omega)$ and d is a greatest common divisor for this set, then there exist $g_1, g_2, \dots, g_n \in A(\Omega)$ such that $f_1g_1 + f_2g_2 + \cdots + f_ng_n = d$.

Proof. Use (6.3.7) and induction. The details are left as an exercise (Problem 1). ♣

Recall that an *ideal* $I \subseteq A(\Omega)$ is a subset that is closed under addition and subtraction and has the property that if $f \in A(\Omega)$ and $g \in I$, then $fg \in I$.

We are now going to show that $A(\Omega)$ is what is referred to in the literature as a *Bezout domain*. This means that each finitely generated ideal in the integral domain $A(\Omega)$ is a principal ideal. A *finitely generated* ideal is an ideal of the form $\{f_1g_1 + \cdots + f_ng_n : g_1, \dots, g_n \in A(\Omega)\}$ where $\{f_1, \dots, f_n\}$ is some fixed finite set of elements in $A(\Omega)$. A *principal ideal* is an ideal that is generated by a *single* element f_1. Most of the work has already been done in preceding two propositions.

6.3.9 Theorem

Let $f_1, \dots, f_n \in A(\Omega)$ and let $I = \{f_1g_1 + \cdots + f_ng_n : g_1, \dots, g_n \in A(\Omega)\}$ be the ideal generated by $f_1, \dots, f_n$. Then there exists $f \in A(\Omega)$ such that $I = \{fg : g \in A(\Omega)\}$. In other words, I is a principal ideal.

Proof. If $f \in I$ then $f = f_1h_1 + \cdots + f_nh_n$ for some $h_1, \dots, h_n \in A(\Omega)$. If d is a greatest common divisor for $\{f_1, \dots, f_n\}$, then d divides each f_j, hence d divides f. Thus f is a multiple of d. On the other hand, by (6.3.8), there exist $g_1, \dots, g_n \in A(\Omega)$ such that

$d = f_1g_1 + \cdots + f_ng_n$. Therefore d and hence every multiple of d belongs to I. Thus I is the ideal generated by the single element d. ♣

A *principal ideal domain* is an integral domain in which every ideal is principal. Problem 2 asks you to show that $A(\Omega)$ is *never* a principal ideal domain, regardless of the region Ω. There is another class of (commutative) rings called *Noetherian*; these are rings in which every ideal is finitely generated. Problem 2, when combined with (6.3.9), also shows that $A(\Omega)$ is never Noetherian.

Problems

1. Supply the details to the proof of (6.3.8). (Hint: Use induction, (6.3.7), and the fact that if d is a greatest common divisor (gcd) for $\{f_1, \dots, f_n\}$ and d_1 is a gcd for $\{f_1, \dots, f_{n-1}\}$, then d is a gcd for the set $\{d_1, f_n\}$. Also note that 1 is a gcd for $\{f_1/d, \dots, f_n/d\}$.)
2. Show that $A(\Omega)$ is never a principal ideal domain. that is, there always exists ideals I that are not principal ideals, and thus by (6.3.9) are not finitely generated. (Hint: Let $\{a_n\}$ be a sequence of distinct points in Ω with no limit point in Ω. For each n, apply (6.2.6) to the set $\{a_n, a_{n+1}, \dots\}$.)