Chapter 6
Factorization of Analytic Functions

In this chapter we will consider the problems of factoring out the zeros of an analytic function $f$ on a region $\Omega$ (à la polynomials), and of decomposing a meromorphic function (à la partial fractions for rational functions). Suppose $f$ is analytic on a region $\Omega$ and $f \neq 0$. What can be said about $Z(f)$? Theorem 2.4.8, the identity theorem, asserts that $Z(f)$ has no limit point in $\Omega$. It turns out that no more can be said in general. That is, if $A$ is any subset of $\Omega$ with no limit point in $\Omega$, then there exists $f \in A(\Omega)$ whose set of zeros is precisely $A$. Furthermore, we can prescribe the order of the zero which $f$ shall have at each point of $A$. Now if $A$ is a finite subset of $\Omega$, say $\{z_1, \ldots, z_n\}$, and $m_1, \ldots, m_n$ are the corresponding desired multiplicities, then the finite product

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n}$$

would be such a function. However, in general the construction of such an $f$ is accomplished using infinite products, which we now study in detail.

6.1 Infinite Products

Let $\{z_n\}$ be a sequence of complex numbers and put $P_n = \prod_{k=1}^n z_k$, the $n$-th partial product. We say that the infinite product $\prod_{n=1}^\infty z_n$ converges if the sequence $\{P_n\}$ is convergent to a complex number $P$, and in this case we write $P = \prod_{n=1}^\infty z_n$.

This particular definition of convergence of infinite products is a natural one if the usual definition of convergence of infinite series is extended directly to products. Many textbook authors, however, find this approach objectionable, primarily for the following two reasons.

(a) If one of the factors is zero, then the product converges to zero, no matter what the other factors are, and a “correct” notion of convergence should presumably depend on all (but possibly finitely many) of the factors.
(b) It is possible for a product to converge to zero without any of the factors being zero, unlike the situation for a finite product. Nevertheless, we have chosen to take the naive approach, and will deal with the above if and when they are relevant.

Note that if \( P_n \rightarrow P \neq 0 \), then \( z_n = P_n/P_{n-1} \rightarrow P/P = 1 \) as \( n \rightarrow \infty \). Thus a necessary (but not sufficient) condition for convergence of the infinite product to a nonzero limit is that \( z_n \rightarrow 1 \).

A natural approach to the study of an infinite product is to formally convert the product into a sum by taking logarithms. In fact this approach is quite fruitful, as the next result shows.

6.1.1 Lemma

Suppose that \( z_n \neq 0, n = 1, 2, \ldots \). Then \( \prod_{n=1}^{\infty} z_n \) converges to a nonzero limit iff the series \( \sum_{n=1}^{\infty} \log z_n \) converges. (Recall that Log denotes the particular branch of the logarithm such that \(-\pi \leq \text{Im}(\log z) < \pi\).)

Proof. Let \( P_n = \prod_{k=1}^{n} z_k \) and \( S_n = \sum_{k=1}^{n} \log z_k \). If \( S_n \rightarrow S \), then \( P_n = e^{S_n} \rightarrow e^S \neq 0 \). Conversely, suppose that \( P_n \rightarrow P \neq 0 \). Choose any \( \theta \) such that \( \arg \theta \) is continuous at \( P \) (see Theorem 3.1.2). Then \( \log_{\theta} P_n = \ln |P_n| + i \arg_{\theta}(P) \rightarrow \ln |P| + i \arg_{\theta}(P) = \log_{\theta} P \). Since \( e^{S_n} = P_n \), we have \( S_n = \log_{\theta} P_n + 2\pi il_n \) for some integer \( l_n \). But \( S_n - S_{n-1} = \log z_n \rightarrow \log 1 = 0 \). Consequently, \( \log_{\theta} P_n - \log_{\theta} P_{n-1} + 2\pi i(l_n - l_{n-1}) \rightarrow 0 \). Since \( \log_{\theta} P_n - \log_{\theta} P_{n-1} \rightarrow \log_{\theta} P - \log_{\theta} P = 0 \) and \( l_n - l_{n-1} \) is an integer, it follows that \( l_n - l_{n-1} \) is eventually zero. Therefore \( l_n \) is eventually a constant \( l \). Thus \( S_n \rightarrow \log_{\theta} P + 2\pi il \). ♣

6.1.2 Lemma

If \( a_n \geq 0 \) for all \( n \), then \( \prod_{n=1}^{\infty} (1 + a_n) \) converges iff \( \sum_{n=1}^{\infty} a_n \) converges.

Proof. Since \( 1 + x \leq e^x \), we have, for every \( n = 1, 2, \ldots \),

\[
a_1 + \cdots + a_n \leq (1 + a_1) \cdots (1 + a_n) \leq e^{a_1 + \cdots + a_n}.
\]

Lemma 6.1.2 suggests the following useful notion of absolute convergence for infinite products.

6.1.3 Definition

The infinite product \( \prod_{n=1}^{\infty} (1 + z_n) \) is said to converge absolutely if \( \prod_{n=1}^{\infty} (1 + |z_n|) \) converges. Thus by (6.1.2), absolute convergence of \( \prod_{n=1}^{\infty} (1 + z_n) \) is equivalent to absolute convergence of the series \( \sum_{n=1}^{\infty} z_n \).

With this definition of absolute convergence, we can state and prove a result analogous to a well known property of infinite series.
6.1. INFINITE PRODUCTS

6.1.4 Lemma
If the infinite product $\prod_{n=1}^{\infty}(1 + z_n)$ converges absolutely, then it converges.

**Proof.** By Lemma 6.1.2, convergence of $\prod_{n=1}^{\infty}(1 + |z_n|)$ implies that of $\sum_{n=1}^{\infty}|z_n|$, hence $|z_n| \to 0$ in particular. So we can assume that $|z_n| < 1$ for all $n$. Now for $|z| < 1$, we have

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = z h(z)$$

where $h(z) = 1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4} + \cdots \to 1$ as $z \to 0$. Consequently, for $m \leq p$,

$$|\sum_{n=m}^{p} \log(1 + z_n)| \leq \sum_{n=m}^{p} |z_n||h(z_n)|.$$

Since $\{h(z_n) : n = 1, 2, \ldots\}$ is a bounded set and $\sum_{n=1}^{\infty}|z_n|$ converges, it follows from the preceding inequality that $|\sum_{n=m}^{p} \log(1 + z_n)| \to 0$ as $m, p \to \infty$. Thus $\sum_{n=1}^{\infty}\log(1 + z_n)$ is convergent, which by (6.1.1) implies that $\prod_{n=1}^{\infty}(1 + z_n)$ converges.

The preceding result may be combined with (6.1.2) to obtain a rearrangement theorem for absolutely convergent products.

6.1.5 Theorem
If $\prod_{n=1}^{\infty}(1 + z_n)$ converges absolutely, then so does every rearrangement, and to the same limit. That is, if $\prod_{n=1}^{\infty}(1 + |z_n|)$ converges and $P = \prod_{n=1}^{\infty}(1 + z_n)$, then for every permutation $k \to n_k$ of the positive integers, $\prod_{k=1}^{\infty}(1 + z_{n_k})$ also converges to $P$.

**Proof.** Since $\prod_{n=1}^{\infty}(1 + |z_n|)$ converges, so does $\sum_{n=1}^{\infty}|z_n|$ by (6.1.2). But then every rearrangement of this series converges, so by (6.1.2) again, $\prod_{n=1}^{\infty}(1 + |z_n|)$ converges. Thus it remains to show that $\prod_{n=1}^{\infty}(1 + z_{n_k})$ converges to the same limit as $\prod_{n=1}^{\infty}(1 + z_n)$. To this end let $\epsilon > 0$ and for $j = 1, 2, \ldots$, let $Q_j$ be the $j$-th partial product of $\prod_{k=1}^{\infty}(1 + z_{n_k})$. Choose $N$ so large that $\sum_{n=N+1}^{\infty}|z_n| < \epsilon$ and $J$ so large that $j \geq J$ implies that $\{1, 2, \ldots, N\} \subseteq \{n_1, n_2, \ldots, n_j\}$. (The latter is possible because $j \to n_j$ is a permutation of the positive integers.) Then for $j \geq J$ we have

$$|Q_j - P| \leq |Q_j - P_N| + |P_N - P|$$

where the product is taken over those $k \leq j$ such that $n_k > N$. Now for any complex numbers $w_1, \ldots, w_n$ we have (by induction) $|\prod_{k=1}^{n}(1 + w_k) - 1| \leq \prod_{k=1}^{n}|1 + |w_k|| - 1$. Using this, we get from (1) that

$$|Q_j - P| \leq |P_N||\prod_{k}(1 + z_{n_k}) - 1| + |P_N - P|$$

$$\leq |P_N|(|\epsilon - 1| + |P_N - P|).$$

But the right side of the above inequality can be made as small as we wish by choosing $\epsilon$ sufficiently small and $N$ sufficiently large. Therefore $Q_j \to P$ also, and the proof is complete. ♦
CHAPTER 6. FACTORIZATION OF ANALYTIC FUNCTIONS

6.1.6 Proposition

Let \( g_1, g_2, \ldots \) be a sequence of bounded complex-valued functions, each defined on a set \( S \). If the series \( \sum_{n=1}^{\infty} |g_n| \) converges uniformly on \( S \), then the product \( \prod_{n=1}^{\infty} (1 + g_n) \) converges absolutely and uniformly on \( S \). Furthermore, if \( f(z) = \prod_{n=1}^{\infty} (1 + g_n(z)), z \in S \), then \( f(z) = 0 \) for some \( z \in S \) iff \( 1 + g_n(z) = 0 \) for some \( n \).

Proof. Absolute convergence of the product follows from (6.1.2). If \( \sum |g_n| \) converges uniformly on \( S \), there exists \( N \) such that \( n \geq N \) implies \( |g_n(z)| < 1 \) for all \( z \in S \). Now for any \( r \geq N \),

\[
\prod_{n=1}^{r} (1 + g_n(z)) = \prod_{n=1}^{N-1} (1 + g_n(z)) \prod_{n=N}^{r} (1 + g_n(z)).
\]

As in the proof of (6.1.4), with the same \( h \) and with \( m, p \geq N \),

\[
|\sum_{n=m}^{p} \log(1 + g_n(z))| \leq \sum_{n=m}^{p} |g_n(z)| |h(g_n(z))| \to 0
\]

uniformly on \( S \) as \( m, p \to \infty \). Therefore \( \sum_{n=N}^{\infty} \log(1 + g_n(z)) \) converges uniformly on \( S \). Since the functions \( g_N, g_{N+1}, \ldots \) are bounded on \( S \), it follows that the series \( \sum_{n=N}^{\infty} |g_n(z)||h(g_n(z))| \) is bounded on \( S \) and thus by the above inequality, the same is true of \( \sum_{n=N}^{\infty} \log(1 + g_n(z)) \). However, the exponential function is uniformly continuous on bounded subsets of \( \mathbb{C} \), so we may infer that

\[
\exp\left\{ \sum_{n=N}^{r} \log(1 + g_n(z)) \right\} \to \exp\left\{ \sum_{n=N}^{\infty} \log(1 + g_n(z)) \right\} \neq 0
\]

uniformly on \( S \) as \( r \to \infty \). This proves uniform convergence on \( S \) of \( \prod_{n=1}^{\infty} (1 + g_n(z)) \).

Now \( 1 + g_n(z) \) is never 0 on \( S \) for \( n \geq N \), so if \( f(z) = \prod_{n=1}^{\infty} (1 + g_n(z)), z \in S \), then \( f(z) = 0 \) for some \( z \in S \) iff \( 1 + g_n(z) = 0 \) for some \( n < N \).

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Remark

The product \( \prod_{n=1}^{\infty} (1 + |g_n|) \) also converges uniformly on \( S \), as follows from the inequality

\[
\prod_{n=m}^{p} (1 + |g_n|) \leq \exp\left\{ \sum_{n=m}^{p} |g_n| \right\}
\]

or by applying (6.1.6) to \( |g_1|, |g_2|, \ldots \).

Proposition (6.1.6) supplies the essential ingredients for an important theorem on products of analytic functions.

6.1.7 Theorem

Let \( f_1, f_2, \ldots \) be analytic on \( \Omega \). If \( \sum_{n=1}^{\infty} |f_n - 1| \) converges uniformly on compact subsets of \( \Omega \), then \( f(z) = \prod_{n=1}^{\infty} f_n(z) \) defines a function \( f \) that is analytic on \( \Omega \). Furthermore, for any \( z \in \Omega \) we have \( f(z) = 0 \) iff \( f_n = 0 \) for some \( n \).
6.2. WEIERSTRASS PRODUCTS

Proof. By (6.1.6) with \( g_n = f_n - 1 \), the product \( \prod_{n=1}^{\infty} f_n(z) \) converges uniformly on compact subsets of \( \Omega \), hence \( f \) is analytic on \( \Omega \). The last statement of the theorem is also a direct consequence of (6.1.6). ♦

Problems

1. Let \( f_1, f_2, \ldots \) and \( f \) be as in Theorem 6.1.7. Assume in addition that no \( f_n \) is identically zero on any component of \( \Omega \). Prove that for each \( z \in \Omega, m(f, z) = \sum_{n=1}^{\infty} m(f_n, z) \). (Recall that \( m(f, z) \) is the order of the zero of \( f \) at \( z; m(f, z) = 0 \) if \( f(z) \neq 0 \).)

2. Show that \( -\ln(1-x) = x + g(x)x^2, |x| < 1 \), where \( g(x) \to 1/2 \) as \( x \to 0 \). Conclude that if \( a_1, a_2, \ldots \) are real numbers and \( \sum_{n=1}^{\infty} a_n \) converges, then the infinite product \( \prod_{n}(1 - a_n) \) converges to a nonzero limit iff \( \sum_{n=1}^{\infty} a_n^2 < \infty \). Also, if \( \sum_{n=1}^{\infty} a_n^2 < \infty \), then \( \prod_{n}(1 - a_n) \) converges to a nonzero limit iff \( \sum_{n=1}^{\infty} a_n \) converges.

3. Determine whether or not the following infinite products are convergent.
   (a) \( \prod_{n}(1 - 2^{-n}) \), (b) \( \prod_{n}(1 - 1/n + 1) \), (c) \( \prod_{n}(1 - (-1)^n) \), \( \prod_{n}(1 - 1/n) \).

4. (a) Give an example of an infinite product \( \prod_{n}(1 + a_n) \) such that \( \sum a_n \) converges but \( \prod_{n}(1 + a_n) \) diverges.
   (b) Give an example of an infinite product \( \prod_{n}(1 + a_n) \) such that \( \sum a_n \) diverges but \( \prod_{n}(1 + a_n) \) converges to a nonzero limit.

5. Show that the following infinite products define entire functions.
   (a) \( \prod_{n=1}^{\infty}(1 + a^n z), |a| < 1 \),
   (b) \( \prod_{n\in\mathbb{Z}, n\neq0}(1 - z/n^{\alpha}/n) \),
   (c) \( \prod_{n=2}^{\infty}[1 + z/n^{\alpha}/n] \).

6. Criticize the following argument. We know that \( \prod_{n}(1 + z_n) \) converges to a nonzero limit iff \( \sum_{n} \log(1 + z_n) \) converges. The Taylor expansion of \( \log(1 + z) \) yields \( \log(1 + z) = zg(z) \), where \( g(z) \to 1 \) as \( z \to 0 \). If \( z_n \to 0 \), then \( g(z_n) \) will be arbitrarily close to 1 for large \( n \), and thus \( \prod_{n} z_n g(z) \) will converge iff \( \sum_{n} z_n \) converges. Consequently, \( \prod_{n}(1 + z_n) \) converges to a nonzero limit iff \( \sum_{n} z_n \) converges.

6.2 Weierstrass Products

In this section we will consider the problem of constructing an analytic function \( f \) with a prescribed sequence of complex numbers as its set of zeros, as was discussed at the beginning of the chapter. A naive approach is simply to write \( \prod_{n}(z - a_n)^{m_n} \) where \( a_1, a_2, \ldots \) is the sequence of (distinct) desired zeros and \( m_n \) is the specified multiplicity of the zero, that is, \( m(f, a_n) = m_n \). But if \( a_1, a_2, \ldots \) is an infinite sequence, then the infinite product \( \prod_{n}(z - a_n)^{m_n} \) need not converge. A more subtle approach is required, one that achieves convergence by using factors more elaborate than \( (z - a_n) \). These “primary factors” were introduced by Weierstrass.
6.2.1 Definition
Define \( E_0(z) = 1 - z \) and for \( m = 1, 2, \ldots \),
\[
E_m(z) = (1 - z) \exp \left[ z + \frac{z^2}{2} + \cdots + \frac{z^m}{m} \right].
\]
Note that if \( |z| < 1 \), then as \( m \to \infty \), \( E_m(z) \to (1 - z) \exp[-\text{Log}(1 - z)] = 1 \). Indeed, \( E_m(z) \to 1 \) uniformly on compact subsets of the unit disk \( D \). Also, the \( E_m \) are entire functions, and \( E_m \) has a zero of order 1 at \( z = 1 \), and no other zeros.

6.2.2 Lemma
\(|1 - E_m(z)| \leq |z|^{m+1} \) for \( |z| \leq 1 \).

Proof. If \( m = 0 \), equality holds, so assume \( m \geq 1 \). Then a calculation shows that
\[
E'_m(z) = -z^m \exp \left[ z + \frac{z^2}{2} + \cdots + \frac{z^m}{m} \right]
\]
so that
\[
(1 - E_m(z))' = z^m \exp \left[ z + \frac{z^2}{2} + \cdots + \frac{z^m}{m} \right].
\] (1)
This shows that the derivative of \( 1 - E_m \) has a zero of order \( m \) at \( 0 \). Since \( 1 - E_m(0) = 0 \), it follows that \( 1 - E_m \) has a zero of order \( m + 1 \) at \( z = 0 \). Thus \( (1 - E_m(z))/z^{m+1} \) has a removable singularity at \( 0 \) and so has a Taylor expansion \( \sum_{n=0}^{\infty} a_n z^n \) valid everywhere on \( \mathbb{C} \). Equation (1) shows also that the derivative of \( 1 - E_m \) has nonnegative Taylor coefficients and hence the same must be true of \( (1 - E_m(z))/z^{m+1} \). Thus \( a_n \geq 0 \) for all \( n \). Consequently,
\[
\left| \frac{1 - E_m(z)}{z^{m+1}} \right| \leq \sum_{n=0}^{\infty} |a_n| |z|^n \leq \sum_{n=0}^{\infty} a_n \text{ if } |z| \leq 1.
\]
But \( \sum_{n=0}^{\infty} a_n = [(1 - E_m(1))/1^{m+1} = 1 \), and the result follows. ♣

Weierstrass’ primary factors \( E_m \) will now be used to construct functions with prescribed zeros. We begin by constructing entire functions with given zeros.

6.2.3 Theorem
Let \( \{z_n\} \) be a sequence of nonzero complex numbers such that \( |z_n| \to \infty \). Then there is a sequence \( \{m_n\} \) of nonnegative integers such that the infinite product \( \prod_{n=1}^{\infty} E_{m_n}(z/z_n) \) defines an entire function \( f \). Furthermore, \( f(z) = 0 \) if \( z = z_n \) for some \( n \). Thus it is possible to construct an entire function having zeros precisely at the \( z_n \), with prescribed multiplicities. (If \( a \) appears \( k \) times in the sequence \( \{z_n\} \), then \( f \) has a zero of order \( k \) at \( a \). Also, a zero at the origin is handled by multiplying the product by \( z^m \).)
6.2. WEIERSTRASS PRODUCTS

Proof. Let \( \{m_n\} \) be a sequence of nonnegative integers with the property that
\[
\sum_{n=1}^{\infty} \left( \frac{r}{|z_n|} \right)^{m_n+1} < \infty
\]
for every \( r > 0 \). (One such sequence is \( m_n = n - 1 \) since for any \( r > 0 \), \( r/|z_n| \) is eventually less than 1/2.) For fixed \( r > 0 \), (6.2.2) implies that
\[
|1 - E_{m_n}(z/z_n)| \leq |z/z_n|^{m_n+1} \leq (r/z_n)^{m_n+1}
\]
for all \( z \in D(0, r) \). Thus the series \( \sum |1 - E_{m_n}(z/z_n)| \) converges uniformly on \( D(0, r) \). Since \( r \) is arbitrary, the series converges uniformly on compact subsets of \( \mathbb{C} \). The result follows from (6.1.7).

6.2.4 Remark

Let \( \{z_n\} \) be as in (6.2.3). If \( |z_n| \) grows sufficiently rapidly, it may be possible to take \( \{m_n\} \) to be a constant sequence. For example, if \( |z_n| = n \), then we may choose \( m_n = 1 \). The corresponding product is \( \prod_{n=1}^{\infty} E_1(z/z_n) = \prod_{n=1}^{\infty} (1 - z/z_n) e^{z/z_n} \). In this case, \( m = 1 \) is the smallest nonnegative integer for which \( \sum_{n=1}^{\infty} (r/|z_n|)^{m+1} < \infty \) for all \( r > 0 \), and \( \prod_{n=1}^{\infty} E_m(z/z_n) \) can be viewed as the canonical product associated with the sequence \( \{z_n\} \). On the other hand, if \( |z_n| = \ln n \), then \( \sum_{n=1}^{\infty} (1/|z_n|)^m = +\infty \) for every nonnegative integer \( m \), so no constant sequence suffices. These concepts arise in the study of the order of growth of entire functions, but we will not pursue this area further.

Theorem 6.2.3 allows us to factor out the zeros of an entire function. Specifically, we have a representation of an entire function as a product involving the primary factors \( E_m \).

6.2.5 Weierstrass Factorization Theorem

Let \( f \) be an entire function, \( f \not\equiv 0 \), and let \( k \geq 0 \) be the order of the zero of \( f \) at 0. Let the remaining zeros of \( f \) be at \( z_1, z_2, \ldots \), where each \( z_n \) is repeated as often as its multiplicity. Then
\[
f(z) = e^{g(z)} z^k \prod_{n} E_{m_n}(z/z_n)
\]
for some entire function \( g \) and nonnegative integers \( m_n \).

Proof. If \( f \) has finitely many zeros, the result is immediate, so assume that there are infinitely many \( z_n \). Since \( f \not\equiv 0 \), \( |z_n| \to \infty \). By (6.2.3) there is a sequence \( \{m_n\} \) such that
\[
h(z) = f(z)/[z^k \prod_{n=1}^{\infty} E_{m_n}(z/z_n)]
\]
has a zero-free extension to an entire function, which we will persist in calling \( h \). But now \( h \) has an analytic logarithm \( g \) on \( \mathbb{C} \), hence \( h(z) = e^{g(z)} \) and we have the desired representation.

More generally, versions of (6.2.3) and its consequence (6.2.5) are available for any proper open subset of \( \hat{\mathbb{C}} \). We begin with the generalization of (6.2.3).
6.2.6 Theorem

Let \( \Omega \) be a proper open subset of \( \mathbb{C} \), \( A = \{ a_n : n = 1, 2, \ldots \} \) a set of distinct points in \( \Omega \) with no limit point in \( \Omega \), and \( \{ m_n \} \) a sequence of positive integers. Then there exists \( f \in A(\Omega) \) such that \( Z(f) = A \) and such that for each \( n \) we have \( m(f, a_n) = m_n \).

**Proof.** We first show that it is sufficient to prove the theorem in the special case where \( \Omega \) is a deleted neighborhood of \( \infty \) in \( \mathbb{C} \) and \( \infty \notin A \). For suppose that the theorem has been established in this special case. Then let \( \Omega_1 \) and \( A_1 \) be arbitrary but as in the hypothesis of the theorem. Choose a point \( a \neq \infty \) in \( \Omega_1 \setminus A_1 \) and define \( T(z) = 1/(z - a) \), \( z \in \mathbb{C} \). Then \( T \) is a linear fractional transformation of \( \mathbb{C} \) onto \( \mathbb{C} \) and thus is a one-to-one continuous map of the open set \( \Omega_1 \) in \( \mathbb{C} \) onto an open set \( \Omega \). Further, if \( A = \{ T(a_n) : n = 1, 2, \ldots \} \) then \( \Omega \) and \( A \) satisfy the hypotheses of the special case. Having assumed the special case, there exists \( f \) analytic on \( \Omega \) such that \( Z(f) = A \) and \( m(f, T(a_n)) = m_n \). Now consider the function \( f_1 = f \circ T \). Since \( T \) is analytic on \( \Omega_1 \setminus \{ a \} \), so is \( f_1 \). But as \( z \to a \), \( T(z) \to \infty \), and since \( f \) is analytic at \( \infty \), \( f(T(z)) \) approaches a nonzero limit as \( z \to a \). Thus \( f_1 \) has a removable singularity at \( a \) with \( f_1(a) \neq 0 \). The statement regarding the zeros of \( f_1 \) and their multiplicities follows from the fact that \( T \) is one-to-one.

Now we must establish the special case. First, if \( A \) is a finite set \( \{ a_1, \ldots, a_n \} \), then we can simply take

\[
f(z) = \frac{(z - a_1)^{m_1} \cdots (z - a_n)^{m_n}}{(z - b)^{m_1 + \cdots + m_n}}
\]

where \( b \in \mathbb{C} \setminus \Omega \). The purpose of the denominator is to assure that \( f \) is analytic and nonzero at \( \infty \).

Now suppose that \( A = \{ a_1, a_2, \ldots \} \) is an infinite set. Let \( \{ z_n \} \) be a sequence whose range is \( A \) but such that for each \( j \), we have \( z_n = a_j \) for exactly \( m_j \) values of \( n \). Since \( \mathbb{C} \setminus \Omega \) is a nonempty compact subset of \( \mathbb{C} \), for each \( n \geq 1 \) there exists a point \( w_n \) in \( \mathbb{C} \setminus \Omega \) such that \( |w_n - z_n| = \text{dist}(z_n, \mathbb{C} \setminus \Omega) \). Note that \( |w_n - z_n| \to 0 \) as \( n \to \infty \) because the sequence \( \{ z_n \} \) has no limit point in \( \Omega \). Let \( \{ f_n \} \) be the sequence of functions on \( \Omega \) defined by

\[
f_n(z) = E_n \left( \frac{z_n - w_n}{z - w_n} \right),
\]

where \( f_n(\infty) = E_n(0) = 1 \). Then \( f_n \) has a simple zero at \( z_n \) and no other zeros. Furthermore, \( \sum |f_n - 1| \) converges uniformly on compact subsets of \( \Omega \). For if \( K \subseteq \Omega \), \( K \) compact, then eventually \( |z_n - w_n|/|z - w_n| \) is uniformly bounded by \( 1/2 \) on \( K \). Thus by Lemma 6.2.2,

\[
|f_n(z) - 1| = \left| 1 - E_n \left( \frac{z_n - w_n}{z - w_n} \right) \right| \leq \left| \frac{z_n - w_n}{z - w_n} \right|^{n+1} \leq (1/2)^{n+1}
\]

for each \( z \in K \). The statement of the theorem then follows from (6.1.7) by setting

\[
f(z) = \prod_{n=1}^{\infty} f_n(z). \quad \blacklozenge
\]
It is interesting to see what the preceding argument yields in the special case $\Omega = \mathbb{C}$, a case which was established directly in (6.2.3). Specifically, suppose that $A = \{a_1, a_2, \ldots\}$ is an infinite set of distinct points in $\mathbb{C}$ (with no limit point in $\mathbb{C}$), and assume that $0 \notin A$. Let $\{m_j\}$ and $\{z_n\}$ be as in the preceding proof. We are going to reconstruct the proof in the case where $\infty \in \Omega \setminus A$. In order to do this, consider the transformation $T(z) = 1/z$. This maps $\mathbb{C}$ onto $\hat{\mathbb{C}} \setminus \{0\}$ and the sequence $\{z_n\}$ in $\mathbb{C} \setminus \{0\}$ onto the sequence $\{1/z_n\}$ in $T(\mathbb{C})$. The points $w_n$ obtained in the proof of (6.2.6) are all 0, and the corresponding functions $f_n$ would be given by
\[
f_n(z) = E_n(1/z_n), \quad z \in \mathbb{C} \setminus \{0\}.
\]
Thus $f(z) = \prod_{n=1}^{\infty} f_n(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ and $f$ has a zero of order $m_j$ at $1/a_j$.

Transforming $\hat{\mathbb{C}} \setminus \{0\}$ back to $\mathbb{C}$, it follows that
\[
F(z) = f(1/z) = \prod_{n=1}^{\infty} E_n(z/z_n)
\]
is an entire function with zeros of order $m_j$ at $a_j$ and no other zeros. That is, we obtain (6.2.3) with $m_n = n$. (Note that this $m_n$ from (6.2.3) is unrelated to the sequence $\{m_j\}$ above.)

The fact that we can construct analytic functions with prescribed zeros has an interesting consequence, which was referred to earlier in (4.2.5).

### 6.2.7 Theorem

Let $h$ be meromorphic on the open set $\Omega \subseteq \mathbb{C}$. Then $h = f/g$ where $f$ and $g$ are analytic on $\Omega$.

**Proof.** Let $A$ be the set of poles of $h$ in $\Omega$. Then $A$ satisfies the hypothesis in (6.2.6). Let $g$ be an analytic function on $\Omega$ with zeros precisely at the points in $A$ and such that for each $a \in A$, the order of the zero of $g$ at $a$ equals the order of the pole of $h$ at $a$. Then $gh$ has only removable singularities in $\Omega$ and thus can be extended to an analytic function $f \in A(\Omega)$. ♦

### Problems

1. Determine the canonical products associated with each of the following sequences. [See the discussion in (6.2.4).]
   (a) $z_n = 2^n$, \hspace{1cm} (b) $z_n = n^b$, $b > 0$, \hspace{1cm} (c) $z_n = n(\ln n)^2$.

2. Apply Theorem 6.2.6 to construct an analytic function $f$ on the unit disk $D$ such that $f$ has no proper analytic extension to a region $\Omega \supset D$. (Hint: Construct a countable set $A = \{a_n : n = 1, 2, \ldots\}$ in $D$ such that every point in $\partial D$ is an accumulation point of $A$.) Compare this approach to that in Theorem 4.9.5, where essentially the same result is obtained by quite different means.
6.3 Mittag-Leffler’s Theorem and Applications

Let Ω be an open subset of ℂ and let \( A = \{a_n : n = 1, 2, \ldots \} \) be a set of distinct points in Ω with no limit point in Ω. If \( \{m_n\} \) is a sequence of positive integers, then Theorem 6.2.6 implies (by using \( 1/f \)) that there is a meromorphic function \( f \) on Ω such that \( f \) has poles of order precisely \( m_n \) at precisely the points \( a_n \). The theorem of Mittag-Leffler, which we will prove next, states that we can actually specify the coefficients of the principal part at each pole \( a_n \). The exact statement follows; the proof requires Runge’s theorem.

6.3.1 Mittag-Leffler’s Theorem

Let Ω be an open subset of ℂ and \( B \) a subset of Ω with no limit point in Ω. Thus \( B = \{b_j : j \in J\} \) where \( J \) is some finite or countably infinite index set. Suppose that to each \( j \in J \) there corresponds a rational function of the form

\[
S_j(z) = \frac{a_{j1}}{z - b_j} + \frac{a_{j2}}{(z - b_j)^2} + \cdots + \frac{a_{jn_j}}{(z - b_j)^{n_j}}.
\]

Then there is a meromorphic function \( f \) on Ω such that \( f \) has poles at precisely the points \( b_j \) and such that the principal part of the Laurent expansion of \( f \) at \( b_j \) is exactly \( S_j \).

Proof. Let \( \{K_n\} \) be the sequence of compact sets defined in (5.1.1). Recall that \( \{K_n\} \) has the properties that \( K_n \subseteq K_{n+1} \) and \( \bigcup K_n = \Omega \). Furthermore, by Problem 5.2.5, each component of \( \mathbb{C} \setminus K_n \) contains a component of \( \mathbb{C} \setminus \Omega \), in particular, \( \mathbb{C} \setminus \Omega \) meets each component of \( \mathbb{C} \setminus K_n \). Put \( K_0 = \emptyset \) and for \( n = 1, 2, \ldots \), define

\[
J_n = \{j \in J : b_j \in K_n \setminus K_{n-1}\}.
\]

The sets \( J_n \) are pairwise disjoint (possibly empty), each \( J_n \) is finite (since \( B \) has no limit point in \( \Omega \)), and \( \bigcup J_n = J \). For each \( n \), define \( Q_n \) by

\[
Q_n(z) = \sum_{j \in J_n} S_j(z)
\]

where \( Q_n = 0 \) if \( J_n \) is empty. Then \( Q_n \) is a rational function whose poles lie in \( K_n \setminus K_{n-1} \). In particular, \( Q_n \) is analytic on a neighborhood of \( K_{n-1} \). Hence by Runge’s theorem (5.2.8) with \( S = \mathbb{C} \setminus \Omega \), there is a rational function \( R_n \) whose poles lie in \( \mathbb{C} \setminus \Omega \) such that

\[
|Q_n(z) - R_n(z)| \leq (1/2)^n, \quad z \in K_{n-1}.
\]

It follows that for any fixed \( m \geq 1 \), the series \( \sum_{n=m+1}^{\infty} (Q_n - R_n) \) converges uniformly on \( K_m \) to a function which is analytic on \( K_m \supseteq K_{m-1} \). Thus it is meaningful to define a function \( f : \Omega \to \mathbb{C} \) by

\[
f(z) = Q_1(z) + \sum_{n=2}^{\infty} (Q_n(z) - R_n(z)), \quad z \in \Omega.
\]

Indeed, note that for any fixed \( m \), \( f \) is the sum of the rational function \( Q_1 + \sum_{n=2}^{m} (Q_n - R_n) \) and the series \( \sum_{n=m+1}^{\infty} (Q_n - R_n) \), which is analytic on \( K_m \). Therefore \( f \) is meromorphic.
on $\Omega$, as well as analytic on $\Omega \setminus B$. It remains to show that $f$ has the required principal part at each point $b \in B$. But for any $b_j \in B$, we have $f(z) = S_j(z)$ plus a function that is analytic on a neighborhood of $b_j$. Thus $f$ has a pole at $b_j$ with the required principal part $S_j$.

### 6.3.2 Remark

Suppose $g$ is analytic at the complex number $b$ and $g$ has a zero of order $m \geq 1$ at $b$. Let $c_1, c_2, \ldots, c_m$ be given complex numbers, and let $R$ be the rational function given by

$$R(z) = \frac{c_1}{z-b} + \cdots + \frac{c_m}{(z-b)^m}.$$  

Then $gR$ has a removable singularity at $b$, so there exist complex numbers $a_0, a_1, a_2, \ldots$ such that for $z$ in a neighborhood of $b$,

$$g(z)R(z) = a_0 + a_1(z-b) + \cdots + a_{m-1}(z-b)^{m-1} + \cdots.$$  

Furthermore, if we write the Taylor series expansion

$$g(z) = b_0(z-b)^m + b_1(z-b)^{m+1} + \cdots + b_{m-1}(z-b)^{2m-1} + \cdots,$$  

then the coefficients $a_0, a_1, \ldots$ for $gR$ must satisfy

$$a_0 = b_0c_m,$$

$$a_1 = b_0c_{m-1} + b_1c_m,$$

$$\vdots$$

$$a_{m-1} = b_0c_1 + b_1c_2 + \cdots + b_{m-1}c_m.$$  

That is, if $c_1, c_2, \ldots, c_m$ are given, then $a_0, a_1, \ldots, a_{m-1}$ are determined by the above equations. Conversely, if $g$ is given as above, and $a_0, a_1, \ldots, a_{m-1}$ are given complex numbers, then since $b_0 \neq 0$, one can sequentially solve the equations to obtain, in order, $c_m, c_{m-1}, \ldots, c_1$. This observation plays a key role in the next result, where it is shown that not only is it possible to construct analytic functions with prescribed zeros and with prescribed orders at these zeros, as in (6.2.3) and (6.2.6), but we can specify the values of $f$ and finitely many of its derivatives in an arbitrary way. To be precise, we have the following extension of (6.2.6).

### 6.3.3 Theorem

Let $\Omega$ be an open subset of $\mathbb{C}$ and $B$ a subset of $\Omega$ with no limit point in $\Omega$. Index $B$ by $J$, as in Mittag-Leffler’s theorem, so $B = \{b_j : j \in J\}$. Suppose that corresponding to each $j \in J$, there is a nonnegative integer $n_j$ and complex numbers $a_{0j}, a_{1j}, \ldots, a_{nj,j}$. Then there exists $f \in A(\Omega)$ such that for each $j \in J$,

$$\frac{f^{(k)}(b_j)}{k!} = a_{kj}, \quad 0 \leq k \leq n_j.$$
Proof. First apply (6.2.6) to produce a function \( g \in A(\Omega) \) such that \( Z(g) = B \) and for each \( j, m(g, b_j) = n_j + 1 = m_j \), say. Next apply the observations made above in (6.3.2) to obtain, for each \( b_j \in B \), complex numbers \( c_{1j}, c_{2j}, \ldots, c_{m_j, j} \) such that

\[
g(z) \sum_{k=1}^{m_j} \frac{c_{kj}}{(z - b_j)^k} = a_{0j} + a_{1j}(z - b_j) + \cdots + a_{n_j, j}(z - b_j)^{n_j} + \cdots
\]

for \( z \) near \( b_j \). Finally, apply Mittag-Leffler’s theorem to obtain \( h \), meromorphic on \( \Omega \), such that for each \( j, \)

\[
h - \sum_{k=1}^{m_j} \frac{c_{kj}}{(z - b_j)^k}
\]

has a removable singularity at \( b_j \). It follows that the analytic extension of \( gh \) to \( \Omega \) is the required function \( f \). (To see this, note that

\[
gh = g \left( h - \sum_{k=1}^{m_j} \frac{c_{kj}}{(z - b_j)^k} \right) + g \sum_{k=1}^{m_j} \frac{c_{kj}}{(z - b_j)^k}
\]

and \( m(g, b_j) > n_j \).) ♦

6.3.4 Remark

Theorem 6.3.3 will be used to obtain a number of algebraic properties of the ring \( A(\Omega) \). This theorem, together with most of results to follow, were obtained (in the case \( \Omega = \mathbb{C} \)) by Olaf Helmer, Duke Mathematical Journal, volume 6, 1940, pp.345-356.

Assume in what follows that \( \Omega \) is connected. Thus by Problem 2.4.11, \( A(\Omega) \) is an integral domain. Recall that in a ring, such as \( A(\Omega) \), \( g \) divides \( f \) if \( f = gq \) for some \( q \in A(\Omega) \). Also, \( g \) is a greatest common divisor of a set \( F \) if \( g \) is a divisor of each \( f \in F \) and if \( h \) divides each \( f \in F \), then \( h \) divides \( g \).

6.3.5 Proposition

Each nonempty subfamily \( F \subseteq A(\Omega) \) has a greatest common divisor, provided \( F \neq \emptyset \).

Proof. Put \( B = \cap \{ Z(f) : f \in F \} \). Apply Theorem 6.2.6 to obtain \( g \in A(\Omega) \) such that \( Z(g) = B \) and for each \( b \in B, m(g, b) = \min \{ m(f, b) : f \in F \} \). Then \( f \in F \) implies that \( g \mid f \) (\( g \) divides \( f \)). Furthermore, if \( h \in A(\Omega) \) and \( h \mid f \) for each \( f \in F \), then \( Z(h) \subseteq B \) and for each \( b \in B, m(h, b) \leq \min \{ m(f, b) : f \in F \} = m(g, b) \). Thus \( h \mid g \), and consequently \( g \) is a greatest common divisor of \( F \). ♦

6.3.6 Definitions

A unit in \( A(\Omega) \) is a function \( f \in A(\Omega) \) such that \( 1/f \in A(\Omega) \). Thus \( f \) is a unit iff \( f \) has no zeros in \( \Omega \). If \( f, g \in A(\Omega) \), we say that \( f \) and \( g \) are relatively prime if each greatest common divisor of \( f \) and \( g \) is a unit. It follows that \( f \) and \( g \) are relatively prime iff \( Z(f) \cap Z(g) = \emptyset \). (Note that \( f \) and \( g \) have a common zero iff they have a nonunit common factor.)
6.3.7 Proposition

If the functions \( f_1, f_2 \in A(\Omega) \) are relatively prime, then there exist \( g_1, g_2 \in A(\Omega) \) such that \( f_1g_1 + f_2g_2 = 1. \)

**Proof.** By the remarks above, \( Z(f_1) \cap Z(f_2) = \emptyset. \) By working backwards, i.e., solving \( f_1g_1 + f_2g_2 = 1 \) for \( g_1, \) we see that it suffices to obtain \( g_2 \) such that \( (1 - f_2g_2)/f_1 \) has only removable singularities. But this entails obtaining \( g_2 \) such that \( Z(f_1) \subseteq Z(1 - f_2g_2) \) and such that for each \( a \in Z(f_1), m(f_1, a) \leq m(1 - f_2g_2, a). \) However, the latter condition may be satisfied by invoking (6.3.3) to obtain \( g_2 \in A(\Omega) \) such that for each \( a \in Z(f_1) \) (recalling that \( f_2(a) \neq 0),
\[
0 = 1 - f_2(a)g_2(a) = (1 - f_2g_2)(a)
\]
\[
0 = f_2(a)g_2'(a) + f_2'(a)g_2(a) = (1 - f_2g_2)'(a)
\]
\[
0 = f_2(a)g_2''(a) + 2f_2'(a)g_2'(a) + f_2''(a)g_2(a) = (1 - f_2g_2)''(a)
\]
\[
\vdots
\]
\[
0 = f_2(a)g_2^{(m-1)}(a) + \cdots + f_2^{(m-1)}(a)g_2(a) = (1 - f_2g_2)^{(m-1)}(a)
\]
where \( m = m(f_1, a). \) [Note that these equations successively determine \( g_2(a), g_2'(a), \ldots, g_2^{(m-1)}(a). \)] This completes the proof of the proposition. □

The preceding result can be generalized to an arbitrary finite collection of functions.

6.3.8 Proposition

If \( \{f_1, f_2, \ldots, f_n\} \subseteq A(\Omega) \) and \( d \) is a greatest common divisor for this set, then there exist \( g_1, g_2, \ldots, g_n \in A(\Omega) \) such that \( f_1g_1 + f_2g_2 + \cdots + f_ng_n = d. \)

**Proof.** Use (6.3.7) and induction. The details are left as an exercise (Problem 1). □

Recall that an ideal \( I \subseteq A(\Omega) \) is a subset that is closed under addition and subtraction and has the property that if \( f \in A(\Omega) \) and \( g \in I, \) then \( fg \in I. \)

We are now going to show that \( A(\Omega) \) is what is referred to in the literature as a **Bezout domain.** This means that each finitely generated ideal in the integral domain \( A(\Omega) \) is a principal ideal. A **finitely generated** ideal is an ideal of the form \( \{f_1g_1 + \cdots + f_ng_n : g_1, \ldots, g_n \in A(\Omega)\} \) where \( \{f_1, \ldots, f_n\} \) is some fixed finite set of elements in \( A(\Omega). \) A **principal ideal** is an ideal that is generated by a single element \( f_1. \) Most of the work has already been done in preceding two propositions.

6.3.9 Theorem

Let \( f_1, \ldots, f_n \in A(\Omega) \) and let \( I = \{f_1g_1 + \cdots + f_ng_n : g_1, \ldots, g_n \in A(\Omega)\} \) be the ideal generated by \( f_1, \ldots, f_n. \) Then there exists \( f \in A(\Omega) \) such that \( I = \{fg : g \in A(\Omega)\}. \) In other words, \( I \) is a principal ideal.

**Proof.** If \( f \in I \) then \( f = f_1h_1 + \cdots + f_nh_n \) for some \( h_1, \ldots, h_n \in A(\Omega). \) If \( d \) is a greatest common divisor for \( \{f_1, \ldots, f_n\}, \) then \( d \) divides each \( f_j, \) hence \( d \) divides \( f. \) Thus \( f \) is a multiple of \( d. \) On the other hand, by (6.3.8), there exist \( g_1, \ldots, g_n \in A(\Omega) \) such that
$d = f_1g_1 + \cdots + f_ng_n$. Therefore $d$ and hence every multiple of $d$ belongs to $I$. Thus $I$ is the ideal generated by the single element $d$. ♣

A principal ideal domain is an integral domain in which every ideal is principal. Problem 2 asks you to show that $A(\Omega)$ is never a principal ideal domain, regardless of the region $\Omega$. There is another class of (commutative) rings called Noetherian; these are rings in which every ideal is finitely generated. Problem 2, when combined with (6.3.9), also shows that $A(\Omega)$ is never Noetherian.

Problems

1. Supply the details to the proof of (6.3.8). (Hint: Use induction, (6.3.7), and the fact that if $d$ is a greatest common divisor (gcd) for $\{f_1, \ldots, f_n\}$ and $d_1$ is a gcd for $\{f_1, \ldots, f_{n-1}\}$, then $d$ is a gcd for the set $\{d_1, f_n\}$. Also note that 1 is a gcd for $\{f_1/d, \ldots, f_n/d\}$.)

2. Show that $A(\Omega)$ is never a principal ideal domain. That is, there always exists ideals $I$ that are not principal ideals, and thus by (6.3.9) are not finitely generated. (Hint: Let $\{a_n\}$ be a sequence of distinct points in $\Omega$ with no limit point in $\Omega$. For each $n$, apply (6.2.6) to the set $\{a_n, a_{n+1}, \ldots\}$.)