Chapter 5

Families of Analytic Functions

In this chapter we consider the linear space $A(\Omega)$ of all analytic functions on an open set Ω and introduce a metric d on $A(\Omega)$ with the property that convergence in the d-metric is uniform convergence on compact subsets of Ω. We will characterize the compact subsets of the metric space $(A(\Omega), d)$ and prove several useful results on convergence of sequences of analytic functions. After these preliminaries we will present a fairly standard proof of the Riemann mapping theorem and then consider the problem of extending the mapping function to the boundary. Also included in this chapter are Runge's theorem on rational approximations and the homotopic version of Cauchy's theorem.

5.1 The Spaces $A(\Omega)$ and $C(\Omega)$

5.1.1 Definitions

Let Ω be an open subset of $\mathbb{C}$. Then $A(\Omega)$ will denote the space of analytic functions on Ω, while $\mathbb{C}(\Omega)$ will denote the space of all continuous functions on Ω. For $n = 1, 2, 3 \dots,$ let

$$K_n = \overline{D}(0, n) \cap \{z : |z - w| \geq 1/n \text{ for all } w \in \mathbb{C} \setminus \Omega\}.$$

By basic topology of the plane, the sequence $\{K_n\}$ has the following three properties:
(1) K_n is compact,
(2) $K_n \subseteq K^o_{n+1}$ (the interior of K_{n+1}),
(3) If $K \subseteq \Omega$ is compact, then $K \subseteq K_n$ for n sufficiently large.

Now fix a nonempty open set Ω, let $\{K_n\}$ be as above, and for $f, g \in \mathbb{C}(\Omega)$, define

$$d(f, g) = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}},$$

where

$$\|f - g\|_{K_n} = \begin{cases} \sup\{|f(z) - g(z)| : z \in K_n\}, & K_n \neq \emptyset \\ 0, & K_n = \emptyset \end{cases}$$

5.1.2 Theorem

The assignment $(f,g) \to d(f,g)$ defines a metric on $C(\Omega)$. A sequence $\{f_j\}$ in $C(\Omega)$ is d-convergent (respectively d-Cauchy) iff $\{f_j\}$ is uniformly convergent (respectively uniformly Cauchy) on compact subsets of Ω. Thus $(C(\Omega), d)$ and $(A(\Omega), d)$ are complete metric spaces.

Proof. That d is a metric on $C(\Omega)$ is relatively straightforward. The only troublesome part of the argument is verification of the triangle inequality, whose proof uses the inequality: If a, b and c are nonnegative numbers and $a \le b + c$, then

$$\frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}.$$

To see this, note that $h(x) = x/(1+x)$ increases with $x \ge 0$, and consequently $h(a) \le h(b+c) = \frac{b}{1+b+c} + \frac{c}{1+b+c} \le \frac{b}{1+b} + \frac{c}{1+c}$. Now let us show that a sequence $\{f_j\}$ is d-Cauchy iff $\{f_j\}$ is uniformly Cauchy on compact subsets of Ω. Suppose first that $\{f_j\}$ is d-Cauchy, and let K be any compact subset of Ω. By the above property (3) of the sequence $\{K_n\}$, we can choose n so large that $K \subseteq K_n$. Since $d(f_j, f_k) \to 0$ as $j,k \to \infty$, the same is true of $\|f_j - f_k\|_{K_n}$. But $\|f_j - f_k\|_K \le \|f_j - f_k\|_{K_n}$, hence $\{f_j\}$ is uniformly Cauchy on K. Conversely, assume that $\{f_j\}$ is uniformly Cauchy on compact subsets of Ω. Let $\epsilon > 0$ and choose a positive integer m such that $\sum_{n=m+1}^{\infty} 2^{-n} < \epsilon$. Since $\{f_j\}$ is uniformly Cauchy on K_m in particular, there exists $N = N(m)$ such that $j,k \ge N$ implies $\|f_j - f_k\|_{K_m} < \epsilon$, hence

$$\begin{aligned}\sum_{n=1}^{m} \left(\frac{1}{2^n}\right) \frac{\|f_j - f_k\|_{K_n}}{1 + \|f_j - f_k\|_{K_n}} &\le \sum_{n=1}^{m} \left(\frac{1}{2^n}\right) \|f_j - f_k\|_{K_n} \\ &\le \|f_j - f_k\|_{K_m} \sum_{n=1}^{m} \frac{1}{2^n} < \epsilon.\end{aligned}$$

It follows that for $j,k \ge N$,

$$d(f_j, f_k) = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) \frac{\|f_j - f_k\|_{K_n}}{1 + \|f_j - f_k\|_{K_n}} < 2\epsilon.$$

The remaining statements in (5.1.2) follow from the above, Theorem 2.2.17, and completeness of $\mathbb{C}$. ♣

If $\{f_n\}$ is a sequence in $A(\Omega)$ and $f_n \to f$ uniformly on compact subsets of Ω, then we know that $f \in \mathbb{A}(\Omega)$ also. The next few theorems assert that certain other properties of the limit function f may be inferred from the possession of these properties by the f_n. The first results of this type relate the zeros of f to those of the f_n.

5.1.3 Hurwitz's Theorem

Suppose that $\{f_n\}$ is a sequence in $A(\Omega)$ that converges to f uniformly on compact subsets of Ω. Let $\overline{D}(z_0, r) \subseteq \Omega$ and assume that $f(z) \neq 0$ for $|z - z_0| = r$. Then there is a positive integer N such that for $n \ge N$, f_n and f have the same number of zeros in $D(z_0, r)$.

Proof. Let $\epsilon = \min\{|f(z)| : |z-z_0| = r\} > 0$. Then for sufficiently large n, $|f_n(z)-f(z)| < \epsilon \le |f(z)|$ for $|z - z_0| = r$. By Rouché's theorem (4.2.8), f_n and f have the same number of zeros in $D(z_0, r)$. ♣

5.1.4 Theorem

Let $\{f_n\}$ be a sequence in $A(\Omega)$ such that $f_n \to f$ uniformly on compact subsets of Ω. If Ω is connected and f_n has no zeros in Ω for infinitely many n, then either f has no zeros in Ω or f is identically zero.

Proof. Assume f is not identically zero, but f has a zero at $z_0 \in \Omega$. Then by the identity theorem (2.4.8), there is $r > 0$ such that the hypothesis of (5.1.3) is satisfied. Thus for sufficiently large n, f_n has a zero in $D(z_0, r)$. ♣

5.1.5 Theorem

Let $\{f_n\}$ be a sequence in $A(\Omega)$ such that f_n converges to f uniformly on compact subsets of Ω. If Ω is connected and the f_n are one-to-one on Ω, then either f is constant on Ω or f is one-to-one.

Proof. Assume that f is not constant on Ω, and choose any $z_0 \in \Omega$. The sequence $\{f_n - f_n(z_0)\}$ satisfies the hypothesis of (5.1.4) on the open connected set $\Omega \setminus \{z_0\}$ (because the f_n are one-to-one). Since $f - f(z_0)$ is not identically zero on $\Omega \setminus \{z_0\}$, it follows from (5.1.4) that $f - f(z_0)$ has no zeros in $\Omega \setminus \{z_0\}$. Since z_0 is an arbitrary point of Ω, we conclude that f is one-to-one on Ω. ♣

The next task will be to identify the compact subsets of the space $A(\Omega)$ (equipped with the topology of uniform convergence on compact subsets of Ω). After introducing the appropriate notion of boundedness for subsets $\mathcal{F} \subseteq A(\Omega)$, we show that each sequence of functions in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of Ω. This leads to the result that a subset of $A(\Omega)$ is compact iff it is closed and bounded.

5.1.6 Definition

A set $\mathcal{F} \subseteq C(\Omega)$ is *bounded* if for each compact set $K \subseteq \Omega$, $\sup\{\|f\|_K : f \in \mathcal{F}\} < \infty$, that is, the functions in $\mathcal{F}$ are uniformly bounded on each compact subset of Ω.

We will also require the notion of equicontinuity for a family of functions.

5.1.7 Definition

A family $\mathcal{F}$ of functions on Ω is *equicontinuous* at $z_0 \in \Omega$ if given $\epsilon > 0$ there exists $\delta > 0$ such that if $z \in \Omega$ and $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \epsilon$ for all $f \in \mathcal{F}$.

We have the following relationship between bounded and equicontinuous subsets of $A(\Omega)$.

5.1.8 Theorem

Let $\mathcal{F}$ be a bounded subset of $A(\Omega)$. Then $\mathcal{F}$ is equicontinuous at each point of Ω.

Proof. Let $z_0 \in \Omega$ and choose $r > 0$ such that $\overline{D}(z_0, r) \subseteq \Omega$. Then for $z \in D(z_0, r)$ and $f \in \mathcal{F}$, we have

$$f(z) - f(z_0) = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(w)}{w - z_0}\, dw.$$

Thus

$$|f(z) - f(z_0)| \le \frac{1}{2\pi} \sup \left\{ \left| \frac{f(w)}{w - z} - \frac{f(w)}{w - z_0} \right| : w \in C(z_0, r) \right\} 2\pi r$$

$$= r|z - z_0| \sup \left\{ \left| \frac{f(w)}{(w - z)(w - z_0)} \right| : w \in C(z_0, r) \right\}.$$

But by hypothesis, there exists M_r such that $|f(w)| \le M_r$ for all $w \in C(z_0, r)$ and all $f \in \mathcal{F}$. Consequently, if $z \in D(z_0, r/2)$ and $f \in \mathcal{F}$, then

$$r|z - z_0| \sup \left\{ \left| \frac{f(w)}{(w - z)(w - z_0)} \right| : w \in C(z_0, r) \right\} \le r|z - z_0| \frac{M_r}{(r/2)^2},$$

proving equicontinuity of $\mathcal{F}$. ♣

We will also need the following general fact about equicontinuous families.

5.1.9 Theorem

Suppose $\mathcal{F}$ is an equicontinuous subset of $C(\Omega)$ (that is, each $f \in \mathcal{F}$ is continuous on Ω and $\mathcal{F}$ is equicontinuous at each point of Ω) and $\{f_n\}$ is a sequence from $\mathcal{F}$ such that f_n converges pointwise to f on Ω. Then f is continuous on Ω and $f_n \to f$ uniformly on compact subsets of Ω. More generally, if $f_n \to f$ pointwise on a dense subset of Ω, then $f_n \to f$ on all of Ω and the same conclusion holds.

Proof. Let $\epsilon > 0$. For each $w \in \Omega$, choose a $\delta_w > 0$ such that $|f_n(z) - f_n(w)| < \epsilon$ for each $z \in D(w, \delta_w)$ and all n. It follows that $|f(z) - f(w)| \le \epsilon$ for all $z \in D(w, \delta_w)$, so f is continuous. Let K be any compact subset of Ω. Since $\{D(w, \delta_w) : w \in K\}$ is an open cover of K, there are $w_1, \dots, w_m \in K$ such that $K \subseteq \cup_{j=1}^m D(w_j, \delta_{w_j})$. Now choose N such that $n \ge N$ implies that $|f(w_j) - f_n(w_j)| < \epsilon$ for $j = 1, \dots, m$. Hence if $z \in D(w_j, \delta_{w_j})$ and $n \ge N$, then

$$|f(z) - f_n(z)| \le |f(z) - f(w_j)| + |f(w_j) - f_n(w_j)| + |f_n(w_j) - f_n(z)| < 3\epsilon.$$

In particular, if $z \in K$ and $n \ge N$, then $|f(z) - f_n(z)| < 3\epsilon$, showing that $f_n \to f$ uniformly on K.

Finally, suppose only that $f_n \to f$ pointwise on a dense subset $S \subseteq \Omega$. Then as before, $|f_n(z) - f_n(w)| < \epsilon$ for all n and all $z \in D(w, \delta_w)$. But since S is dense, $D(w, \delta_w)$ contains a point $z \in S$, and for m and n sufficiently large,

$$|f_m(w) - f_n(w)| \le |f_m(w) - f_m(z)| + |f_m(z) - f_n(z)| + |f_n(z) - f_n(w)| < 3\epsilon.$$

Thus $\{f_n(w)\}$ is a Cauchy sequence and therefore converges, hence $\{f_n\}$ converges pointwise on all of Ω and the first part of the theorem applies. ♣

5.1.10 Montel's Theorem

Let $\mathcal{F}$ be a bounded subset of $A(\Omega)$, as in (5.1.6). Then each sequence $\{f_n\}$ from $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ which converges uniformly om compact subsets of Ω.

Remark

A set $\mathcal{F} \subseteq C(\Omega$ is said to be *relatively compact* if the closure of $\mathcal{F}$ in $C(\Omega)$ is compact. The conclusion of (5.1.10) is equivalent to the statement that $\mathcal{F}$ is a relatively compact subset of $C(\Omega)$, and hence of $A(\Omega)$.

Proof. Let $\{f_n\}$ be any sequence from $\mathcal{F}$ and choose any countable dense subset $S = \{z_1, z_2, \dots\}$ of Ω. The strategy will be to show that $f_n\}$ has a subsequence which converges pointwise on S. Since $\mathcal{F}$ is a bounded subset of $A(\Omega)$, it is equicontinuous on Ω by (5.1.8). Theorem 5.1.9 will then imply that this subsequence converges uniformly on compact subsets of Ω, thus completing the proof. So consider the following bounded sequences of complex numbers:

$$\{f_j(z_1)\}_{j=1}^{\infty}, \ \{f_j(z_2)\}_{j=1}^{\infty}, \dots .$$

There is a subsequence $\{f_{1j}\}_{j=1}^{\infty}$ of $\{f_j\}_{j=1}^{\infty}$ which converges at z_1. There is a subsequence $\{f_{2j}\}_{j=1}^{\infty}$ of $\{f_{1j}\}_{j=1}^{\infty}$ which converges at z_2 and (necessarily) at z_1 as well. Proceeding inductively, for each $n \geq 1$ and each $k = 1, \dots, n$ we construct sequences $\{f_{kj}\}_{j=1}^{\infty}$ converging at $z_1, \dots, z_k$, each a subsequence of the preceding sequence.

Put $g_j = f_{jj}$. Then $\{g_j\}$ is a subsequence of $\{f_j\}$, and $\{g_j\}$ converges pointwise on $\{z_1, z_2, \dots\}$ since for each n, $\{g_j\}$ is eventually a subsequence of $\{f_{nj}\}_{j=1}^{\infty}$. ♣

5.1.11 Theorem (Compactness Criterion)

Let $\mathcal{F} \subseteq A(\Omega)$. Then $\mathcal{F}$ is compact iff $\mathcal{F}$ is closed and bounded. Also, $\mathcal{F}$ is relatively compact iff $\mathcal{F}$ is bounded.

(See Problem 3 for the second part of this theorem.)

Proof.
If $\mathcal{F}$ is compact, then $\mathcal{F}$ is closed (a general property that holds in any metric space). In order to show that $\mathcal{F}$ is bounded, we will use the following device. Let K be any compact subset of Ω. Then $f \to \|f\|_K$ is a continuous map from $A(\Omega)$ into $\mathbb{R}$. Hence $\{\|f\|_K : f \in \mathcal{F}\}$ is a compact subset of $\mathbb{R}$ and thus is bounded. Conversely, if $\mathcal{F}$ is closed and bounded, then $\mathcal{F}$ is closed and, by Montel's theorem, relatively compact. Therefore $\mathcal{F}$ is compact. ♣

Remark

Problem 6 gives an example which shows that the preceding compactness criterion fails in the larger space $C(\Omega)$. That is, there are closed and bounded subsets of $C(\Omega)$ that are not compact.

5.1.12 Theorem

Suppose $\mathcal{F}$ is a nonempty compact subset of $A(\Omega)$. Then given $z_0 \in \Omega$, there exists $g \in \mathcal{F}$ such that $|g'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$.

Proof. Just note that the map $f \to |f'(z_0)|, f \in A(\Omega)$, is continuous. ♣

Here is a compactness result that will be needed for the proof of the Riemann mapping theorem in the next section.

5.1.13 Theorem

Assume that Ω is connected, $z_0 \in \Omega$, and $\epsilon > 0$. Define

$$\mathcal{F} = \{f \in A(\Omega) : f \text{ is a one-to-one map of } \Omega \text{ into } \overline{D}(0,1) \text{ and } |f'(z_0)| \geq \epsilon\}.$$

Then $\mathcal{F}$ is compact. The same conclusion holds with $\overline{D}(0,1)$ replaced by $D(0,1)$.

Proof. By its definition, $\mathcal{F}$ is bounded, and $\mathcal{F}$ is closed by (5.1.5). Thus by (5.1.11), $\mathcal{F}$ is compact. To prove the last statement of the theorem, note that if $f_n \in \mathcal{F}$ and $f_n \to f$ uniformly on compact subsets of Ω, then (5.1.5) would imply that $f \in \mathcal{F}$, were it not for the annoying possibility that $|f(w)| = 1$ for some $w \in \Omega$. But if this happens, the maximum principle implies that f is constant, contradicting $|f'(z_0)| \geq \epsilon > 0$. ♣

The final result of this section shows that if Ω is connected, then any bounded sequence in $A(\Omega)$ that converges pointwise on a set having a limit point in Ω, must in fact converge uniformly on compact subsets of Ω.

5.1.14 Vitali's Theorem

Let $\{f_n\}$ be a bounded sequence in $A(\Omega)$ where Ω is connected. Suppose that $\{f_n\}$ converges pointwise on $S \subseteq \Omega$ and S has a limit point in Ω. Then $\{f_n\}$ is uniformly Cauchy on compact subsets of Ω, hence uniformly convergent on compact subsets of Ω to some $f \in A(\Omega)$.

Proof. Suppose, to the contrary, that there is a compact set $K \subseteq \Omega$ such that $\{f_n\}$ is not uniformly Cauchy on K. Then for some $\epsilon > 0$, we can find sequences $\{m_j\}$ and $\{n_j\}$ of positive integers such that $m_1 < n_1 < m_2 < n_2 < \cdots$ and for each j, $\|f_{m_j} - f_{n_j}\|_K \geq \epsilon$. Put $\{g_j\} = \{f_{m_j}\}$ and $\{h_j\} = \{f_{n_j}\}$. Now apply Montel's theorem (5.1.10) to $\{g_j\}$ to obtain a subsequence $\{g_{j_r}\}$ converging uniformly on compact subsets of Ω to some $g \in A(\Omega)$, and then apply Montel's theorem to $\{h_{j_r}\}$ to obtain a subsequence converging uniformly on compact subsets of Ω to some $h \in A(\Omega)$. To prevent the notation from getting out of hand, we can say that without loss of generality, we have $g_n \to g$ and $h_n \to h$ uniformly on compact subsets, and $\|g_n - h_n\|_K \geq \epsilon$ for all n, hence $\|g - h\|_K \geq \epsilon$. But by hypothesis, $g = h$ on S and therefore, by (2.4.9), $g = h$ on Ω, a contradiction. ♣

Problems

1. Let $\mathcal{F} = \{f \in A(D(0,1)) : \operatorname{Re} f > 0 \text{ and } |f(0)| \leq 1\}$. Prove that $\mathcal{F}$ is relatively compact. Is $\mathcal{F}$ compact? (See Section 4.6, Problem 2.)

2. Let Ω be (open and) connected and let $\mathcal{F} = \{f \in A(\Omega) : |f(z) - a| \geq r$ for all $z \in \Omega\}$, where $r > 0$ and $a \in \mathbb{C}$ are fixed. Show that $\mathcal{F}$ is a *normal family*, that is, if $f_n \in \mathcal{F}, n = 1, 2, \ldots$, then either there is a subsequence $\{f_{n_j}\}$ converging uniformly on compact subsets to a function $f \in A(\Omega)$ or there is a subsequence $\{f_{n_j}\}$ converging uniformly on compact subsets to ∞. (Hint: Look at the sequence $\{1/(f_n - a)\}$.)
3. (a) If $\mathcal{F} \subseteq C(\Omega)$, show that $\mathcal{F}$ is relatively compact iff each sequence in $\mathcal{F}$ has a convergent subsequence (whose limit need not be in $\mathcal{F}$).
(b) Prove the last statement in Theorem 5.1.11.
4. Let $\mathcal{F} \subseteq A(D(0,1))$. Show that $\mathcal{F}$ is relatively compact iff there is a sequence of nonnegative real numbers M_n with $\limsup_{n\to\infty}(M_n)^{1/n} \leq 1$ such that for all $f \in \mathcal{F}$ and all $n = 0, 1, 2, \ldots$, we have $|f^{(n)}(0)/n!| \leq M_n$.
5. (a) Suppose that f is analytic on Ω and $\overline{D}(a, R) \subseteq \Omega$. Prove that

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{it})|^2 r\, dr\, dt.$$

(b) Let $M > 0$ and define $\mathcal{F}$ to be the set

$$\{f \in A(\Omega) : \int_\Omega \int |f(x + iy)|^2\, dx\, dy \leq M\}.$$

Show that $\mathcal{F}$ is relatively compact.
6. Let Ω be open and $K = \overline{D}(a, R) \subseteq \Omega$. Define $\mathcal{F}$ to be the set of all $f \in C(\Omega)$ such that $|f(z)| \leq 1$ for all $z \in \Omega$ and $f(z) = 0$ for $z \in \Omega \setminus K\}$. Show that $\mathcal{F}$ is a closed and bounded subset of $C(\Omega)$, but $\mathcal{F}$ is not compact. (Hint: Consider the map from $\mathcal{F}$ to the reals given by

$$f \to \left[\int_K \int (1 - |f(x + iy)|)\, dx\, dy\right]^{-1}.$$

Show that this map is continuous but not bounded on $\mathcal{F}$.)

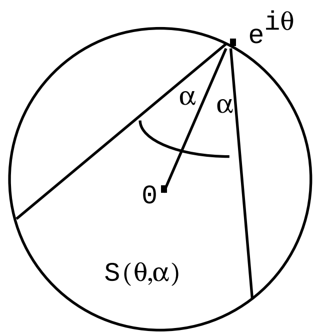

Figure 5.1.1

7. (An application of Vitali's theorem.) Let f be a *bounded* analytic function on $D(0,1)$ with the property that for some θ, $f(re^{i\theta})$ approaches a limit L as $r \to 1^-$. Fix $\alpha \in (0, \pi/2)$ and consider the region $S(\theta, \alpha)$ in Figure 5.1.1. Prove that if $z \in S(\theta, \alpha)$ and $z \to e^{i\theta}$, then $f(z) \to L$. (Suggestion: Look at the sequence of functions defined by $f_n(z) = f(e^{i\theta} + \frac{1}{n}(z - e^{i\theta})), z \in D(0,1)$.)
8. Let L be a multiplicative linear functional on $A(\Omega)$, that is, $L : A(\Omega) \to \mathbb{C}$ such that $L(af + bg) = aL(f) + bL(g)$ and $L(fg) = L(f)L(g)$ for all $a, b \in \mathbb{C}, f, g \in A(\Omega)$. Assume $L \not\equiv 0$. Show that L is a point evaluation, that is, there is some $z_0 \in \Omega$ such that $L(f) = f(z_0)$ for all $f \in A(\Omega)$.
   Outline: First show that for $f \equiv 1$, $L(f) = 1$. Then apply L to the function $I(z) = z$, the identity on Ω, and show that if $L(I) = z_0$, then $z_0 \in \Omega$. Finally, if $f \in A(\Omega)$, apply L to the function
$$g(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0. \end{cases}$$
9. (Osgood's theorem). Let $\{f_n\}$ be a sequence in $A(\Omega)$ such that $f_n \to f$ pointwise on Ω. Show that there is an open set U, dense in Ω, such that $f_n \to f$ uniformly on compact subsets of U. In particular, f is analytic on a dense open subset of Ω.
   (Let $A_n = \{z \in \Omega : |f_k(z)| \leq n \text{ for all } k = 1, 2, \dots\}$. Recall the Baire category theorem: If a complete metric space X is the union of a sequence $\{S_n\}$ of closed subsets, then some S_n contains a nonempty open ball. Use this result to show that some A_n contains a disk D. By Vitali's theorem, $f_n \to f$ uniformly on compact subsets of D. Take U to be the union of all disks D such that $f_n \to f$ uniformly on compact subsets of D.)

5.2 Riemann Mapping Theorem

Throughout this section, Ω will be a nonempty open connected proper subset of $\mathbb{C}$ with the property that every zero-free analytic function has an analytic square root. Later in the section we will prove that *any* open subset Ω such that every zero-free analytic function on Ω has an analytic square root must be (homotopically) simply connected, and conversely. Thus we are considering open, connected and simply connected proper subsets of $\mathbb{C}$. Our objective is to prove the Riemann mapping theorem, which states that there is a one-to-one analytic map of Ω onto the open unit disk D. The proof given is due to Fejer and F.Riesz.

5.2.1 Lemma

There is a one-to-one analytic map of Ω *into* D.

Proof. Fix $a \in \mathbb{C} \setminus \Omega$. Then the function $z - a$ satisfies our hypothesis on Ω and hence there exists $h \in A(\Omega)$ such that $(h(z))^2 = z - a, z \in \Omega$. Note that h is one-to-one and $0 \notin h(\Omega)$. Furthermore, $h(\Omega)$ is open by (4.3.1), the open mapping theorem, hence so is $-h(\Omega) = \{-h(z) : z \in \Omega\}$, and $[h(\Omega] \cap [-h(\Omega] = \emptyset$ (because $0 \notin h(\Omega)$). Now choose $w \in -h(\Omega)$. Since $-h(\Omega)$ is open, there exists $r > 0$ such that $D(w, r) \subseteq -h(\Omega)$, hence $h(\Omega) \cap D(w, r) = \emptyset$. The function $f(z) = 1/(h(z) - w)$, $z \in \Omega$, is one-to-one, and its magnitude is less than $1/r$ on Ω. Thus rf is a one-to-one map of Ω into D. ♣

5.2.2 Riemann Mapping Theorem

Let Ω be as in (5.2.1), that is, a nonempty, proper, open and connected subset of $\mathbb{C}$ such that every zero-free analytic function on Ω has an analytic square root. Then there is a one-to-one analytic map of Ω *onto* D.

Proof. Fix $z_0 \in \Omega$ and a one-to-one analytic map f_0 of Ω into D [f_0 exists by (5.2.1)]. Let $\mathcal{F}$ be the set of all $f \in A(\Omega)$ such that f is a one-to-one analytic map of Ω into D and $|f'(z_0)| \geq |f_0'(z_0)|$. Note that $|f_0'(z_0)| > 0$ by (4.3.1).

Then $\mathcal{F} \neq \emptyset$ (since $f_0 \in \mathcal{F}$) and $\mathcal{F}$ is bounded. Also, $\mathcal{F}$ is closed, for if $\{f_n\}$ is a sequence in $\mathcal{F}$ such that $f_n \to f$ uniformly on compact subsets of Ω, then by (5.1.5), either f is constant on Ω or f is one-to-one. But since $f_n' \to f'$, it follows that $|f'(z_0)| \geq |f_0'(z_0)| > 0$, so f is one-to-one. Also, f maps Ω into D (by the maximum principle), so $f \in \mathcal{F}$. Since $\mathcal{F}$ is closed and bounded, it is compact (Theorem 5.1.1). Hence by (5.1.2), there exists $g \in \mathcal{F}$ such that $|g'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$. We will now show that such a g must map Ω *onto* D. For suppose that there is some $a \in D \setminus g(\Omega)$. Let φ_a be as in (4.6.1), that is,

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z},\ z \in D.$$

Then $\varphi_a \circ g : \Omega \to D$ and $\varphi_a \circ g$ is one-to-one with no zeros in Ω. By hypothesis, there is an analytic square root h for $\varphi_a \circ g$. Note also that $h^2 = \varphi_a \circ g$ is one-to-one, and therefore so is h. Set $b = h(z_0)$ and define $f = \varphi_b \circ h$. Then $f(z_0) = \varphi_b(b) = 0$ and we can write

$$g = \varphi_{-a} \circ h^2 = \varphi_{-a} \circ (\varphi_{-b} \circ f)^2 = \varphi_{-a} \circ (\varphi_{-b}^2 \circ f) = (\varphi_{-a} \circ \varphi_{-b}^2) \circ f.$$

Now

$$\begin{aligned} g'(z_0) &= (\varphi_{-a} \circ \varphi_{-b}^2)'(f(z_0))f'(z_0) \\ &= (\varphi_{-a} \circ \varphi_{-b}^2)'(0)f'(z_0). \end{aligned} \tag{1}$$

The function $\varphi_{-a} \circ \varphi_{-b}^2$ *is* an analytic map of D into D, but it is *not* one-to-one; indeed, it is two-to-one. Hence by the Schwarz-Pick theorem (4.6.3), part (ii), it must be the case that

$$|(\varphi_{-a} \circ \varphi_{-b}^2)'(0)| < 1 - |\varphi_{-a} \circ \varphi_{-b}^2(0)|^2.$$

Since $f'(z_0) \neq 0$, it follows from (1) that

$$|g'(z_0)| < (1 - |\varphi_{-a} \circ \varphi_{-b}^2(0)|^2)|f'(z_0)| \leq |f'(z_0)|.$$

This contradicts our choice of $g \in \mathcal{F}$ as maximizing the numbers $|f'(z_0)|, f \in \mathcal{F}$. Thus $g(\Omega) = D$ as desired. ♣

5.2.3 Remarks

(a) Any function g that maximizes the numbers $\{|f'(z_0)| : f \in \mathcal{F}\}$ must send z_0 to 0.

Proof. Let $a = g(z_0)$. Then $\varphi_a \circ g$ is a one-to-one analytic map of Ω into D. Moreover,

$$\begin{aligned} |(\varphi_a \circ g)'(z_0)| &= |\varphi_a'(g(z_0))g'(z_0)| \\ &= |\varphi_a'(a)g'(z_0)| \\ &= \frac{1}{1-|a|^2}|g'(z_0)| \\ &\geq |g'(z_0)| \geq |f_0'(z_0)|. \end{aligned}$$

Thus $\varphi_a \circ g \in \mathcal{F}$, and since $|g'(z_0)|$ maximizes $|f'(z_0)|$ for $f \in \mathcal{F}$, it follows that equality must hold in the first inequality. Therefore $1/(1-|a|^2) = 1$, so $0 = a = g(z_0)$. ♣

(b) Let f and h be one-to-one analytic maps of Ω onto D such that $f(z_0) = h(z_0) = 0$ and $f'(z_0) = h'(z_0)$ (it is enough that $\operatorname{Arg} f'(z_0) = \operatorname{Arg} h'(z_0)$). Then $f = h$.

Proof. The function $h \circ f^{-1}$ is a one-to-one analytic map of D onto D, and $h \circ f^{-1}(0) = h(z_0) = 0$. Hence by Theorem 4.6.4 (with $a = 0$), there is a unimodular complex number λ such that $h(f^{-1}(z)) = \lambda z, z \in D$. Thus $h(w) = \lambda f(w), w \in D$. But if $h'(z_0) = f'(z_0)$ (which is equivalent to $\operatorname{Arg} h'(z_0) = \operatorname{Arg} f'(z_0)$ since $|h'(z_0)| = |\lambda||f'(z_0)| = |f'(z_0)|$), we have $\lambda = 1$ and $f = h$. ♣

(c) Let f be any analytic map of Ω *into* D (not necessarily one-to-one or onto) with $f(z_0) = 0$. Then with g as in the theorem, $|f'(z_0)| \leq |g'(z_0)|$. Also, equality holds iff $f = \lambda g$ with $|\lambda| = 1$.

Proof. The function $f \circ g^{-1}$ is an analytic map of D into D such that $f \circ g^{-1}(0) = 0$. By Schwarz's lemma (2.4.16), $|f(g^{-1}(z))| \leq |z|$ and $|f'(g^{-1}(0)) \cdot \frac{1}{g'(z_0)}| \leq 1$. Thus $|f'(z_0)| \leq |g'(z_0)|$. Also by (2.4.16), equality holds iff for some unimodular λ we have $f \circ g^{-1}(z) = \lambda z$, that is, $f(z) = \lambda g(z)$, for all $z \in D$. ♣

If we combine (a), (b) and (c), and observe that $\lambda g'(z_0)$ will be real and greater than 0 for appropriately chosen unimodular λ, then we obtain the following existence and uniqueness result.

(d) Given $z_0 \in \Omega$, there is a unique one-to-one analytic map g of Ω onto D such that $g(z_0) = 0$ and $g'(z_0)$ is real and positive.

As a corollary of (d), we obtain the following result, whose proof will be left as an exercise; see Problem 1.

(e) Let Ω_1 and Ω_2 be regions that satisfy the hypothesis of the Riemann mapping theorem. Let $z_1 \in \Omega_1$ and $z_2 \in \Omega_2$. Then there is a unique one-to-one analytic map f of Ω_1 onto Ω_2 such that $f(z_1) = z_2$ and $f'(z_1)$ is real and positive.

Recall from (3.4.6) that if $\Omega \subseteq \mathbb{C}$ and Ω satisfies any one of the six equivalent conditions listed there, then Ω is called (homologically) simply connected. Condition (6) is that every zero-free analytic function on Ω have an analytic n-th root for $n = 1, 2, \ldots$. Thus if Ω is homologically simply connected, then in particular, assuming $\Omega \neq \mathbb{C}$, the Riemann mapping theorem implies that Ω is *conformally equivalent* to D, in other words, there is a one-to-one analytic map of Ω onto D. The converse is also true, but before showing this, we need to take a closer look at the relationship between homological simple connectedness and homotopic simple connectedness [see (4.9.12)].

5.2.4 Theorem

Let γ_0 and γ_1 be closed curves in an open set $\Omega \subseteq \mathbb{C}$. If γ_0 and γ_1 are Ω-homotopic (in other words, homotopic in Ω), then they are Ω-homologous, that is, $n(\gamma_0, z) = n(\gamma_1, z)$ for every $z \in \mathbb{C} \setminus \Omega$.

Proof. We must show that $n(\gamma_0, z) = n(\gamma_1, z)$ for each $z \in \mathbb{C} \setminus \Omega$. Thus let $z \in \mathbb{C} \setminus \Omega$, let H be a homotopy of γ_0 to γ_1, and let θ be a continuous argument of $H - z$. (See Problem 6 of Section 3.2.) That is, θ is a real continuous function on $[a, b] \times [0, 1]$ such that

$$H(t, s) - z = |H(t, s) - z|e^{i\theta(t,s)}$$

for $(t, s) \in [a, b] \times [0, 1]$. Then for each $s \in [0, 1]$, the function $t \to \theta(t, s)$ is a continuous argument of $H(\cdot, s) - z$ and hence

$$n(H(\cdot, s), z) = \frac{\theta(b, s) - \theta(a, s)}{2\pi}.$$

This shows that the function $s \to n(H(\cdot, s), z)$ is continuous, and since it is integer valued, it must be constant. In particular,

$$n(H(\cdot, 0), z) = n(H(\cdot, 1), z).$$

In other words, $n(\gamma_0, z) = n(\gamma_1, z)$. ♣

The above theorem implies that if γ is Ω-homotopic to a point in Ω, then γ must be Ω-homologous to 0. Thus if γ is a closed path in Ω such that γ is Ω-homotopic to a point, then $\int_\gamma f(z)\,dz = 0$ for every analytic function f on Ω. We will state this result formally.

5.2.5 The Homotopic Version of Cauchy's Theorem

Let γ be a closed path in Ω such that γ is Ω-homotopic to a point. Then $\int_\gamma f(z)\,dz = 0$ for every analytic function f on Ω.

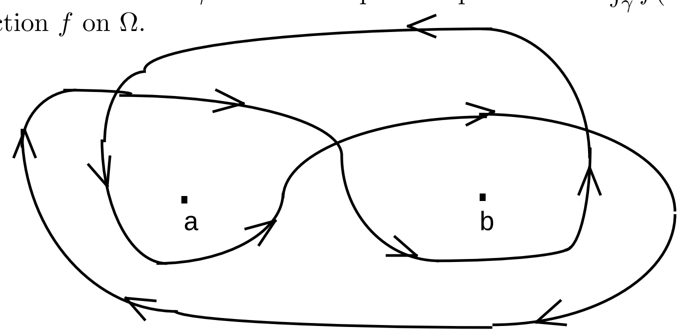

Figure 5.2.1

Remark

The converse of Theorem 5.2.5 is *not* true. In particular, there are closed curves γ and open sets Ω such that γ is Ω-homologous to 0 but γ is not homotopic to a point. Take $\Omega = \mathbb{C} \setminus \{a, b\}$, $a \neq b$, and consider the closed path γ of Figure 5.2.1. Then $n(\gamma, a) = n(\gamma, b) = 0$, hence γ is Ω-homologous to 0. But (intuitively at least) we see that γ cannot be shrunk to a point without passing through a or b. It follows from this example and Theorem 5.2.4 that the homology version of Cauchy's theorem (3.3.1) is actually stronger than the homotopy version (5.2.5). That is, if γ is a closed path to which the homotopy version applies, then so does the homology version, while the homology version applies to the above path, but the homotopy version does not. However, if *every* closed path in Ω is homologous to zero, then every closed path is homotopic to a point, as we now show.

5.2.6 Theorem

Let Ω be an open connected subset of $\mathbb{C}$. The following are equivalent.

(1) Every zero-free $f \in A(\Omega)$ has an analytic square root.

(2) If $\Omega \neq \mathbb{C}$, then Ω is conformally equivalent to D.

(3) Ω is homeomorphic to D.

(4) Ω is homotopically simply connected.

(5) Each closed *path* in Ω is homotopic to a point.

(6) Ω is homologically simply connected.

Proof.

(1) implies (2): This is the Riemann mapping theorem.

(2) implies (3): If $\Omega \neq \mathbb{C}$, this follows because a conformal equivalence is a homeomorphism., while if $\Omega = \mathbb{C}$, then the map $h(z) = z/(1 + |z|)$ is a homeomorphism of $\mathbb{C}$ onto D (see Problem 2).

(3) implies (4): Let $\gamma : [a, b] \to \Omega$ be any closed curve in Ω. By hypothesis there is a homeomorphism f of Ω onto D. Then $f \circ \gamma$ is a closed curve in D, and there is a homotopy H (in D) of $f \circ \gamma$ to the point $f(\gamma(a))$ (see Problem 4). Therefore $f^{-1} \circ H$ is a homotopy in Ω of γ to $\gamma(a)$.

(4) implies (5): Every closed path is a closed curve.

(5) implies (6): Let γ be any closed path in Ω. If γ is Ω-homotopic to a point, then by Theorem 5.2.4, γ is Ω-homologous to zero.

(6) implies (1): This follows from part (6) of (3.4.6).

Remark

If Ω is any open set (not necessarily connected) then the statement of the preceding theorem applies to each component of Ω. Therefore (1), (4), (5) and (6) are equivalent for arbitrary open sets.

Here is yet another condition equivalent to simple connectedness of an open set Ω.

5.2.7 Theorem

Let Ω be a simply connected open set. Then every harmonic function on Ω has a harmonic conjugate. Conversely, if Ω is an open set such that every harmonic function on Ω has a harmonic conjugate, then Ω is simply connected.

Proof. The first assertion was proved as Theorem 4.9.14 using the method of analytic continuation. However, we can also give a short proof using the Riemann mapping theorem, as follows. First note that we can assume that Ω is connected by applying this case to components. If $\Omega = \mathbb{C}$ then every harmonic function on Ω has a harmonic conjugate as in Theorem 1.6.2. Suppose then that $\Omega \neq \mathbb{C}$. By the Riemann mapping theorem, there is a conformal equivalence f of Ω onto D. Let u be harmonic on Ω. Then $u \circ f^{-1}$ is harmonic on D and thus by (1.6.2), there is a harmonic function V on D such that $u \circ f^{-1} + iV$ is analytic on D. Since $(u \circ f^{-1} + iV) \circ f$ is analytic on Ω, there is a harmonic conjugate of u on Ω, namely $v = V \circ f$.

Conversely, suppose that Ω is not simply connected. Then Ω is not homologically simply connected, so there exists $z_0 \in \mathbb{C} \setminus \Omega$ and a closed path γ in Ω such that $n(\gamma, z_0) \neq 0$. Thus by (3.1.9) and (3.2.3), the function $z \to z - z_0$ does not have an analytic logarithm on Ω, hence $z \to \ln|z - z_0|$ does not have a harmonic conjugate. ♣

The final result of this section is Runge's theorem on rational and polynomial approximation of analytic functions. One consequence of the development is another condition that is equivalent to simple connectedness.

5.2.8 Runge's Theorem

Let K be a compact subset of $\mathbb{C}$, and S a subset of $\hat{\mathbb{C}} \setminus K$ that contains at least one point in each component of $\hat{\mathbb{C}} \setminus K$. Define $B(S) = \{f : f$ is a uniform limit on K of rational functions whose poles lie in $S\}$. Then every function f that is analytic on a neighborhood of K is in $B(S)$. That is, there is a sequence $\{R_n\}$ of rational functions whose poles lie in S such that $R_n \to f$ uniformly on K.

Before giving the proof, let us note the conclusion in the special case where $\hat{\mathbb{C}} \setminus K$ is connected. In this case, we can take $S = \{\infty\}$, and our sequence of rational functions will actually be a sequence of polynomials. The proof given is due to Sandy Grabiner (Amer. Math. Monthly, 83 (1976), 807-808) and is based on three lemmas.

5.2.9 Lemma

Suppose K is a compact subset of the open set $\Omega \subseteq \mathbb{C}$. If $f \in A(\Omega)$, then f is a uniform limit on K of rational functions whose poles (in the extended plane!) lie in $\Omega \setminus K$.

5.2.10 Lemma

Let U and V be open subsets of $\mathbb{C}$ with $V \subseteq U$ and $\partial V \cap U = \emptyset$. If H is any component of U and $V \cap H \neq \emptyset$, then $H \subseteq V$.

5.2.11 Lemma

If K is a compact subset of $\mathbb{C}$ and $\lambda \in \mathbb{C} \setminus K$, then $(z-\lambda)^{-1} \in B(S)$.

Let us see how Runge's theorem follows from these three lemmas, and then we will prove the lemmas. First note that if f and g belong to $B(S)$, then so do $f+g$ and fg. Thus by Lemma 5.2.11 (see the partial fraction decomposition of Problem 4.1.7), every rational function with poles in $\hat{\mathbb{C}} \setminus K$ belongs to $B(S)$. Runge's theorem is then a consequence of Lemma 5.2.9. (The second of the three lemmas is used to prove the third.)

Proof of Lemma 5.2.9

Let Ω be an open set containing K. By (3.4.7), there is a cycle γ in $\Omega \setminus K$ such that for every $f \in A(\Omega)$ and $z \in K$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw.$$

Let $\epsilon > 0$ be given. Then $\delta = \operatorname{dist}(\gamma^*, K) > 0$ because γ^* and K are disjoint compact sets. Assume $[0,1]$ is the domain of γ and let $s, t \in [0,1], z \in K$. Then

$$\begin{aligned}
&\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z} \right| \\
&= \left| \frac{f(\gamma(t))(\gamma(s)-z) - f(\gamma(s))(\gamma(t)-z)}{(\gamma(t)-z)(\gamma(s)-z)} \right| \\
&= \left| \frac{f(\gamma(t))(\gamma(s)-\gamma(t)) + \gamma(t)(f(\gamma(t))-f(\gamma(s))) - z(f(\gamma(t))-f(\gamma(s)))}{(\gamma(t)-z)(\gamma(s)-z)} \right| \\
&\le \frac{1}{\delta^2}(|f(\gamma(t))||\gamma(s)-\gamma(t)| + |\gamma(t)||f(\gamma(t))-f(\gamma(s))| + |z||f(\gamma(t))-f(\gamma(s))|).
\end{aligned}$$

Since γ and $f \circ \gamma$ are bounded functions and K is a compact set, there exists $C > 0$ such that for $s, t \in [0,1]$ and $z \in K$, the preceding expression is bounded by

$$\frac{C}{\delta^2}(|\gamma(s)-\gamma(t)| + |f(\gamma(t))-f(\gamma(s))|.$$

Thus by uniform continuity of γ and $f \circ \gamma$ on the interval $[0,1]$, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for $t \in [t_{j-1}, t_j]$ and $z \in K$,

$$\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_j))}{\gamma(t_j)-z} \right| < \epsilon.$$

Define

$$R(z) = \sum_{j=1}^{n} \frac{f(\gamma(t_j))}{\gamma(t_j)-z}(\gamma(t_j)-\gamma(t_{j-1})),\ \ z \neq \gamma(t_j).$$

Then $R(z)$ is a rational function whose poles are included in the set $\{\gamma(t_1), \dots, \gamma(t_n)\}$, in particular, the poles are in $\Omega \setminus K$. Now for all $z \in K$,

$$\begin{aligned}
|2\pi i f(z) - R(z)| &= \left| \int_\gamma \frac{f(w)}{w-z}\, dw - \sum_{j=1}^n \frac{f(\gamma(t_j))}{\gamma(t_j)-z}(\gamma(t_j) - \gamma(t_{j-1})) \right| \\
&= \left| \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left(\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_j))}{\gamma(t_j)-z} \right) \gamma'(t)\, dt \right| \\
&\le \epsilon \int_0^1 |\gamma'(t)|\, dt = \epsilon \cdot \text{length of } \gamma.
\end{aligned}$$

Since the length of γ is independent of ϵ, the lemma is proved. ♣

Proof of Lemma 5.2.10

Let H be any component of U such that $V \cap H \neq \emptyset$. We must show that $H \subseteq V$. let $s \in V \cap H$ and let G be that component of V that contains s. It suffices to show that $G = H$. Now $G \subseteq H$ since G is a connected subset of U containing s and H is the union of all subsets with this property. Write

$$H = G \cup (H \setminus G) = G \cup [(\partial G \cap H) \cup (H \setminus \overline{G})].$$

But $\partial G \cap H = \emptyset$, because otherwise the hypothesis $\partial V \cap U = \emptyset$ would be violated. Thus $H = G \cup (H \setminus \overline{G})$, the union of two disjoint open sets. Since H is connected and $G \neq \emptyset$, we have $G = H$ as required. ♣

Proof of Lemma 5.2.11

Suppose first that $\infty \in S$. Then for sufficiently large $|\lambda_0|$, with λ_0 in the unbounded component of $\mathbb{C} \setminus K$, the Taylor series for $(z - \lambda_0)^{-1}$ converges uniformly on K. Thus $(z - \lambda_0)^{-1} \in B(S)$, and it follows that

$$B((S \setminus \{\infty\}) \cup \{\lambda_0\}) \subseteq B(S).$$

(If $f \in B((S \setminus \{\infty\}) \cup \{\lambda_0\})$ and R is a rational function with poles in $(S \setminus \{\infty\}) \cup \{\lambda_0\}$ that approximates f, write $R = R_1 + R_2$ where all the poles (if any) of R_1 lie in $S \setminus \{\infty\}$ and the pole (if any) of R_0 is at λ_0. But R_0 can be approximated by a polynomial P_0, hence $R_1 + P_0$ approximates f and has its poles in S, so $f \in B(S)$.) Thus it is sufficient to establish the lemma for sets $S \subseteq \mathbb{C}$. We are going to apply Lemma 5.2.10. Put $U = \mathbb{C} \setminus K$ and define

$$V = \{\lambda \in U : (z - \lambda)^{-1} \in B(S)\}.$$

Recall that by hypothesis, $S \subseteq U$ and hence $S \subseteq V \subseteq U$. To apply (5.2.10) we must first show that V is open. Suppose $\lambda \in V$ and μ is such that $0 < |\lambda - \mu| < \text{dist}(\lambda, K)$. Then $\mu \in \mathbb{C} \setminus K$ and for all $z \in K$,

$$\frac{1}{z - \mu} = \frac{1}{(z - \lambda)[1 - \frac{\mu - \lambda}{z - \lambda}]}.$$

Since $(z-\lambda)^{-1} \in B(S)$, it follows from the remarks preceding the proof of Lemma 5.2.9 that $(z-\mu)^{-1} \in B(S)$. Thus $\mu \in V$, proving that V is open. Next we'll show that $\partial V \cap U = \emptyset$. Let $w \in \partial V$ and let $\{\lambda_n\}$ be a sequence in V such that $\lambda_n \to w$. Then as we noted earlier in this proof, $|\lambda_n - w| < \text{dist}(\lambda_n, K)$ implies $w \in V$, so it must be the case that $|\lambda_n - w| \geq \text{dist}(\lambda_n, K)$ for all n. Since $|\lambda_n - w| \to 0$, the distance from w to K must be 0, so $w \in K$. Thus $w \notin U$, proving that $\partial V \cap U = \emptyset$, as desired. Consequently, V and U satisfy the hypotheses of (5.2.10).

Let H be any component of U. By definition of S, there exists $s \in S$ such that $s \in H$. Now $s \in V$ because $S \subseteq V$. Thus $H \cap V \neq \emptyset$, and Lemma 5.2.10 implies that $H \subseteq V$. We have shown that every component of U is a subset of V, and consequently $U \subseteq V$. Since $V \subseteq U$, we conclude that $U = V$. ♣

5.2.12 Remarks

Theorem 5.2.8 is often referred to as Runge's theorem for compact sets. Other versions of Runge's theorem appear as Problems 6(a) and 6(b).

We conclude this section by collecting a long list of conditions, all equivalent to simple connectedness.

5.2.13 Theorem

If Ω is an open subset of $\mathbb{C}$, the following are equivalent.

(a) $\hat{\mathbb{C}} \setminus \Omega$ is connected.

(b) $n(\gamma, z) = 0$ for each closed path (or cycle) γ in Ω and each point $z \in \mathbb{C} \setminus \Omega$.

(c) $\int_\gamma f(z)\,dz = 0$ for each $f \in A(\Omega)$ and each closed path γ in Ω.

(d) $n(\gamma, z) = 0$ for each closed curve γ in Ω and each $z \in \mathbb{C} \setminus \Omega$.

(e) Every analytic function on Ω has a primitive.

(f) Every zero-free analytic function on Ω has an analytic logarithm.

(g) Every zero-free analytic function on Ω has an analytic n-th root for $n = 1, 2, 3, \ldots$.

(h) Every zero-free analytic function on Ω has an analytic square root.

(i) Ω is homotopically simply connected.

(j) Each closed path in Ω is homotopic to a point.

(k) If Ω is connected and $\Omega \neq \mathbb{C}$, then Ω is conformally equivalent to D.

(l) If Ω is connected, then Ω is homeomorphic to D.

(m) Every harmonic function on Ω has a harmonic conjugate.

(n) Every analytic function on Ω can be uniformly approximated on compact sets by polynomials.

Proof. See (3.4.6), (5.2.4), (5.2.6), (5.2.7), and Problem 6(b) in this section. ♣

Problems

1. Prove (5.2.3e).
2. Show that $h(z) = z/(1 + |z|)$ is a homeomorphism of $\mathbb{C}$ onto D.
3. Let $\gamma : [a, b] \to \Omega$ be a closed curve in a convex set Ω. Prove that

$$H(t, s) = s\gamma(a) + (1 - s)\gamma(t),\ t \in [a, b],\ s \in [0, 1]$$

is an Ω-homotopy of γ to the point $\gamma(a)$.
4. Show directly, using the techniques of Problem 3, that a starlike open set is homotopically simply connected.
5. This problem is in preparation for other versions of Runge's theorem that appear in Problem 6. Let Ω be an open subset of $\mathbb{C}$, and let $\{K_n\}$ be as in (5.1.1). Show that in addition to the properties (1), (2) and (3) listed in (5.1.1), the sequence $\{K_n\}$ has an additional property:
(4) Each component of $\hat{\mathbb{C}} \setminus K_n$ contains a component of $\hat{\mathbb{C}} \setminus \Omega$.
6. Prove the following versions of Runge's theorem:
(a) Let Ω be an open set and let S be a set containing at least one point in each component of $\hat{\mathbb{C}} \setminus \Omega$. Show that if $f \in A(\Omega)$, then there is a sequence $\{R_n\}$ of rational functions with poles in S such that $R_n \to f$ uniformly on compact subsets of Ω.
(b) Let Ω be an open subset of $\mathbb{C}$. Show that Ω is simply connected if and only if for each $f \in A(\Omega)$, there is a sequence $\{P_n\}$ of polynomials converging to f uniformly on compact subsets of Ω.

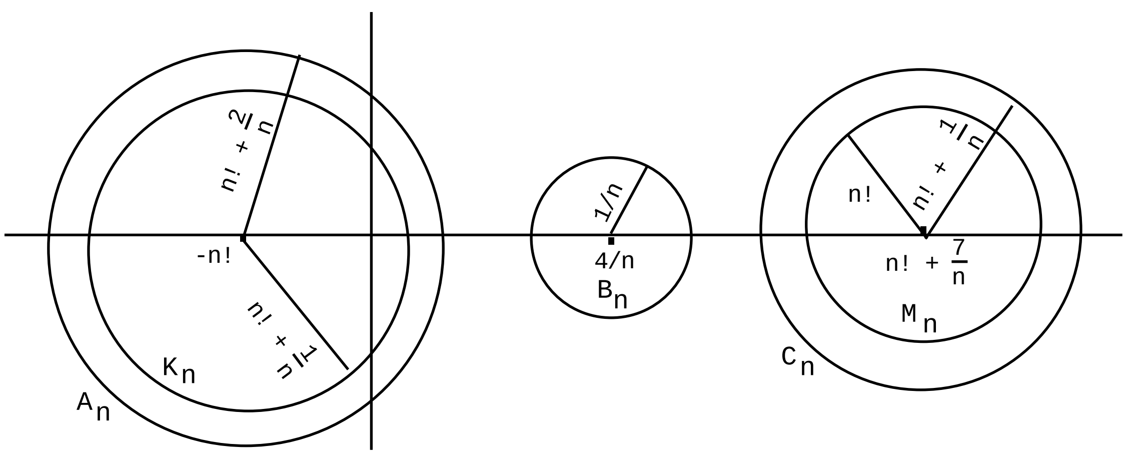

Figure 5.2.2

7. Define sequences of sets as follows:

$$A_n = \{z : |z + n!| < n! + \frac{2}{n}\}, \quad B_n = \{z : |z - \frac{4}{n}| < \frac{1}{n}\},$$

$$C_n = \{z : |z - (n! + \frac{7}{n})| < n! + \frac{1}{n}\}, \quad K_n = \{z : |z + n!| \le n! + \frac{1}{n}\},$$

$$L_n = \{\frac{4}{n}\}, \quad M_n = \{z : |z - (n! + \frac{7}{n})| \le n!\}$$

(see Figure 5.2.2). Define

$$f_n(z) = \begin{cases} 0, & z \in A_n \\ 1, & z \in B_n \\ 0, & z \in C_n \end{cases} \quad \text{and} \quad g_n(z) = \begin{cases} 0, & z \in A_n \\ 1, & z \in C_n \end{cases}$$

(a) By approximating f_n by polynomials (see Problem 6), exhibit a sequence of polynomials converging pointwise to 0 on all of $\mathbb{C}$, but not uniformly on compact subsets.
(b) By approximating g_n by polynomials, exhibit a sequence of polynomials converging pointwise on all of $\mathbb{C}$ to a discontinuous limit.

5.3 Extending Conformal Maps to the Boundary

Let Ω be a proper simply connected region in $\mathbb{C}$. By the Riemann mapping theorem, there is a one-to-one analytic map of Ω onto the open unit disk D. In this section we will consider the problem of extending f to a homeomorphism of the closure $\overline{\Omega}$ of Ω onto $\overline{D}$. Note that if f is extended, then $\overline{\Omega}$ must be compact. Thus we assume in addition that Ω is bounded. We will see that $\partial\Omega$ plays an essential role in determining whether such an extension is possible. We begin with some results of a purely topological nature.

5.3.1 Theorem

Suppose Ω is an open subset of $\mathbb{C}$ and f is a homeomorphism of Ω onto $f(\Omega) = V$. Then a sequence $\{z_n\}$ in Ω has no limit point in Ω iff the sequence $\{f(z_n)\}$ has no limit point in V.

Proof. Assume $\{z_n\}$ has a limit point $z \in \Omega$. There is a subsequence $\{z_{n_j}\}$ in Ω such that $z_{n_j} \to z$. By continuity, $f(z_{n_j}) \to f(z)$, and therefore the sequence $\{f(z_n)\}$ has a limit point in V. The converse is proved by applying the preceding argument to f^{-1}. ♣

5.3.2 Corollary

Suppose f is a conformal equivalence of Ω onto D. If $\{z_n\}$ is a sequence in Ω such that $z_n \to \beta \in \partial\Omega$, then $|f(z_n)| \to 1$.

Proof. Since $\{z_n\}$ has no limit point in Ω, $\{f(z_n)\}$ has no limit point in D, hence $|f(z_n)| \to 1$. ♣

Let us consider the problem of extending a conformal map f to a single boundary point $\beta \in \partial\Omega$. As the following examples indicate, the relationship of Ω and β plays a crucial role.

5.3.3 Examples

(1) Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and let $\sqrt{z}$ denote the analytic square root of z such that $\sqrt{1} = 1$. Then $\sqrt{z}$ is a one-to-one analytic map of Ω onto the right half plane. The linear fractional transformation $T(z) = (z-1)/(z+1)$ maps the right half plane onto the unit disk D, hence $f(z) = (\sqrt{z}-1)/(\sqrt{z}+1)$ is a conformal equivalence of Ω and D. Now T maps $\operatorname{Re} z = 0$ onto $\partial D \setminus \{1\}$, so if $\{z_n\}$ is a sequence in $\operatorname{Im} z > 0$ that converges to $\beta \in (-\infty, 0)$, then $\{f(z_n)\}$ converges to a point $w \in \partial D$ with $\operatorname{Im} w > 0$. On the other hand, if $\{z_n\}$ lies in $\operatorname{Im} z < 0$ and $z_n \to \beta$, then $\{f(z_n)\}$ converges to a point $w \in \partial D$ with $\operatorname{Im} w < 0$. Thus f does not have a continuous extension to $\Omega \cup \{\beta\}$ for any β on the negative real axis.

(2) To get an example of a *bounded* simply connected region Ω with boundary points to which the mapping functions are not extendible, let

$$\Omega = [(0,1) \times (0,1)] \setminus \{\{1/n\} \times (0, 1/2] : n = 2, 3, \dots\}.$$

Thus Ω is the open unit square with vertical segments of height 1/2 removed at each of the points $1/2, 1/3, \dots$ on the real axis; see Figure 5.3.1. Then $\hat{\mathbb{C}} \setminus \Omega$ is seen to be

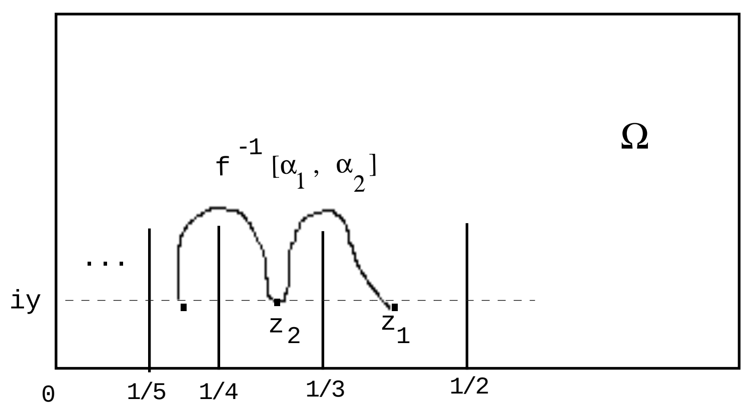

Figure 5.3.1

connected, so that Ω is simply connected. Let $\beta = iy$ where $0 < y < 1/2$, and choose a sequence $\{z_n\}$ in Ω such that $z_n \to iy$ and $\operatorname{Im} z_n = y, n = 1, 2, 3, \dots$. Let f be any conformal map of Ω onto D. Since by (5.3.2), $|f(z_n)| \to 1$, there is a subsequence $\{z_{n_k}\}$ such that $\{f(z_{n_k})\}$ converges to a point $w \in \partial D$. For simplicity assume that $\{f(z_n)\}$ converges to w. Set $\alpha_n = f(z_n)$ and in D, join α_n to α_{n+1} with the straight line segment $[\alpha_n, \alpha_{n+1}], n = 1, 2, 3, \dots$. Then $f^{-1}([\alpha_n, \alpha_{n+1}])$ is a curve in Ω joining z_n to $z_{n+1}, n = 1, 2, 3, \dots$. It follows that every point of $[iy, i/2]$ is a limit point of $\cup_n f^{-1}([\alpha_n, \alpha_{n+1}])$. Hence f^{-1}, in this case, cannot be extended to be continuous at $w \in \partial D$.

As we now show, if $\beta \in \partial\Omega$ is such that sequences of the type $\{z_n\}$ in the previous example are ruled out, then any mapping function can be extended to $\Omega \cup \{\beta\}$.

5.3.4 Definition

A point $\beta \in \partial\Omega$ is called *simple* if to each sequence $\{z_n\}$ in Ω such that $z_n \to \beta$, there corresponds a curve $\gamma : [0,1] \to \Omega \cup \{\beta\}$ and a strictly increasing sequence $\{t_n\}$ in [0,1) such that $t_n \to 1, \gamma(t_n) = z_n$, and $\gamma(t) \in \Omega$ for $0 \le t < 1$.

Thus a boundary point is simple iff for any sequence $\{z_n\}$ that converges to β, there is a curve γ in Ω that contains the points z_n and terminates at β. In Examples 1 and 2 of (5.3.3), none of the boundary points β with $\beta \in (-\infty, 0)$ or $\beta \in (0, i/2)$ is simple.

5.3.5 Theorem

Let Ω be a bounded simply connected region in $\mathbb{C}$, and let $\beta \in \partial\Omega$ be simple. If f is a conformal equivalence of Ω onto D, then f has a continuous extension to $\Omega \cup \{\beta\}$.

To prove this theorem, we will need a lemma due to Lindelöf.

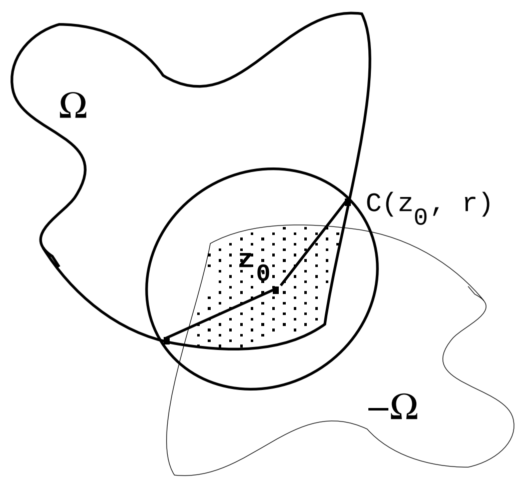

Figure 5.3.2

5.3.6 Lemma

Suppose Ω is an open set in $\mathbb{C}$, $z_0 \in \Omega$, and the circle $C(z_0, r)$ has an arc lying in the complement of $\overline{\Omega}$ which subtends an angle greater than π at z_0 (see Figure 5.3.2). Let g be any continuous function on $\overline{\Omega}$ which is analytic on Ω. If $|g(z)| \le M$ for all $z \in \overline{\Omega}$ while $|g(z)| \le \epsilon$ for all $z \in D(z_0, r) \cap \partial\Omega$, then $|g(z_0)| \le \sqrt{\epsilon M}$.

Proof. Assume without loss of generality that $z_0 = 0$. Put $U = \Omega \cap (-\Omega) \cap D(0, r)$. (This is the shaded region in Figure 5.3.2.) Define h on $\overline{U}$ by $h(z) = g(z)g(-z)$. We claim first that $\overline{U} \subseteq \overline{\Omega} \cap (-\overline{\Omega}) \cap D(0, r)$. For by general properties of the closure operation, $\overline{U} \subseteq \overline{\Omega} \cap (-\overline{\Omega}) \cap \overline{D}(0, r)$. Thus it is enough to show that if $z \in \partial\overline{D}(0, r)$, that is, $|z| = r$, then $z \notin \overline{\Omega}$ or $z \notin (-\overline{\Omega})$. But this is a consequence of our assumption that $C(0, r)$ has an arc lying in the complement of $\overline{\Omega}$ that subtends an angle greater than π at $z_0 = 0$, from which it follows that the entire circle $C(0, r)$ lies in the complement of $\overline{\Omega} \cap (-\overline{\Omega})$. Consequently, we conclude that if $z \in \partial U$, then $z \in \partial\Omega \cap D(0, r)$ or $z \in \partial(-\Omega) \cap D(0, r)$. Therefore, for all $z \in \partial U$, hence for all $z \in \overline{U}$ by the maximum principle, we have

$$|h(z)| = |g(z)||g(-z)| \le \epsilon M.$$

In particular, $|h(0)| = |g(0)|^2 \le \epsilon M$, and the lemma is proved. ♣

We now proceed to prove Theorem 5.3.5. Assume the statement of the theorem is false. This implies that there is a sequence $\{z_n\}$ in Ω converging to β, and distinct complex numbers w_1 and w_2 of modulus 1, such that $f(z_{2j-1}) \to w_1$ while $f(z_{2j}) \to w_2$. (Proof: There is a sequence $\{z_n\}$ in Ω such that $z_n \to \beta$ while $\{f(z_n)\}$ does not converge. But $\{f(z_n)\}$ is bounded, hence it has at least two convergent subsequences with different limits w_1, w_2 and with $|w_1| = |w_2| = 1$.) Let p be the midpoint of the positively oriented arc of ∂D from w_1 to w_2. Choose points a and b, interior to this arc, equidistant from p and close enough to p for Figure 5.3.3 to obtain. Let γ and $\{t_n\}$ be as in the definition

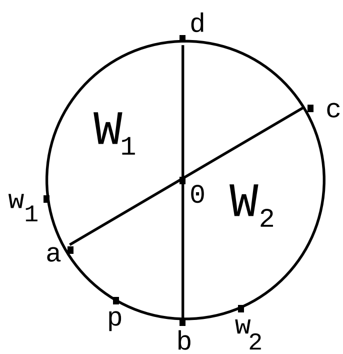

Figure 5.3.3

of simple boundary point. No loss of generality results if we assume that $f(z_{2j-1}) \in W_1$ and $f(z_{2j}) \in W_2$ for all j, and that $|f(\gamma(t))| > 1/2$ for all t. Since $f(\gamma(t_{2j-1})) \in W_1$ and $f(\gamma(t_{2j})) \in W_2$ for each j, there exist x_j and y_j with $t_{2j-1} < x_j < y_j < t_{2j}$ such that one of the following holds:

(1) $f(\gamma(x_j)) \in (0, a)$, $f(\gamma(y_j)) \in (0, b)$, and $f(\gamma(t))$ is in the open sector $a0ba$ for all t such that $x_j < t < y_j$, or

(2) $f(\gamma(x_j)) \in (0, d)$, $f(\gamma(y_j)) \in (0, c)$, and $f(\gamma(t))$ is in the open sector $d0cd$ for all t such that $x_j < t < y_j$.

See Figure 5.3.4 for this and details following. Thus (1) holds for infinitely many j or (2) holds for infinitely many j. Assume that the former is the case, and let J be the set of all j such that (1) is true. For $j \in J$ define γ_j on $[0,1]$ by

$$\gamma_j(t) = \begin{cases} \frac{t}{x_j} f(\gamma(x_j)), & 0 \le t \le x_j, \\ f(\gamma(t)), & x_j \le t \le y_j, \\ \frac{1-t}{1-y_j} f(\gamma(y_j)), & y_j \le t \le 1. \end{cases}$$

Thus γ_j is the closed path whose trajectory γ_j^* consists of

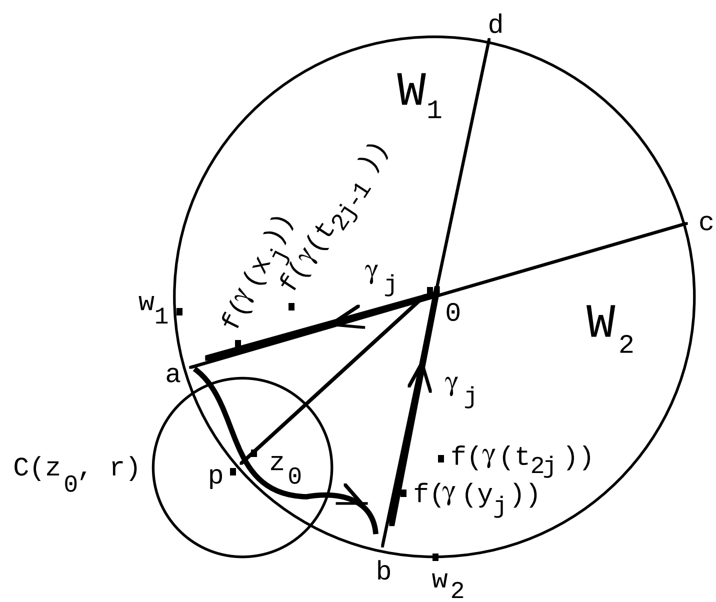

Figure 5.3.4

$$[0, f(\gamma(x_j))] \cup \{f(\gamma(t) : x_j \le t \le y_j\} \cup [f(\gamma(y_j)), 0].$$

Let Ω_j be that component of $\mathbb{C} \setminus \gamma_j^*$ such that $\frac{1}{2}p \in \Omega_j$. Then $\partial\Omega_j \subseteq \gamma_j^*$. Furthermore, $\Omega_j \subseteq D$, for if we compute the index $n(\gamma_j, \frac{1}{2}p)$, we get 1 because $|\gamma_j(t)| > \frac{1}{2}$ for $x_j \le t \le y_j$, while the index of any point in $\mathbb{C} \setminus D$ is 0. Let r be a positive number with $r < \frac{1}{2}|a-b|$ and choose a point z_0 on the open radius $(0,p)$ so close to p that the circle $C(0,r)$ meets the complement of $\overline{D}$ in an arc of length greater than πr. For sufficiently large $j \in J$, $|f(\gamma(t))| > |z_0|$ for all $t \in [t_{2j-1}, t_{2j}]$; so for these j we have $z_0 \in \Omega_j$. Further, if $z \in \partial\Omega_j \cap D(z_0, r)$, then $z \in \{f(\gamma(t)) : t_{2j-1} \le t \le t_{2j}\}$ and hence $f^{-1}(z) \in \gamma([t_{2j-1}, t_{2j}]\}$. Define

$$\epsilon_j = \sup\{|f^{-1}(z) - \beta| : z \in \partial\Omega_j \cap D(z_0, r)\} \le \sup\{|\gamma(t) - \beta| : t \in [t_{2j-1}, t_{2j}]\}$$

and

$$M = \sup\{|f^{-1}(z) - \beta| : z \in D\}.$$

Since $M \geq \sup\{|f^{-1}(z) - \beta| : z \in \overline{\Omega}_j\}$, Lemma 5.3.6 implies that $|f^{-1}(z_0) - \beta| \leq \sqrt{\epsilon_j M}$. Since ϵ_j can be made as small as we please by taking $j \in J$ sufficiently large, we have $f^{-1}(z_0) = \beta$. This is a contradiction since $f^{-1}(z_0) \in \Omega$, and the proof is complete. ♣

We next show that if β_1 and β_2 are simple boundary points and $\beta_1 \neq \beta_2$, then any continuous extension f to $\Omega \cup \{\beta_1, \beta_2\}$ that results from the previous theorem is one-to-one, that is, $f(\beta_1) \neq f(\beta_2)$. The proof requires a lemma that expresses the area of the image of a region under a conformal map as an integral. (Recall that a one-to-one analytic function is conformal.)

5.3.7 Lemma

Let g be a conformal map of an open set Ω. Then the area (Jordan content) of $g(\Omega)$ is $\int\int_\Omega |g'|^2\,dx\,dy$.

Proof. Let $g = u + iv$ and view g as a transformation from $\Omega \subseteq \mathbb{R}^2$ into $\mathbb{R}^2$. Since g is analytic, u and v have continuous partial derivatives (of all orders). Also, the Jacobian determinant of the transformation g is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 = |g'|^2$$

by the Cauchy-Riemann equations. Since the area of $g(\Omega)$ is $\int\int_\Omega \frac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$, the statement of the lemma follows. ♣

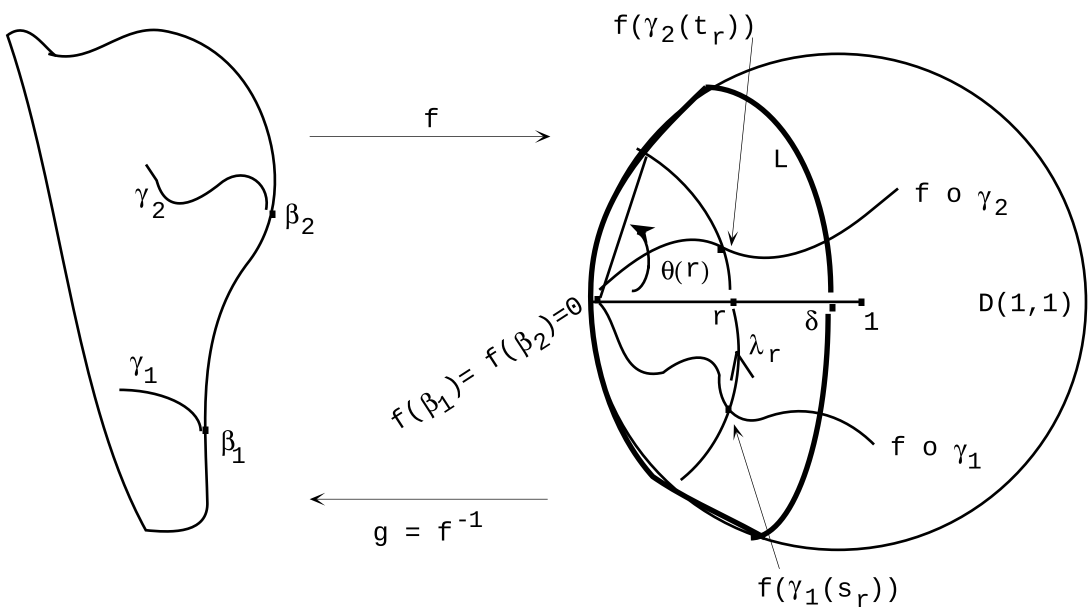

Figure 5.3.5

5.3.8 Theorem

Let Ω be a bounded, simply connected region and f a conformal map of Ω onto D. If β_1 and β_2 are distinct simple boundary points of Ω and f is extended continuously to $\Omega \cup \{\beta_1, \beta_2\}$, then $f(\beta_1) \neq f(\beta_2)$.

Proof. Assume that β_1 and β_2 are simple boundary points of Ω and $f(\beta_1) = f(\beta_2)$. We will show that $\beta_1 = \beta_2$. It will simplify the notation but result in no loss of generality if we replace D by $D(1,1)$ and assume that $f(\beta_1) = f(\beta_2) = 0$.

Since β_1 and β_2 are simple boundary points, for $j = 1, 2$ there are curves γ_j in $\Omega \cup \{\beta_j\}$ such that $\gamma_j([0,1)) \subseteq \Omega$ and $\gamma_j(1) = \beta_j$. Put $g = f^{-1}$. By continuity, there exists $\tau < 1$ such that $\tau < s, t < 1$ implies

$$|\gamma_2(t) - \gamma_1(s)| \geq \frac{1}{2}|\beta_2 - \beta_1| \tag{1}$$

and there exists δ, $0 < \delta < 1$, such that for $t \leq \tau$ we have $f(\gamma_j(t)) \notin \overline{D}(0, \delta), j = 1, 2$. Also, for each r such that $0 < r \leq \delta$, we can choose s_r and $t_r > \tau$ such that $f(\gamma_1(s_r))$ and $f(\gamma_2(t_r))$ meet the circle $C(0, r)$; see Figure 5.3.5. Let $\theta(r)$ be the principal value of the argument of the point of intersection in the upper half plane of $C(0, r)$ and $C(1, 1)$. Now $g(f(\gamma_2(t_r))) - g(f(\gamma_1(s_r)))$ is the integral of g' along the arc λ_r of $C(0, r)$ from $f(\gamma_1(s_r))$ to $f(\gamma_2(t_r))$. It follows from this and (1) that

$$\begin{aligned}
\frac{1}{2}|\beta_2 - \beta_1| &\leq |\gamma_2(t_r) - \gamma_1(s_r)| \\
&= |g(f(\gamma_2(t_r))) - g(f(\gamma_1(s_r)))| \\
&= |\int_{\lambda_r} g'(z)\, dz| \\
&\leq \int_{-\theta(r)}^{\theta(r)} |g'(re^{i\theta})| r\, d\theta. \qquad (2)
\end{aligned}$$

(Note: The function $\theta \to |g'(re^{i\theta})|$ is positive and continuous on the open interval $(-\theta(r), \theta(r))$, but is not necessarily bounded. Thus the integral in (2) may need to be treated as an improper Riemann integral. In any case (2) remains correct and the calculations that follow are also seen to be valid.)

Squaring in (2) and applying the Cauchy-Schwarz inequality for integrals we get

$$\frac{1}{4}|\beta_2 - \beta_1|^2 \leq 2\theta(r) r^2 \int_{-\theta(r)}^{\theta(r)} |g'(re^{i\theta})|^2\, d\theta.$$

(The factor $2\theta(r)$ comes from integrating $1^2 d\theta$ from $-\theta(r)$ to $\theta(r)$.) Since $\theta(r) \leq \pi/2$, we have

$$\frac{|\beta_2 - \beta_1|^2}{4\pi r} \leq r \int_{-\theta(r)}^{\theta(r)} |g'(re^{i\theta})|^2\, d\theta. \tag{3}$$

Now integrate the right hand side of (3) with respect to r from $r = 0$ to $r = \delta$. We obtain

$$\int_0^\delta \int_{-\theta(r)}^{\theta(r)} |g'(re^{i\theta})|^2 r\, d\theta\, dr \leq \int\int_L |g'(x + iy)|^2\, dx\, dy$$

where L is the lens-shaped open set whose boundary is formed by arcs of $C(0,\delta)$ and $C(1,1)$; see Figure 5.3.5. By (5.3.7), $\int\int_L |g'|^2\,dx\,dy$ is the area (or Jordan content) of $g(L)$. Since $g(L)\subseteq\Omega$ and Ω is bounded, $g(L)$ has finite area. But the integral from 0 to δ of the left hand side of (3) is $+\infty$ unless $\beta_1=\beta_2$. Thus $f(\beta_1)=f(\beta_2)$ implies that $\beta_1=\beta_2$. ♣

We can now prove that $f:\Omega\to D$ extends to a homeomorphism of $\overline{\Omega}$ and $\overline{D}$ if *every* boundary point of Ω is simple.

5.3.9 Theorem

Suppose Ω is a bounded, simply connected region with the property that every boundary point of Ω is simple. If $f:\Omega\to D$ is a conformal equivalence, then f extends to a homeomorphism of $\overline{\Omega}$ onto $\overline{D}$.

Proof. By Theorem 5.3.5, for each $\beta\in\partial\Omega$ we can extend f to $\Omega\cup\{\beta\}$ so that f is continuous on $\Omega\cup\{\beta\}$. . Assume this has been done. Thus (the extension of) f is a map of $\overline{\Omega}$ into $\overline{D}$, and Theorem 5.3.8 implies that f is one-to-one. Furthermore, f is continuous at each point $\beta\in\partial\Omega$, for if $\{z_n\}$ is any sequence in $\overline{\Omega}$ such that $z_n\to\beta$ then for each n there exists $w_n\in\Omega$ with $|z_n-w_n|<1/n$ and also $|f(z_n)-f(w_n)|<1/n$, by Theorem 5.3.5. But again by (5.3.5), $f(w_n)\to f(\beta)$ because $w_n\to\beta$ and $w_n\in\Omega$. Hence $f(z_n)\to\beta$, proving that f is continuous on $\overline{\Omega}$. Now $D\subseteq f(\overline{\Omega})\subseteq\overline{D}$, and since $f(\overline{\Omega})$ is compact, hence closed, $f(\overline{\Omega})=\overline{D}$. Consequently, f is a one-to-one continuous map of $\overline{\Omega}$ onto $\overline{D}$, from which it follows that f^{-1} is also continuous. ♣

Theorem 5.3.9 has various applications, and we will look at a few of these in the sequel.

In the proof of (5.3.8), we used the fact that for open subsets $L\subseteq D$,

$$\int\int_L |g'|^2\,dx\,dy \tag{1}$$

is precisely the area of $g(L)$, where g is a one-to-one analytic function on D. Suppose that $g(z)=\sum_{n=0}^\infty a_nz^n,\ z\in D$. Then $g'(z)=\sum_{n=1}^\infty na_nz^{n-1}$. Now in polar coordinates the integral in (1), with L replaced by D, is given by

$$\int\int_D |g'(re^{i\theta})|^2 r\,dr\,d\theta=\int_0^1 r\,dr\int_{-\pi}^{\pi}|g'(re^{i\theta})|^2\,d\theta.$$

But for $0\le r<1$,

$$\begin{aligned}|g'(re^{i\theta})|^2&=g'(re^{i\theta})\overline{g'(re^{i\theta})}\\&=\sum_{n=1}^\infty na_nr^{n-1}e^{i(n-1)\theta}\sum_{m=1}^\infty m\overline{a_m}r^{m-1}e^{-i(m-1)\theta}\\&=\sum_{j=1}^\infty\sum_{m+n=j}nma_n\overline{a_m}r^{m+n-2}e^{i(n-m)\theta}.\end{aligned}\tag{2}$$

Since

$$\int_{-\pi}^{\pi}e^{i(n-m)\theta}\,d\theta=\begin{cases}2\pi, & n=m\\0, & n\neq m\end{cases}$$

and the series in (2) converges uniformly in θ, we can integrate term by term to get

$$\int_{-\pi}^{\pi} |g'(re^{i\theta})|^2 \, d\theta = 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-2}.$$

Multiplying by r and integrating with respect to r, we have

$$\int_0^1 r \, dr \int_{-\pi}^{\pi} |g'(re^{i\theta})|^2 \, d\theta = \lim_{\rho \to 1^-} 2\pi \sum_{k=1}^{\infty} \frac{k^2 |a_k|^2 \rho^{2k}}{2k}$$

If this limit, which is the area of $g(D)$, is finite, then

$$\pi \sum_{k=1}^{\infty} k|a_k|^2 < \infty.$$

We have the following result.

5.3.10 Theorem

Suppose $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is one-to-one and analytic on D. If $g(D)$ has finite area, then $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$.

Now we will use the preceding result to study the convergence of the power series for $g(z)$ when $|z| = 1$. Here is a result on uniform convergence.

5.3.11 Theorem

Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be a one-to-one analytic map of D onto a bounded region Ω such that every boundary point of Ω is simple. Then the series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\overline{D}$ to (the extension of) g on $\overline{D}$.

Proof. By the maximum principle, it is sufficient to show that $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly to $g(z)$ for $|z| = 1$; in other words, $\sum_{n=0}^{\infty} a_n e^{in\theta}$ converges uniformly in θ to $g(e^{i\theta})$. So let $\epsilon > 0$ be given. Since g is uniformly continuous on $\overline{D}$, $|g(e^{i\theta}) - g(re^{i\theta})| \to 0$ uniformly in θ as $r \to 1^-$. If m is any positive integer and $0 < r < 1$, we have

$$|g(e^{i\theta}) - \sum_{n=0}^{m} a_n e^{in\theta}| \leq |g(e^{i\theta}) - g(re^{i\theta})| + |g(re^{i\theta}) - \sum_{n=0}^{m} a_n e^{in\theta}|.$$

The first term on the right hand side tends to 0 as $r \to 1^-$, uniformly in θ, so let us consider the second term. If k is any positive integer less than m, then since $g(re^{i\theta}) =$

$\sum_{n=0}^{\infty} a_n r^n e^{in\theta}$, we can write the second term as

$$\begin{aligned}
&|\sum_{n=0}^{k} a_n(r^n-1)e^{in\theta} + \sum_{n=k+1}^{m} a_n(r^n-1)e^{in\theta} + \sum_{n=m+1}^{\infty} a_n r^n e^{in\theta}| \\
&\leq \sum_{n=0}^{k}(1-r^n)|a_n| + \sum_{n=k+1}^{m}(1-r^n)|a_n| + \sum_{n=m+1}^{\infty}|a_n|r^n \\
&\leq \sum_{n=0}^{k} n(1-r)|a_n| + \sum_{n=k+1}^{m} n(1-r)|a_n| + \sum_{n=m+1}^{\infty}|a_n|r^n
\end{aligned}$$

(since $\frac{1-r^n}{1-r} = 1 + r + \cdots + r^{n-1} < n$). We continue the bounding process by observing that in the first of the three above terms, we have $n \leq k$. In the second term, we write $n(1-r)|a_n| = [\sqrt{n}(1-r)][\sqrt{n}|a_n|]$ and apply Schwarz's inequality. In the third term, we write $|a_n|r^n = [\sqrt{n}|a_n|][r^n/\sqrt{n}]$ and again apply Schwarz's inequality. Our bound becomes

$$\begin{aligned}
&k(1-r)\sum_{n=0}^{k}|a_n| + \left\{\sum_{n=k+1}^{m} n(1-r)^2\right\}^{1/2}\left\{\sum_{n=k+1}^{m} n|a_n|^2\right\}^{1/2} \\
&+ \left\{\sum_{n=m+1}^{\infty} n|a_n|^2\right\}^{1/2}\left\{\sum_{n=m+1}^{\infty}\frac{r^{2n}}{n}\right\}^{1/2}. \qquad (1)
\end{aligned}$$

Since $\sum_{n=0}^{\infty} n|a_n|^2$ is convergent, there exists $k > 0$ such that $\{\sum_{n=k+1}^{\infty} n|a_n|^2\}^{1/2} < \epsilon/3$. Fix such a k. For $m > k$ put $r_m = (m-1)/m$. Now the first term in (1) is less than $\epsilon/3$ for m sufficiently large and $r = r_m$. Also, since $\{\sum_{n=k+1}^{m} n(1-r_m)^2\}^{1/2} = \{\sum_{n=k+1}^{m} n(1/m)^2\}^{1/2} = (1/m)\{\sum_{n=k+1}^{m} n\}^{1/2} < (1/m)\{m(m+1)/2\}^{1/2} < 1$, the middle term in (1) is also less than $\epsilon/3$. Finally, consider

$$\left\{\sum_{n=m+1}^{\infty}\frac{r_m^{2n}}{n}\right\}^{1/2} \leq \left\{\frac{r_m^{2(m+1)}}{m+1}\sum_{n=0}^{\infty} r_m^{2n}\right\}^{1/2} \leq \left\{\frac{1}{m+1}\sum_{n=0}^{\infty} r_m^{n}\right\}^{1/2}$$

which evaluates to

$$\left\{\frac{1}{m+1}\frac{1}{1-r_m}\right\}^{1/2} = \left\{\frac{m}{m+1}\right\}^{1/2} < 1.$$

Thus the last term in (1) is also less than $\epsilon/3$ for all sufficiently large $m > k$ and $r = r_m = (m-1)/m$. Thus $|g(e^{i\theta}) - \sum_{n=0}^{m} a_n e^{in\theta}| \to 0$ uniformly in θ as $m \to \infty$. ♣

The preceding theorem will be used to produce examples of uniformly convergent power series that are not absolutely convergent. That is, power series that converge uniformly, but to which the Weierstrass M-test does not apply. One additional result will be needed.

5.3.12 Theorem

Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in D$, and $\sum_{n=0}^{\infty} |a_n| < +\infty$. Then for each θ,

$$\int_0^{1^-} |f'(re^{i\theta})|\, dr = \lim_{\rho \to 1^-} \int_0^{\rho} |f'(re^{i\theta})|\, dr < +\infty.$$

Proof. If $0 \le r \le \rho < 1$ then for any θ we have $f'(re^{i\theta}) = \sum_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}$. Thus

$$\int_0^{\rho} |f'(re^{i\theta})|\, dr \le \sum_{n=1}^{\infty} |a_n| \rho^n \le \sum_{n=1}^{\infty} |a_n| < +\infty. \quad \clubsuit$$

5.3.13 Remark

For each θ, $\int_0^1 |f'(re^{i\theta})|\, dr$ is the length of the image under f of the radius $[0, e^{i\theta}]$ of $\overline{D}$. For if $\gamma(r) = re^{i\theta}, 0 \le r \le 1$, then the length of $f \circ \gamma$ is given by

$$\int_0^1 |(f \circ \gamma)'(r)|\, dr = \int_0^1 |f'(re^{i\theta})e^{i\theta}|\, dr = \int_0^1 |f'(re^{i\theta})|\, dr.$$

Thus in geometric terms, the conclusion of (5.3.12) is that f maps every radius of $\overline{D}$ onto an arc of finite length.

We can now give a method for constructing uniformly convergent power series that are not absolutely convergent. Let Ω be the bounded, connected, simply connected region that appears in Figure 5.3.6. Then each boundary point of Ω is simple with the possible exception of 0, and the following argument shows that 0 is also a simple boundary point. Let $\{z_n\}$ be any sequence in Ω such that $z_n \to 0$. For $n = 1, 2, \ldots$ put $t_n = (n-1)/n$. Then for each n there is a polygonal path $\gamma_n : [t_n, t_{n+1}] \to \Omega$ such that $\gamma_n(t_n) = z_n, \gamma_n(t_{n+1}) = z_{n+1}$, and such that for $t_n \le t \le t_{n+1}, \operatorname{Re} \gamma_n(t)$ is between $\operatorname{Re} z_n$ and $\operatorname{Re} z_{n+1}$. If we define $\gamma = \cup \gamma_n$, then γ is a continuous map of $[0, 1)$ into Ω, and $\gamma(t) = \gamma_n(t)$ for $t_n \le t \le t_{n+1}$. Furthermore, $\gamma(t) \to 0$ as $t \to 1^-$. Thus by definition, 0 is simple boundary point of Ω.

Hence by (5.3.9) and the Riemann mapping theorem (5.2.2), there is a homeomorphism f of $\overline{D}$ onto $\overline{\Omega}$ such that f is analytic on D. Write $f(z) = \sum a_n z^n, \ z \in D$. By (5.3.11), this series converges uniformly on $\overline{D}$. Now let $e^{i\theta}$ be that point in ∂D such that $f(e^{i\theta}) = 0$. Since f is a homeomorphism, f maps the radius of $\overline{D}$ that terminates at $e^{i\theta}$ onto an arc in $\Omega \cup \{0\}$ that terminates at 0. Further, the image arc in $\Omega \cup \{0\}$ cannot have finite length. Therefore by (5.3.12) we have $\sum |a_n| = +\infty$.

Additional applications of the results in this section appear in the exercises.

Problems

1. Let Ω be a bounded simply connected region such that every boundary point of Ω is simple. Prove that the Dirichlet problem is solvable for Ω. That is, if u_0 is a real-valued continuous function on $\partial\Omega$, then u_0 has a continuous extension u to $\overline{\Omega}$ such that u is harmonic on Ω.

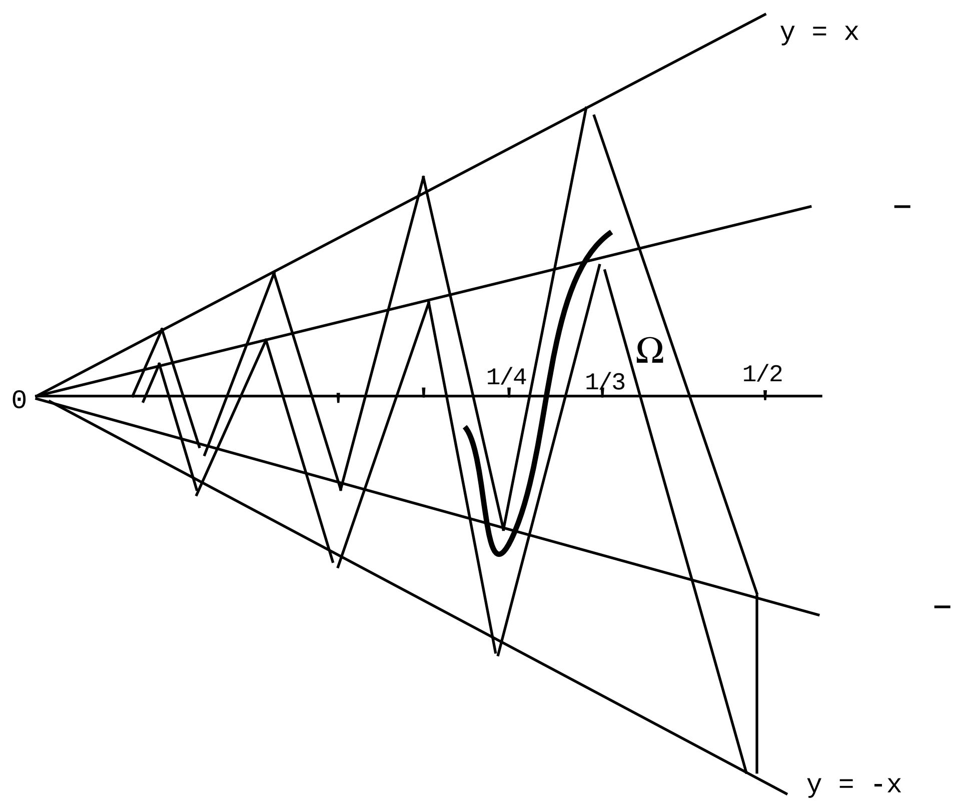

Figure 5.3.6

2. Let $\Omega = \{x + iy : 0 < x < 1 \text{ and } -x^2 < y < x^2\}$. Show that
(a) The identity mapping $z \to z$ has a continuous argument u on $\overline{\Omega}$ (necessarily harmonic on Ω).
(b) There is a homeomorphism f of $\overline{D}$ onto $\overline{\Omega}$ which is analytic on D.
(c) $u \circ f$ is continuous on $\overline{D}$ and harmonic on D.
(d) No harmonic conjugate V for $u \circ f$ can be bounded on D.

3. Let Ω be a bounded, simply connected region such that every boundary point of Ω is simple. Show that every zero-free continuous function f on $\overline{\Omega}$ has a continuous logarithm g. In addition, show that if f is analytic on Ω, then so is g.

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