Chapter 4

Applications Of The Cauchy Theory

This chapter contains several applications of the material developed in Chapter 3. In the first section, we will describe the possible behavior of an analytic function near a singularity of that function.

4.1 Singularities

We will say that f has an *isolated singularity* at z_0 if f is analytic on $D(z_0, r) \setminus \{z_0\}$ for some r. What, if anything, can be said about the behavior of f near z_0? The basic tool needed to answer this question is the *Laurent series*, an expansion of $f(z)$ in powers of $z - z_0$ in which negative as well as positive powers of $z - z_0$ may appear. In fact, the number of negative powers in this expansion is the key to determining how f behaves near z_0.

From now on, the punctured disk $D(z_0, r) \setminus \{z_0\}$ will be denoted by $D'(z_0, r)$. We will need a consequence of Cauchy's integral formula.

4.1.1 Theorem

Let f be analytic on an open set Ω containing the *annulus* $\{z : r_1 \le |z - z_0| \le r_2\}$, $0 < r_1 < r_2 < \infty$, and let γ_1 and γ_2 denote the positively oriented inner and outer boundaries of the annulus. Then for $r_1 < |z - z_0| < r_2$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w - z}\, dw.$$

Proof. Apply Cauchy's integral formula [part (ii) of (3.3.1)] to the cycle $\gamma_2 - \gamma_1$. ♣

4.1.2 Definition

For $0 \le s_1 < s_2 \le +\infty$ and $z_0 \in \mathbb{C}$, we will denote the open annulus $\{z : s_1 < |z-z_0| < s_2\}$ by $A(z_0, s_1, s_2)$.

4.1.3 Laurent Series Representation

If f is analytic on $\Omega = A(z_0, s_1, s_2)$, then there is a unique two-tailed sequence $\{a_n\}_{n=-\infty}^{\infty}$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, z \in \Omega.$$

In fact, if r is such that $s_1 < r < s_2$, then the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw, \;\; n = 0, \pm 1, \pm 2, \dots .$$

Also, the above series converges absolutely on Ω and uniformly on compact subsets of Ω.

Proof. Choose r_1 and r_2 such that $s_1 < r_1 < r_2 < s_2$ and consider the Cauchy type integral

$$\frac{1}{2\pi i} \int_{C(z_0,r_2)} \frac{f(w)}{w - z} \, dw, \;\; z \in D(z_0, r_2).$$

Then proceeding just as we did in the proof of Theorem 2.2.16, we obtain

$$\frac{1}{2\pi i} \int_{C(z_0,r_2)} \frac{f(w)}{w - z} \, dw = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{C(z_0,r_2)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw.$$

The series converges absolutely on $D(z_0, r_2)$, and uniformly on compact subsets of $D(z_0, r)$. Next, consider the Cauchy type integral

$$-\frac{1}{2\pi i} \int_{C(z_0,r_1)} \frac{f(w)}{w - z} \, dw, \;\; |z - z_0| > r_1.$$

This can be written as

$$\frac{1}{2\pi i} \int_{C(z_0,r_1)} \frac{f(w)}{(z - z_0)[1 - \frac{w-z_0}{z-z_0}]} \, dw = \frac{1}{2\pi i} \int_{C(z_0,r_1)} \left[\sum_{n=1}^{\infty} f(w) \frac{(w - z_0)^{n-1}}{(z - z_0)^n} \right] dw.$$

By the Weierstrass M-test, the series converges absolutely and uniformly for $w \in C(z_0, r_1)$. Consequently, we may integrate term by term to obtain the series

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \text{ where } b_n = \frac{1}{2\pi i} \int_{C(z_0,r_1)} \frac{f(w)}{(w - z_0)^{-n+1}} \, dw.$$

This is a power series in $1/(z-z_0)$, and it converges for $|z-z_0| > r_1$, and hence uniformly on sets of the form $\{z : |z-z_0| \geq 1/\rho\}$ where $(1/\rho) > r_1$. It follows that the convergence is uniform on compact (indeed on closed) subsets of $\{z : z-z_0| > r_1$.

The existence part of the theorem now follows from (4.1.1) and the above computations, if we note two facts. First, if $s_1 < r < s_2$ and $k = 0, \pm 1, \pm 2, \dots,$

$$\int_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{k+1}}\, dw = \int_{C(z_0,r_1)} \frac{f(w)}{(w-z_0)^{k+1}}\, dw = \int_{C(z_0,r_2)} \frac{f(w)}{(w-z_0)^{k+1}}\, dw.$$

Second, any compact subset of $A(z_0, s_1, s_2)$ is contained in $\{z : \rho_1 \leq |z-z_0| \leq \rho_2\}$ for some ρ_1 and ρ_2 with $s_1 < \rho_1 < \rho_2 < s_2$.

We turn now to the question of uniqueness. Let $\{b_n\}$ be a sequence such that $f(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ for $z \in A(z_0, s_1, s_2)$. As in the above argument, this series must converge uniformly on compact subsets of $A(z_0, s_1, s_2)$. Therefore if k is any integer and $s_1 < r < s_2$, then

$$\begin{aligned}\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f(w)}{(w-z_0)^{k+1}}\, dw &= \frac{1}{2\pi i}\int_{C(z_0,r)} \left[\sum_{n=-\infty}^{\infty} b_n(w-z_0)^{n-k-1}\right] dw \\ &= \sum_{n=-\infty}^{\infty} b_n \frac{1}{2\pi i}\int_{C(z_0,r)} (w-z_0)^{n-k-1}\, dw \\ &= b_k,\end{aligned}$$

because

$$\frac{1}{2\pi i}\int_{C(z_0,r)} (w-z_0)^{n-k-1}\, dw = \begin{cases} 1 & \text{if } n-k-1 = -1 \\ 0 & \text{otherwise.}\end{cases}$$

The theorem is completely proved. ♣

We are now in a position to analyze the behavior of f near an isolated singularity. As the preceding discussion shows, if f has an isolated singularity at z_0, then f can be represented uniquely by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

in some deleted neighborhood of z_0.

4.1.4 Definition

Suppose f has an isolated singularity at z_0, and let $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the *Laurent expansion of f about z_0*, that is, the series given in (4.1.3). We say that f has a *removable singularity* at z_0 if $a_n = 0$ for all $n < 0$; f has a *pole of order m* at z_0 if m is the largest positive integer such that $a_{-m} \neq 0$. (A pole of order 1 is called a *simple pole.*) Finally, if $a_n \neq 0$ for infinitely many $n < 0$, we say that f has an *essential singularity* at z_0.

The next theorem relates the behavior of $f(z)$ for z near z_0 to the type of singularity that f has at z_0.

4.1.5 Theorem

Suppose that f has an isolated singularity at z_0. Then

(a) f has a removable singularity at z_0 iff $f(z)$ approaches a finite limit as $z \to z_0$ iff $f(z)$ is bounded on the punctured disk $D'(z_0, \delta)$ for some $\delta > 0$.

(b) For a given positive integer m, f has a pole of order m at z_0 iff $(z-z_0)^m f(z)$ approaches a finite nonzero limit as $z \to z_0$. Also, f has a pole at z_0 iff $|f(z)| \to +\infty$ as $z \to z_0$.

(c) f has an essential singularity at z_0 iff $f(z)$ does not approach a finite or infinite limit as $z \to z_0$, that is, $f(z)$ has no limit in $\hat{\mathbb{C}}$ as $z \to z_0$.

Proof. Let $\{a_n\}_{n=-\infty}^{\infty}$ and $r > 0$ be such that $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$ for $0 < |z-z_0| < r$.

(a) If $a_n = 0$ for all $n < 0$, then $\lim_{z\to z_0} f(z) = a_0$. Conversely, if $\lim_{z\to z_0} f(z)$ exists (in $\mathbb{C}$), then f can be defined (or redefined) at z_0 so that f is analytic on $D(z_0, r)$. It follows that there is a sequence $\{b_n\}_{n=0}^{\infty}$ such that $f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ for $z \in D'(z_0, r)$. By uniqueness of the Laurent expansion, we conclude that $a_n = 0$ for $n < 0$ and $a_n = b_n$ for $n \geq 0$. (Thus in this case, the Laurent and Taylor expansions coincide.) The remaining equivalence stated in (a) is left as an exercise (Problem 1).

(b) If f has a pole of order m at z_0, then for $0 < |z - z_0| < r$,

$$f(z) = a_{-m}(z - z_0)^{-m} + \cdots + a_{-1}(z - z_0)^{-1} + \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

where $a_{-m} \neq 0$. Consequently, $(z - z_0)^m f(z) \to a_{-m} \neq 0$ as $z \to z_0$. Conversely, if $\lim_{z\to z_0}(z - z_0)^m f(z) \neq 0$, then by (a) applied to $(z - z_0)^m f(z)$, there is a sequence $\{b_n\}_{n=0}^{\infty}$ such that

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n, z \in D'(z_0, r).$$

Let $z \to z_0$ to obtain $b_0 = \lim_{z\to z_0}(z - z_0)^m f(z) \neq 0$. Thus $f(z)$ can be written as $b_0(z - z_0)^{-m} + b_1(z - z_0)^{-m+1} + \cdots$, showing that f has a pole of order m at z_0. The remaining equivalence in (b) is also left as an exercise (Problem 1).

(c) If $f(z)$ does not have a limit in $\hat{\mathbb{C}}$ as $z \to z_0$, then by (a) and (b), f must have an essential singularity at z_0. Conversely, if f has an essential singularity at z_0, then (a) and (b) again imply that $\lim_{z\to z_0} f(z)$ cannot exist in $\hat{\mathbb{C}}$. ♣

The behavior of a function near an essential singularity is much more pathological even than (4.1.5c) suggests, as the next theorem shows.

4.1.6 Casorati-Weierstrass Theorem

Let f have an isolated essential singularity at z_0. Then for any complex number w, $f(z)$ comes arbitrarily close to w in every deleted neighborhood of z_0. That is, for any $\delta > 0, f(D'(z_0, \delta))$ is a dense subset of $\mathbb{C}$.

Proof. Suppose that for some $\delta > 0, f(D'(z_0, \delta))$ is not dense in $\mathbb{C}$. Then for some $w \in \mathbb{C}$, there exists $\epsilon > 0$ such that $D(w, \epsilon)$ does not meet $f(D'(z_0, \delta))$. For $z \in D'(z_0, \delta)$, put

$g(z) = 1/(f(z)-w)$). Then g is bounded and analytic on $D'(z_0, \delta)$, and hence by (4.1.5a), g has a removable singularity at z_0. Let m be the order of the zero of g at z_0 (set $m = 0$ if $g(z_0) \neq 0$) and write $g(z) = (z-z_0)^m g_1(z)$ where g_1 is analytic on $D(z_0, \delta)$ and $g_1(z_0) \neq 0$ [see (2.4.4)]. Then $(z - z_0)^m g_1(z) = 1/(f(z) - w)$, so as z approaches z_0,

$$(z-z_0)^m f(z) = (z-z_0)^m w + \frac{1}{g_1(z)} \longrightarrow \begin{cases} w + 1/g_1(z_0) & \text{if } m = 0 \\ 1/g_1(z_0) & \text{if } m \neq 0. \end{cases}$$

Thus f has a removable singularity or a pole at z_0. ♣

4.1.7 Remark

The Casorati-Weierstrass theorem is actually a weak version of a much deeper result called the "big Picard theorem", which asserts that if f has an isolated essential singularity at z_0, then for any $\delta > 0, f(D'(z_0, \delta))$ is either the complex plane $\mathbb{C}$ or $\mathbb{C}$ minus one point. We will not prove this result.

The behavior of a complex function f at ∞ may be studied by considering $g(z) = f(1/z)$ for z near 0. This allows us to talk about isolated singularities at ∞. Here are the formal statements.

4.1.8 Definition

We say that f has an *isolated singularity at* ∞ if f is analytic on $\{z : |z| > r\}$ for some r; thus the function $g(z) = f(1/z)$ has an isolated singularity at 0. The type of singularity of f at ∞ is then defined as that of g at 0.

4.1.9 Remark

Liouville's theorem implies that if an entire function f has a removable singularity at ∞, then f is constant. (By (4.1.5a), f is bounded on $\mathbb{C}$.)

Problems

1. Complete the proofs of (a) and (b) of (4.1.5). (Hint for (a): If f is bounded on $D'(z_0, \delta)$, consider $g(z) = (z - z_0)f(z)$.)
2. Classify the singularities of each of the following functions (include the point at ∞). (a) $z/\sin z$ (b) $\exp(1/z)$ (c) $z \cos 1/z$ (d) $1/[z(e^z - 1)]$ (e) cot z
3. Obtain three different Laurent expansions of $(7z - 2)/z(z + 1)(z - 2)$ about $z = -1$. (Use partial fractions.)
4. Obtain all Laurent expansions of $f(z) = z^{-1} + (z - 1)^{-2} + (z + 2)^{-1}$ about $z = 0$, and indicate where each is valid.
5. Find the first few terms in the Laurent expansion of $\frac{1}{z^2(e^z - e^{-z})}$ valid for $0 < |z| < \pi$.
6. Without carrying out the computation in detail, indicate a relatively easy procedure for finding the Laurent expansion of $1/\sin z$ valid for $\pi < |z| < 2\pi$.

7. (Partial Fraction Expansion). Let $R(z) = P(z)/Q(z)$, where P and Q are polynomials and $\deg P < \deg Q$. (If this is not the case, then by long division we may write $P(z)/Q(z) = a_n z^n + \cdots + a_1 z + a_0 + P_1(z)/Q(z)$ where $\deg P_1 < \deg Q$.) Suppose that the zeros of Q are at $z_1, \ldots, z_k$ with respective orders $n_1, \ldots, n_k$. Show that $R(z) = \sum_{j=1}^{k} B_j(z)$, where $B_j(z)$ is of the form

$$\frac{A_{j,0}}{(z-z_j)^{n_j}} + \cdots + \frac{A_{j,(n_j-1)}}{(z-z_j)},$$

with

$$A_{j,r} = \lim_{z \to z_j} \frac{1}{r!} \frac{d^r}{dz^r} [(z-z_j)^{n_j} R(z)]$$

($\frac{d^r}{dz^r} f(z)$ is interpreted as $f(z)$ when $r = 0$).

Apply this result to $R(z) = 1/[z(z+i)^3]$.

8. Find the sum of the series $\sum_{n=0}^{\infty} e^{-n} \sin nz$ (in closed form), and indicate where the series converges. Make an appropriate statement about uniform convergence. (Suggestion: Consider $\sum_{n=0}^{\infty} e^{-n} e^{inz}$ and $\sum_{n=0}^{\infty} e^{-n} e^{-inz}$.)

9. (a) Show that if f is analytic on $\hat{\mathbb{C}}$, then f is constant.
(b) Suppose f is entire and there exists $M > 0$ and $k > 0$ such that $|f(z)| \le M|z|^k$ for $|z|$ sufficiently large. Show that $f(z)$ is a polynomial of degree at most k. (This can also be done without series; see Problem 2.2.13.)
(c) Prove that if f is entire and has a nonessential singularity at ∞, then f is a polynomial.
(d) Prove that if f is meromorphic on $\hat{\mathbb{C}}$ (that is, any singularity of f in $\hat{\mathbb{C}}$ is a pole), then f is a rational function.

10. Classify the singularities of the following functions (include the point at ∞).

$$\text{(a)} \frac{\sin^2 z}{z^4} \quad \text{(b)} \frac{1}{z^2(z+1)} + \sin\frac{1}{z} \quad \text{(c)} \csc z - \frac{k}{z} \quad \text{(d)} \exp(\tan\frac{1}{z}) \quad \text{(e)} \frac{1}{\sin(\sin z)}.$$

11. Suppose that a and b are distinct complex numbers. Show that $(z-a)/(z-b)$ has an analytic logarithm on $\mathbb{C} \setminus [a,b]$, call it g. Then find the possible Laurent expansions of $g(z)$ about $z = 0$.

12. Suppose f is entire and $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Show that f is constant.

13. Assume f has a pole of order m at α, and P is a polynomial of degree n. Prove that the composition $P \circ f$ has a pole of order mn at α.

4.2 Residue Theory

We now develop a technique that often allows for the rapid evaluation of integrals of the form $\int_\gamma f(z)\,dz$, where γ is a closed path (or cycle) in Ω and f is analytic on Ω except possibly for isolated singularities.

4.2.1 Definition

Let f have an isolated singularity at z_0, and let the Laurent expansion of f about z_0 be $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. The *residue* of f at z_0, denoted by $\operatorname{Res}(f, z_0)$, is defined to be a_{-1}.

4.2.2 Remarks

In many cases, the evaluation of an integral can be accomplished by the computation of residues. This is illustrated by (a) and (b) below.

(a) Suppose f has an isolated singularity at z_0, so that f is analytic on $D'(z_0, \rho)$ for some $\rho > 0$. Then for any r such that $0 < r < \rho$, we have

$$\int_{C(z_0,r)} f(w)\,dw = 2\pi i \operatorname{Res}(f, z_0).$$

Proof. Apply the integral formula (4.1.3) for a_{-1}. ♣

(b) More generally, if γ is a closed path or cycle in $D'(z_0, \rho)$ such that $n(\gamma, z_0) = 1$ and $n(\gamma, z) = 0$ for every $z \notin D(z_0, \rho)$, then

$$\int_{\gamma} f(w)\,dw = 2\pi i \operatorname{Res}(f, z_0).$$

Proof. This follows from (3.3.7). ♣

(c) $\operatorname{Res}(f, z_0)$ is that number k such that $f(z) - [k/(z - z_0)]$ has a primitive on $D'(z_0, \rho)$.

Proof. Note that if $0 < r < \rho$, then by (a),

$$\int_{C(z_0,r)} \left(f(w) - \frac{k}{w - z_0} \right) dw = 2\pi i[\operatorname{Res}(f, z_0) - k].$$

Thus if $f(z) - [k/(z - z_0)]$ has a primitive on $D'(z_0, \rho)$, then the integral is zero, and hence $\operatorname{Res}(f, z_0) = k$. Conversely, if $\operatorname{Res}(f, z_0) = k$, then

$$f(z) - \frac{k}{z - z_0} = \sum_{\substack{n=-\infty \\ n\neq -1}}^{\infty} a_n(z - z_0)^n,$$

which has a primitive on $D'(z_0, \rho)$, namely

$$\sum_{\substack{n=-\infty \\ n\neq -1}}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1}. \quad ♣$$

(d) If f has a pole of order m at z_0, then

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}.$$

In particular, if f has a simple pole at z_0, then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} [(z - z_0) f(z)].$$

Proof. Let $\{a_n\}$ be the Laurent coefficient sequence for f about z_0, so that $a_n = 0$ for $n < -m$ and $a_{-m} \neq 0$. Then for $z \in D'(z_0, \rho)$,

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots ,$$

hence

$$\frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)] = (m-1)! a_{-1} + (z - z_0) g(z)$$

where g has a removable singularity at z_0. The result follows. ♣

(e) Suppose f is analytic at z_0 and has a zero of order k at z_0. Then f'/f has a simple pole at z_0 and $\operatorname{Res}(f'/f, z_0) = k$.

Proof. There exists $\rho > 0$ and a zero-free analytic function g on $D(z_0, \rho)$ such that $f(z) = (z - z_0)^k g(z)$ for $z \in D(z_0, \rho)$. Then $f'(z) = k(z - z_0)^{k-1} g(z) + (z - z_0)^k g'(z)$, and hence for $z \in D'(z_0, \rho)$,

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}.$$

Since g'/g is analytic on $D(z_0, \rho)$, it follows that f'/f has a simple pole at z_0 and $\operatorname{Res}(f'/f, z_0) = k$. ♣

We are now ready for the main result of this section.

4.2.3 Residue Theorem

Let f be analytic on $\Omega \setminus S$, where S is a subset of Ω with no limit point in Ω. In other words, f is analytic on Ω except for isolated singularities. Then for any closed path (or cycle) γ in $\Omega \setminus S$ such that γ is Ω-homologous to 0, we have

$$\int_\gamma f(w)\, dw = 2\pi i \sum_{w \in S} n(\gamma, w) \operatorname{Res}(f, w).$$

Proof. Let $S_1 = \{w \in S : n(\gamma, w) \neq 0\}$. Then $S_1 \subseteq Q = \mathbb{C} \setminus \{z \notin \gamma^* : n(\gamma, z) = 0\}$. Since γ is Ω-homologous to 0, Q is a subset of Ω. Furthermore, by (3.2.5), Q is closed and bounded. Since S has no limit point in Ω, S_1 has no limit points at all. Thus S_1 is a finite set. Consequently, the sum that appears in the conclusion of the theorem is the finite sum obtained by summing over S_1. Let $w_1, w_2, \ldots, w_k$ denote the distinct points of S_1. [If S_1 is empty, we are finished by Cauchy's theorem (3.3.1).] Choose positive numbers $r_1, r_2, \ldots, r_k$ so small that

$$D'(w_j, r_j) \subseteq \Omega \setminus S, \quad j = 1, 2, \ldots, k.$$

Let σ be the cycle $\sum_{j=1}^k n(\gamma, w_j)\gamma_j$, where γ_j is the positively oriented boundary of $D(w_j, r_j)$. Then σ is cycle in the open set $\Omega \setminus S$, and you can check that if $z \notin \Omega \setminus S$, then

$n(\gamma, z) = n(\sigma, z)$. Since f is analytic on $\Omega \setminus S$, it follows from (3.3.7) that $\int_\gamma f(w)\,dw = \int_\sigma f(w)\,dw$. But by definition of σ,

$$\int_\gamma f(w)\,dw = \sum_{j=1}^{k} n(\gamma, w_j) \int_{\gamma_j} f(w)\,dw = 2\pi i \sum_{j=1}^{k} n(\gamma, w_j)\,\mathrm{Res}(f, w_j)$$

by part (a) of (4.2.2). ♣

In many applications of the residue theorem, the integral $\int_\gamma f(w)\,dw$ is computed by evaluating the sum $2\pi i \sum_{w \in S} n(\gamma, w)\,\mathrm{Res}(f, w)$. Thus it is important to have methods available for calculating residues. For example, (4.2.2d) is useful when f is a rational function, since the only singularities of f are poles. The residue theorem can also be applied to obtain a basic geometric property of analytic functions called the argument principle. Before discussing the general result, let's look at a simple special case. Suppose z traverses the unit circle once in the positive sense, that is, $z = e^{it}, 0 \le t \le 2\pi$. Then the argument of z^2, namely $2t$, changes by 4π, so that z^2 makes two revolutions around the origin. Thus the number of times that z^2 winds around the origin as z traverses the unit circle is the number of zeros of z^2 inside the circle, counting multiplicity.

The index of a point with respect to a closed path allows us to formalize the notion of the number of times that $f(z)$ winds around the origin as z traverses a path γ. For we are looking at the net number of revolutions about 0 of $f(\gamma(t)), a \le t \le b$, and this, as we have seen, is $n(f \circ \gamma, 0)$. We may now state the general result.

4.2.4 Argument Principle

Let f be analytic on Ω, and assume that f is not identically zero on any component of Ω. If $Z(f) = \{z : f(z) = 0\}$ and γ is any closed path in $\Omega \setminus Z(f)$ such that γ is Ω-homologous to 0,then

$$n(f \circ \gamma, 0) = \sum_{z \in Z(f)} n(\gamma, z) m(f, z)$$

where $m(f, z)$ is the order of the zero of f at z.

Proof. The set $S = Z(f)$ and the function f'/f satisfy the hypothesis of the residue theorem. Applying it, we get

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{z \in Z(f)} n(\gamma, z)\,\mathrm{Res}(f'/f, z).$$

But the left side equals $n(f \circ \gamma, 0)$ by (3.2.3), and the right side equals $\sum_{z \in Z(f)} n(\gamma, z) m(f, z)$ by (4.2.2e). ♣

4.2.5 Remarks

Assuming that for each $z \in Z(f)$, $n(\gamma, z) = 1$ or 0, the argument principle says that the net increase in the argument of $f(z)$ as z traverses γ^* in the positive direction is equal to the number of zeros of f "inside γ" ($n(\gamma, z) = 1$) with multiplicities taken into account.

There is a useful generalization of (4.2.4) to meromorphic functions. A function f is *meromorphic* on Ω if f is analytic on Ω except possibly for poles. That is, there is a subset $S \subseteq \Omega$ with no limit points in Ω such that f is analytic on $\Omega \setminus S$ and f has a pole at each point of S. For example, any rational function is meromorphic on $\mathbb{C}$. More generally, the quotient f/g of two analytic functions is meromorphic, provided g is not identically zero on any component of Ω. (This follows from (2.4.8) and (4.1.5).) Conversely, every meromorphic function is a quotient of two analytic functions. (This is a much deeper result, which will be proved in a later chapter.)

4.2.6 Definition

For f meromorphic on Ω, let $Z(f)$ denote the set of zeros of f, and $P(f)$ the set of poles of f. If $z \in Z(f) \cup P(f)$, let $m(f, z)$ be the order of the zero or pole of f at z.

4.2.7 Argument Principle for Meromorphic Functions

Suppose f is meromorphic on Ω. Then for any closed path (or cycle) γ in $\Omega \setminus (Z(f) \cup P(f))$ such that γ is Ω-homologous to 0, we have

$$n(f \circ \gamma, 0) = \sum_{z \in Z(f)} n(\gamma, z) m(f, z) - \sum_{z \in P(f)} n(\gamma, z) m(f, z).$$

Proof. Take $S = Z(f) \cup P(f)$, and apply the residue theorem to f'/f. The analysis is the same as in the proof of (4.2.4), if we note that if $z_0 \in P(f)$, then $\operatorname{Res}(f'/f, z_0) = -m(f, z_0)$. To see this, write $f(z) = g(z)/(z - z_0)^k$ where $k = m(f, z_0)$ and g is analytic at z_0, with $g(z_0) \neq 0$. Then $f'(z)/f(z) = [g'(z)/g(z)] - [k/(z - z_0)]$. ♣

Under certain conditions, the argument principle allows a very useful comparison of the number of zeros of two functions.

4.2.8 Rouché's Theorem

Suppose f and g are analytic on Ω, with neither f nor g identically zero on any component of Ω. Let γ be a closed path in Ω such that γ is Ω-homologous to 0. If

$$|f(z) + g(z)| < |f(z)| + |g(z)| \text{ for each } \ z \in \gamma^*, \tag{1}$$

then

$$\sum_{z \in Z(f)} n(\gamma, z) m(f, z) = \sum_{z \in Z(g)} n(\gamma, z) m(g, z).$$

Thus f and g have the same number of zeros, counting multiplicity and index.

Proof. The inequality (1) implies that $\gamma^* \subseteq \Omega \setminus [Z(f) \cup P(f)]$, and hence by the argument principle, applied to each of f and g, we obtain

$$n(f \circ \gamma, 0) = \sum_{z \in Z(f)} n(\gamma, z) m(f, z) \quad \text{and} \quad n(g \circ \gamma, 0) = \sum_{z \in Z(g)} n(\gamma, z) m(g, z).$$

But again by (1), $|f(\gamma(t)) + g(\gamma(t))| < |f(\gamma(t))| + |g(\gamma(t))|$ for all t in the domain of the closed path γ. Therefore by the generalized dog-walking theorem (Problem 3.2.4), $n(f \circ \gamma, 0) = n(g \circ \gamma, 0)$. The result follows. ♣

4.2.9 Remarks

Rouché's theorem is true for cycles as well. To see this, suppose that γ is the formal sum $k_1\gamma_1 + \cdots + k_r\gamma_r$. Then just as in the proof of (4.2.8), we have $n(f \circ \gamma, 0) = \sum_{z \in Z(f)} n(\gamma, z)m(f, z)$ and $n(g \circ \gamma, 0) = \sum_{z \in Z(g)} n(\gamma, z)m(g, z)$. But now $|f(z) + g(z)| < |f(z)| + |g(z)|$ for each $z \in \gamma^* = \cup_{j=1}^r \gamma_j^*$ implies, as before, that $n(f \circ \gamma_j, 0) = n(g \circ \gamma_j, 0)$ for $j = 1, \ldots, r$, hence $n(f \circ \gamma, 0) = n(g \circ \gamma, 0)$ and the proof is complete. ♣

In the hypothesis of (4.2.8), (1) is often replaced by

$$|f(z) - g(z)| < |f(z)| \text{ for each } z \in \gamma^*. \qquad (2)$$

But now if (2) holds, then $|f(z) + (-g(z))| < |f(z)| \leq |f(z)| + |-g(z)|$ on γ^*, so f and $-g$, hence f and g, have the same number of zeros.

Problems

1. Let $f(z) = (z - 1)(z - 3 + 4i)/(z + 2i)^2$, and let γ be as shown in Figure 4.2.1. Find $n(f \circ \gamma, 0)$, and interpret the result geometrically.

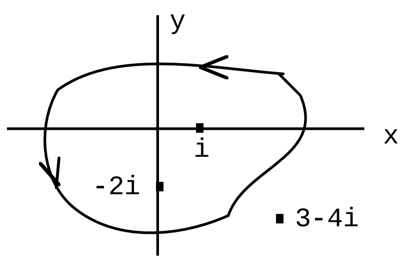

Figure 4.2.1

2. Use the argument principle to find (geometrically) the number of zeros of $z^3 - z^2 + 3z + 5$ in the right half plane.
3. Use Rouché's theorem to prove that any polynomial of degree $n \geq 1$ has exactly n zeros, counting multiplicity.
4. Evaluate the following integrals using residue theory or Cauchy's theorem.
   (a) $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\, dx,\ a > 0$ (b) $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\, dx$

   (c) $\int_{-\infty}^{\infty} \frac{1}{(x^2-4x+5)^2}\, dx$ (d) $\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\, d\theta$ (e) $\int_0^{\infty} \frac{1}{x^4+a^4}\, dx,\ a > 0$

   (f) $\int_0^{\infty} \frac{\cos x}{x^2+1}\, dx$ (g) $\int_0^{2\pi} (\sin\theta)^{2n}\, d\theta$

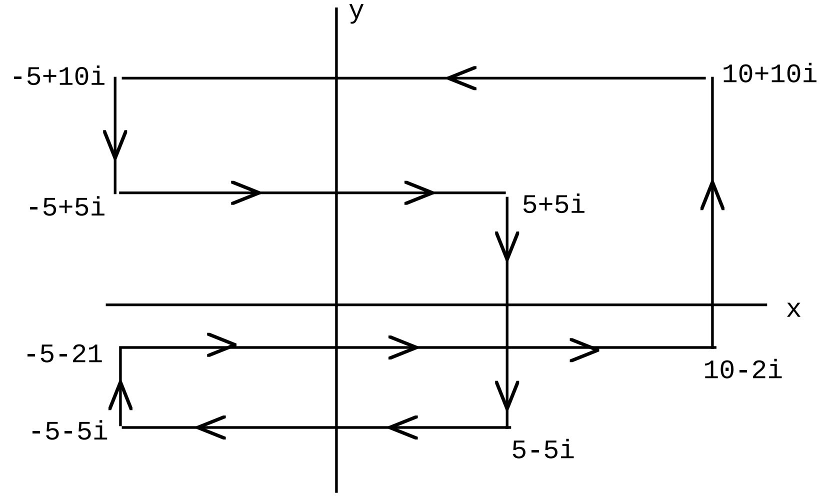

Figure 4.2.2

5. Evaluate $\int_\gamma \frac{\operatorname{Log} z}{1+e^z}\,dz$ along the path γ indicated in Figure 4.2.2.
6. Find the residue at $z=0$ of (a) $\csc^2 z$, (b) $z^{-3}\csc(z^2)$, (c) $z\cos(1/z)$.
7. Find the residue of $\sin(e^z/z)$ at $z=0$. (Leave the answer in the form of an infinite series.)
8. The results of this exercise are necessary for the calculations that are to be done in Problem 9.
(a) Show that for any $r>0$,

$$\int_0^{\pi/2} e^{-r\sin\theta}\,d\theta \le \frac{\pi}{2r}(1-e^{-r}).$$

(Hint: $\sin\theta \ge 2\theta/\pi$ for $0\le\theta\le\pi/2$.)
(b) Suppose f has a simple pole at z_0, and let γ_ϵ be a circular arc with center z_0 and radius ϵ which subtends an angle α at $z_0, 0<\alpha\le 2\pi$ (see Figure 4.2.3). Prove that

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)\,dz = \alpha i\operatorname{Res}(f,z_0).$$

In particular, if the γ_ϵ are semicircular arcs ($\alpha=\pi$), then

$$\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} f(z)\,dz = \pi i\operatorname{Res}(f,z_0) = (1/2)2\pi i\operatorname{Res}(f,z_0).$$

(Hint: $f(z)-[\operatorname{Res}(f,z_0)/(z-z_0)]$ has a removable singularity at z_0.)
9. (a) Show that $\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\pi$ by integrating e^{iz}/z on the closed path $\gamma_{R,\epsilon}$ indicated in Figure 4.2.4.

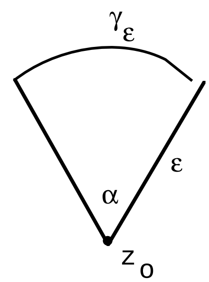

Figure 4.2.3

(b) Show that $\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\pi/2}$. (Integrate e^{iz^2} around the closed path indicated in Figure 4.2.5; assume as known the result that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$.)
(c) Compute $\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx$ by integrating $\frac{\mathrm{Log}(z+i)}{z^2+1}$ around the closed path of Figure 4.2.6.
(d) Derive formulas for $\int_0^{\pi/2} \ln\cos\theta\,d\theta$ and $\int_0^{\pi/2} \ln\sin\theta\,d\theta$ by making the change of variable $x = \tan\theta$ in (c).

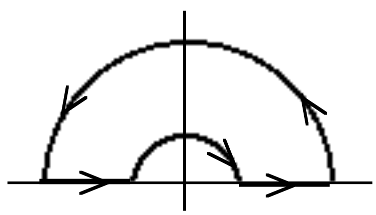

Figure 4.2.4

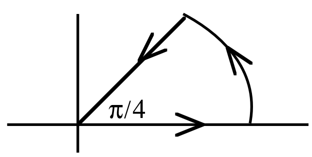

Figure 4.2.5

10. Use Rouché's theorem to show that all the zeros of $z^4 + 6z + 3$ are in $|z| < 2$, and three of them are in $1 < |z| < 2$.
11. Suppose f is analytic on an open set $\Omega \supset \overline{D}(0,1)$, and $|f(z)| < 1$ for $|z| = 1$. Show that for each n, the function $f(z) - z^n$ has exactly n zeros in $D(0,1)$, counting multiplicity. In particular, f has exactly one fixed point in $D(0,1)$.
12. Prove the following version of Rouché's theorem. Suppose K is compact, Ω is an open subset of K, f and g are continuous on K and analytic on Ω, and we have the

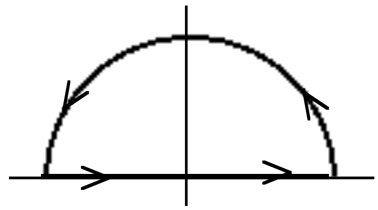

Figure 4.2.6

inequality $|f(z)+g(z)| < |f(z)|+|g(z)|$ for every $z \in K \setminus \Omega$. Show that f and g have the same number of zeros in Ω, that is,

$$\sum_{z \in Z(f)} m(f,z) = \sum_{z \in Z(g)} m(g,z).$$

[Hint: Note that $Z(f) \cup Z(g) \subseteq \{z : |f(z)+g(z)| = |f(z)|+|g(z)|$, and the latter set is a compact subset of Ω. Now apply the hexagon lemma and (4.2.9).]

13. Show that $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{iux}}{1+x^2}\,dx = e^{-|u|}$ for real u.
14. Evaluate the integral of $\exp[\sin(1/z)]$ around the unit circle $|z| = 1$.
15. Suppose f and g are analytic at z_0. Establish the following:
(a) If f has a zero of order k and g has a zero of order $k+1$ at z_0, then f/g has a simple pole at z_0 and

$$\operatorname{Res}(f/g, z_0) = (k+1)f^{(k)}(z_0)/g^{(k+1)}(z_0).$$

(The case $k = 0$ is allowed.)
(b) If $f(z_0) \neq 0$ and g has a zero of order 2 at z_0, then f/g has a pole of order 2 at z_0 and

$$\operatorname{Res}(f/g, z_0) = 2\frac{f'(z_0)}{g''(z_0)} - \frac{2}{3}\frac{f(z_0)g'''(z_0)}{[g''(z_0)]^2}.$$

16. Show that the equation $3z = e^{-z}$ has exactly one root in $|z| < 1$.
17. Let f be analytic on $D(0,1)$ with $f(0) = 0$. Suppose $\epsilon > 0, 0 < r < 1$, and $\min_{|z|=r} |f(z)| \geq \epsilon$. Prove that $D(0,\epsilon) \subseteq f(D(0,r))$.
18. Evaluate

$$\int_{C(1+i,2)} \left[\frac{e^{\pi z}}{z^2+1} + \cos\frac{1}{z} + \frac{1}{e^z}\right] dz.$$

19. Suppose that P and Q are polynomials, the degree of Q exceeds that of P by at least 2, and the rational function P/Q has no poles on the real axis. Prove that $\int_{-\infty}^{\infty}[P(x)/Q(x)]\,dx$ is $2\pi i$ times the sum of the residues of P/Q at its poles in the upper half plane. Then compute this integral with $P(x) = x^2$ and $Q(x) = 1 + x^4$.

20. Prove that the equation $e^z - 3z^7 = 0$ has seven roots in the unit disk $|z| < 1$. More generally, if $|a| > e$ and n is a positive integer, prove that $e^z - az^n$ has exactly n roots in $|z| < 1$.
21. Prove that $e^z = 2z + 1$ for exactly one $z \in D(0,1)$.
22. Show that $f(z) = z^7 - 5z^4 + z^2 - 2$ has exactly 4 zeros inside the unit circle.
23. If $f(z) = z^5 + 15z + 1$, prove that all zeros of f are in $\{z : |z| < 2\}$, but only one zero of f is in $\{z : |z| < 1/2\}$.
24. Show that all the roots of $z^5 + z + 1 = 0$ satisfy $|z| < 5/4$.
25. Let $\{f_n\}$ be a sequence of analytic functions on an open connected set Ω such that $f_n \to f$ uniformly on compact subsets of Ω. Assume that f is not identically zero, and let $z_0 \in \Omega$. Prove that $f(z_0) = 0$ iff there is a subsequence $\{f_{n_k}\}$ and a sequence $\{z_k\}$ such that $z_k \to z_0$ and $f_{n_k}(z_k) = 0$ for all k. (Suggestion: Rouché's theorem.)
26. Let $p(z) = a_n z^n + \cdots + a_1 z + a_0, a_n \neq 0$, define $q(z) = \overline{a}_0 z^n + \cdots + \overline{a}_{n-1} z + \overline{a}_n$, and put

$$f(z) = \overline{a}_0 p(z) - a_n q(z).$$

Assume that p has $k \geq 0$ zeros in $|z| < 1$, but no zeros on $|z| = 1$. Establish the following.
(a) For $z \neq 0, q(z) = z^n \overline{p}(1/\overline{z})$.
(b) q has $n - k$ zeros in $|z| < 1$.
(c) $|p(z)| = |q(z)|$ for $|z| = 1$.
(d) If $|a_0| > |a_n|$, then f also has k zeros in $|z| < 1$, while if $|a_0| < |a_n|$, then f has $n - k$ zeros in $|z| < 1$.
(e) If $|a_0| > |a_n|$, then p has at least one zero in $|z| > 1$, while if $|a_0| < |a_n|$, then p has at least one zero in $|z| < 1$.

4.3 The Open Mapping Theorem for Analytic Functions

Our aim in this section is to show that a non-constant analytic function on a region Ω maps Ω to a region, and that a one-to-one analytic function has an analytic inverse. These facts, among others, are contained in the following theorem.

4.3.1 Open Mapping Theorem

Let f be a non-constant analytic function on an open connected set Ω. Let $z_0 \in \Omega$ and $w_0 = f(z_0)$, and let $k = m(f - w_0, z_0)$ be the order of the zero which $f - w_0$ has at z_0.

(a) There exists $\epsilon > 0$ such that $\overline{D}(z_0, \epsilon) \subseteq \Omega$ and such that neither $f - w_0$ nor f' has a zero in $\overline{D}(z_0, \epsilon) \setminus \{z_0\}$.

(b) Let γ be the positively oriented boundary of $\overline{D}(z_0, \epsilon)$, let W_0 be the component of $\mathbb{C} \setminus (f \circ \gamma)^*$ that contains w_0, and let $\Omega_1 = D(z_0, \epsilon) \cap f^{-1}(W_0)$. Then f is a k-to-one map of $\Omega_1 \setminus \{z_0\}$ onto $W_0 \setminus \{w_0\}$.

(c) f is a one-to-one map of Ω_1 onto W_0 iff $f'(z_0) \neq 0$.

(d) $f(\Omega)$ is open.

(e) $f : \Omega \to \mathbb{C}$ maps open subsets of Ω onto open sets.

(f) If f is one-to-one, then f^{-1} is analytic.

Proof.

(a) This follows from the identity theorem; the zeros of a non-constant analytic function and its derivative have no limit point in Ω.

(b) If $w \in W_0$, then by the argument principle, $n(f \circ \gamma)$ is the number of zeros of $f - w$ in $D(z_0, \epsilon)$. But $n(f \circ \gamma, w) = n(f \circ \gamma, w_0)$, because the index is constant on components of the complement of $(f \circ \gamma)^*$. Since $n(f \circ \gamma, w_0) = k$, and f' has no zeros in $D'(z_0, \epsilon)$, it follows that for $w \neq w_0$, $f - w$ has exactly k zeros in $D(z_0, \epsilon)$, all simple. This proves (b).

(c) If $f'(z_0) \neq 0$, then $k = 1$. Conversely, if $f'(z_0) = 0$, then $k > 1$.

(d) This is a consequence of (a) and (b), as they show that $f(z_0)$ is an interior point of the range of f.

(e) This is a consequence of (d) as applied to an arbitrary open subdisk of Ω.

(f) Assume that f is one-to-one from Ω onto $f(\Omega)$. Since f maps open subsets of Ω onto open subsets of $f(\Omega)$, f^{-1} is continuous on $f(\Omega)$. By (c), f' has no zeros in Ω, and Theorem 1.3.2 then implies that f^{-1} is analytic. ♣

4.3.2 Remarks

If Ω is not assumed to be connected, but f is non-constant on each component of Ω, then the conclusions of (4.3.1) are again true. In particular, if f is one-to-one, then surely f is non-constant on components of Ω and hence f^{-1} is analytic on $f(\Omega)$. Finally, note that the maximum principle is an immediate consequence of the open mapping theorem. (Use (4.3.1d), along with the observation that given any disk $D(w_0, r)$, there exists $w \in D(w_0, r)$ with $|w| > |w_0|$.)

The last result of this section is an integral representation theorem for f^{-1} in terms of the given function f. It can also be used to give an alternative proof that f^{-1} is analytic.

4.3.3 Theorem

Let f and g be analytic on Ω and assume that f is one-to-one. Then for each $z_0 \in \Omega$ and each r such that $\overline{D}(z_0, r) \subseteq \Omega$, we have

$$g(f^{-1}(w)) = \frac{1}{2\pi i} \int_{C(z_0,r)} g(z) \frac{f'(z)}{f(z) - w} \, dw$$

for every $w \in f(D(z_0, r))$. In particular, with $g(z) = z$, we have

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{C(z_0,r)} z \frac{f'(z)}{f(z) - w} \, dw.$$

Proof. Let $w \in f(D(z_0, r))$. The function $h(z) = g(z)\frac{f'(z)}{f(z)-w}$ is analytic on $\Omega \setminus \{f^{-1}(w)\}$, and hence by the residue theorem [or even (4.2.2a)],

$$\frac{1}{2\pi i}\int_{C(z_0,r)} g(z)\frac{f'(z)}{f(z)-w}\,dz = \operatorname{Res}(h, w).$$

But g is analytic at $f^{-1}(w)$ and $f - w$ has a simple zero at $f^{-1}(w)$ (because f is one-to-one), hence (Problem 1)

$$\operatorname{Res}(h, w) = g(f^{-1}(w))\operatorname{Res}(\frac{f'}{f-w}, w) = g(f^{-1}(w)) \text{ by (4.2.2e).} \ ♣$$

In Problem 2, the reader is asked to use the above formula to give another proof that f^{-1} is analytic on $f(\Omega)$.

Problems

1. Suppose g is analytic at z_0 and f has a simple pole at z_0. show that $\operatorname{Res}(gf, z_0) = g(z_0)\operatorname{Res}(f, z_0)$. Show also that the result is false if the word "simple" is omitted.
2. Let f be as in Theorem 4.3.3. Use the formula for f^{-1} derived therein to show that f^{-1} is analytic on $f(\Omega)$. (Show that f^{-1} is representable in $f(\Omega)$ by power series.)
3. The goal of this problem is an open mapping theorem for meromorphic functions. Recall from (4.2.5) that f is meromorphic on Ω if f is analytic on $\Omega \setminus P$ where P is a subset of Ω with no limit point in Ω such that f has a pole at each point of P. Define $f(z) = \infty$ if $z \in P$, so that by (4.1.5b), f is a continuous map of Ω into the extended plane $\hat{\mathbb{C}}$. Prove that if f is non-constant on each component of Ω, then $f(\Omega)$ is open in $\hat{\mathbb{C}}$.
4. Suppose f is analytic on Ω, $\overline{D}(z_0, r) \subseteq \Omega$, and f has no zeros on $C(z_0, r)$. Let $a_1, a_2, \ldots, a_n$ be the zeros of f in $D(z_0, r)$. Prove that for any g that is analytic on Ω,

$$\frac{1}{2\pi i}\int_{C(z_0,r)} \frac{f'(z)}{f(z)}g(z)\,dz = \sum_{j=1}^{n} m(f, a_j)g(a_j)$$

   where (as before) $m(f, a_j)$ is the order of the zero of f at a_j.
5. Let f be a non-constant analytic function on an open connected set Ω. How does the open mapping theorem imply that neither $|f|$ nor $\operatorname{Re} f$ nor $\operatorname{Im} f$ takes on a local maximum in Ω?

4.4 Linear Fractional Transformations

In this section we will study the mapping properties of a very special class of functions on $\mathbb{C}$, the *linear fractional transformations* (also known as *Möbius transformations*).

4.4.1 Definition

If a, b, c, d are complex numbers such that $ad - bc \neq 0$, the *linear fractional transformation* $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ associated with a, b, c, d is defined by

$$T(z) = \begin{cases} \frac{az+b}{cz+d}, & z \neq \infty, z \neq -d/c \\ a/c, & z = \infty \\ \infty, & z = -d/c. \end{cases}$$

Note that the condition $ad - bc \neq 0$ guarantees that T is not constant. Also, if $c = 0$, then $a \neq 0$ and $d \neq 0$, so that the usual agreements regarding ∞ can be made, that is,

$$T(\infty) = \begin{cases} a/c & \text{if } c \neq 0, \\ \infty & \text{if } c = 0 \end{cases} \qquad \text{and} \qquad T(-d/c) = \infty \text{ if } c \neq 0.$$

It follows from the definition that T is a one-to-one continuous map of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$. Moreover, T is analytic on $\hat{\mathbb{C}} \setminus \{-d/c\}$ with a simple pole at the point $-d/c$. Also, each such T is a composition of maps of the form

(i) $z \to z + B$ (translation)
(ii) $z \to \lambda z$, where $|\lambda| = 1$ (rotation)
(iii) $z \to \rho z,\ \rho > 0$ (dilation)
(iv) $z \to 1/z$ (inversion).

To see that T is always such a composition, recall that if $c = 0$, then $a \neq 0 \neq d$, so

$$T(z) = |a/d| \frac{a/d}{|a/d|} z + \frac{b}{d},$$

and if $c \neq 0$, then

$$T(z) = \frac{(bc - ad)/c^2}{z + (d/c)} + \frac{a}{c}.$$

Linear fractional transformations have the important property of mapping the family of lines and circles in $\mathbb{C}$ onto itself. This is most easily seen by using complex forms of equations for lines and circles.

4.4.2 Theorem

Let $L = \{z : \alpha z\overline{z} + \overline{\beta} z + \beta \overline{z} + \gamma = 0\}$ where α and γ are real numbers, β is complex, and $s^2 = \beta\overline{\beta} - \alpha\gamma > 0$. If $\alpha \neq 0$, then L is a circle, while if $\alpha = 0$, then L is a line. Conversely, each line or circle can be expressed as one of the sets L for appropriate α, γ, β.

Proof. First let us suppose that $\alpha \neq 0$. Then the equation defining L is equivalent to $|z + (\beta/\alpha)|^2 = (\beta\overline{\beta} - \alpha\gamma)/\alpha^2$, which is the equation of a circle with center at $-\beta/\alpha$ and radius $s/|\alpha|$. Conversely, the circle with center z_0 and radius $r > 0$ has equation $|z - z_0|^2 = r^2$, which is equivalent to $z\overline{z} - \overline{z}_0 z - z_0\overline{z} + |z_0|^2 - r^2 = 0$. This has the required form with $\alpha = 1,\ \beta = -z_0,\ \gamma = |z_0|^2 - r^2$. On the other hand, if $\alpha = 0$, then $\beta \neq 0$, and the equation defining L becomes $\overline{\beta}z+\beta\overline{z}+\gamma=0$, which is equivalent to $\text{Re}(\overline{\beta}z)+\gamma/2=0$. This has the form $Ax+By+\gamma/2=0$ where $z=x+iy$ and $\beta=A+iB$, showing that L is a line in this case. Conversely, an equation of the form $Ax+By+C=0$, where A and B are not both zero, can be written in complex form as $\text{Re}(\overline{\beta}z)+\gamma/2=0$, where $\beta=A+iB$ and $\gamma=2C$. ♣

4.4.3 Theorem

Suppose L is a line or circle, and T is a linear fractional transformation. Then $T(L)$ is a line or circle.

Proof. Since T is a composition of maps of the types (i)-(iv) of (4.4.1), it is sufficient to show that $T(L)$ is a line or circle if T is any one of these four types. Now translations, dilations, and rotations surely map lines to lines and circles to circles, so it is only necessary to look at the case where $T(z)=1/z$. But if z satisfies $\alpha z\overline{z}+\overline{\beta}z+\beta\overline{z}+\gamma=0$, then $w=1/z$ satisfies $\gamma w\overline{w}+\beta w+\overline{\beta}\overline{w}+\alpha=0$, which is also an equation of a line or circle. ♣

Note, for example, that if $T(z)=1/z$, $\gamma=0$ and $\alpha\neq 0$, then L is a circle through the origin, but $T(L)$, with equation $\beta w+\overline{\beta}\overline{w}+\alpha=0$, is a line not through the origin. This is to be expected because inversion interchanges 0 and ∞.

Linear fractional transformations also have an angle-preserving property that is possessed, more generally, by all analytic functions with non-vanishing derivatives. This will be discussed in the next section. Problems on linear fractional transformations will be postponed until the end of Section 4.5.

4.5 Conformal Mapping

We saw in the open mapping theorem that if $f'(z_0)\neq 0$, then f maps small neighborhoods of z_0 onto neighborhoods of $f(z_0)$ in a one-to-one fashion. In particular, f maps smooth arcs (that is, continuously differentiable arcs) through z_0 onto smooth arcs through $f(z_0)$. Our objective now is to show that f preserves angles between any two such arcs. This is made precise as follows.

4.5.1 Definition

Suppose f is a complex function defined on a neighborhood of z_0, with $f(z)\neq f(z_0)$ for all z near z_0 but not equal to z_0. If there exists a unimodular complex number $e^{i\varphi}$ such that for all θ,

$$\frac{f(z_0+re^{i\theta})-f(z_0)}{|f(z_0+re^{i\theta})-f(z_0)|}\to e^{i\varphi}e^{i\theta}$$

as $r\to 0^+$, then we say that f *preserves angles* at z_0.

To gain some insight and intuitive feeling for the meaning of the above condition, note that for any θ and small $r_0>0$, $\frac{f(z_0+re^{i\theta})-f(z_0)}{|f(z_0+r_0e^{i\theta})-f(z_0)|}$ is a unit vector from $f(z_0)$ to $f(z_0+r_0e^{i\theta})$. The vectors from z_0 to $z_0+re^{i\theta}, 0<r\leq r_0$, have argument θ, so the

condition states that f maps these vectors onto an arc from $f(z_0)$ whose unit tangent vector at $f(z_0)$ has argument $\varphi+\theta$. Since φ is to be the same for all θ, f rotates all short vectors from z_0 through the fixed angle φ. Thus we see that f preserves angles between tangent vectors to smooth arcs through z_0.

4.5.2 Theorem

Suppose f is analytic at z_0. Then f preserves angles at z_0 iff $f'(z_0) \neq 0$.

Proof. If $f'(z_0) \neq 0$, then for any θ,

$$\lim_{r\to 0^+} \frac{f(z_0+re^{i\theta})-f(z_0)}{|f(z_0+re^{i\theta})-f(z_0)|} = e^{i\theta} \lim_{r\to 0^+} \frac{[f(z_0+re^{i\theta})-f(z_0)]/re^{i\theta}}{|[f(z_0+re^{i\theta})-f(z_0)]|/r} = e^{i\theta}\frac{f'(z_0)}{|f'(z_0)|}.$$

Thus the required unimodular complex number of Definition 4.5.1 is $f'(z_0)/|f'(z_0)|$. Conversely, suppose that $f'(z_0)=0$. Assuming that f is not constant, $f-f(z_0)$ has a zero of some order $m>1$ at z_0, hence we may write $f(z)-f(z_0)=(z-z_0)^m g(z)$ where g is analytic at z_0 and $g(z_0)\neq 0$. For any θ and small $r>0$,

$$\frac{f(z_0+re^{i\theta})-f(z_0)}{|f(z_0+re^{i\theta})-f(z_0)|} = e^{im\theta}\frac{g(z_0+re^{i\theta})}{|g(z_0+re^{i\theta})|} = e^{i\theta}e^{i(m-1)\theta}\frac{g(z_0+re^{i\theta})}{|g(z_0+re^{i\theta})|}$$

and the expression on the right side approaches $e^{i\theta}e^{i(m-1)\theta}g(z_0)/|g(z_0)|$ as $r\to 0^+$. Since the factor $e^{i(m-1)\theta}g(z_0)/|g'(z_0)|$ depends on θ, f does *not* preserve angles at z_0. Indeed, the preceding shows that angles are increased by a factor of m, the order of the zero of $f-f(z_0)$ at z_0. ♣

A function f on Ω that is analytic and has a nonvanishing derivative will be called a *conformal map*; it is locally one-to-one and preserves angles. Examples are the exponential function and the linear fractional transformation (on their domains of analyticity). The angle-preserving property of the exponential function was illustrated in part (i) of (2.3.1), where it was shown that exp maps any pair of vertical and horizontal lines onto, respectively, a circle with center 0 and an open ray emanating from 0. Thus the exponential function preserves the orthogonality of vertical and horizontal lines.

Problems

1. Show that the inverse of a linear fractional transformation and the composition of two linear fractional transformations is again a linear fractional transformation.
2. Consider the linear fractional transformation $T(z)=(1+z)/(1-z)$.
   (a) Find a formula for the inverse of T.
   (b) Show that T maps $|z|<1$ onto $\operatorname{Re} z>0$, $|z|=1$ onto $\{z : \operatorname{Re} z = 0\}\cup\{\infty\}$, and $|z|>1$ onto $\operatorname{Re} z<0$.
3. Find linear fractional transformations that map
   (a) $1, i, -1$ to $1, 0, -1$ respectively.
   (b) $1, i, -1$ to $-1, i, 1$ respectively.

4. Let (z_1, z_2, z_3) be a triple of distinct complex numbers.
(a) Prove that there is a unique linear fractional transformation T with the property that $T(z_1) = 0,\ T(z_2) = 1,\ T(z_3) = \infty$.
(b) Prove that if one of z_1, z_2, z_3 is ∞, then the statement of (a) remains true.
(c) Let each of (z_1, z_2, z_3) and (w_1, w_2, w_3) be triples of distinct complex numbers (or extended complex numbers in $\hat{\mathbb{C}}$). Show that there is a unique linear fractional transformation such that $T(z_j) = w_j, j = 1, 2, 3$.

5. Let f be meromorphic on $\mathbb{C}$ and assume that f is one-to-one. Show that f is a linear fractional transformation. In particular, if f is entire, then f is linear, that is, a first degree polynomial in z. Here is a suggested outline:
(a) f has at most one pole in $\mathbb{C}$, consequently ∞ is an isolated singularity of f.
(b) $f(D(0, 1))$ and $f(\mathbb{C} \setminus \overline{D}(0, 1))$ are disjoint open sets in $\hat{\mathbb{C}}$.
(c) f has a pole or removable singularity at ∞, so f is meromorphic on $\hat{\mathbb{C}}$.
(d) f has exactly one pole in $\hat{\mathbb{C}}$.
(e) Let the pole of f be at z_0. If $z_0 = \infty$, then f is a polynomial, which must be of degree 1. If $z_0 \in \mathbb{C}$, consider $g(z) = 1/f(z), z \neq z_0;\ g(z_0) = 0$. Then g is analytic at z_0 and $g'(z_0) \neq 0$.
(f) f has a simple pole at z_0.
(g) $f(z) - [\text{Res}(f, z_0))/(z - z_0)]$ is constant, hence f is a linear fractional transformation.

4.6 Analytic Mappings of One Disk to Another

In this section we will investigate the behavior of analytic functions that map one disk into another. The linear fractional transformations are examples which are, in addition, one-to-one. Schwarz's lemma (2.4.16) is an important illustration of the type of conclusion that can be drawn about such functions, and will be generalized in this section. We will concentrate on the special case of maps of the unit disk $D = D(0, 1)$ into itself. The following lemma supplies us with an important class of examples.

4.6.1 Lemma

Fix $a \in D$, and define a function φ_a on $\hat{\mathbb{C}}$ by

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z},$$

where the usual conventions regarding ∞ are made: $\varphi_a(\infty) = -1/\overline{a}$ and $\varphi_a(1/\overline{a}) = \infty$. Then φ_a is a one-to-one continuous map of $\hat{\mathbb{C}}$ into $\hat{\mathbb{C}}$ whose inverse is φ_{-a}. Also, φ_a is analytic on $\hat{\mathbb{C}} \setminus \{1/\overline{a}\}$ with a simple pole at $1/\overline{a}$ (and a zero of order 1 at a). Thus φ_a is analytic on a neighborhood of the closed disk $\overline{D}$. Finally,

$$\varphi_a(D) = D, \quad \varphi_a(\partial D) = \partial D, \quad \varphi_a'(z) = \frac{1 - |a|^2}{(1 - \overline{a}z)^2}$$

hence

$$\varphi_a'(a) = \frac{1}{1 - |a|^2} \quad \text{and} \quad \varphi_a'(0) = 1 - |a|^2.$$

Proof. Most of the statements follow from the definition of φ_a and the fact that it is a linear fractional transformation. To see that φ_a maps $|z| = 1$ into itself, we compute, for $|z| = 1$,

$$\left|\frac{z-a}{1-\overline{a}z}\right| = \left|\frac{z-a}{\overline{z}(1-\overline{a}z)}\right| = \left|\frac{z-a}{\overline{z}-\overline{a}}\right| = 1.$$

Thus by the maximum principle, φ_a maps D into D. Since $\varphi_a^{-1} = \varphi_{-a}$ (a computation shows that $\varphi_{-a}(\varphi_a(z)) = z$), and $|a| < 1$ iff $|-a| < 1$, it follows that φ_a maps D *onto* D and maps ∂D onto ∂D. The formulas involving the derivative of φ_a are verified by a direct calculation. ♣

4.6.2 Remark

The functions φ_a are useful in factoring out the zeros of a function g on D, because $g(z)$ and $\varphi_a(g(z))$ have the same maximum modulus on D, unlike $g(z)$ and $(z-a)g(z)$. In fact, if g is defined on the closed disk $\overline{D}$, then

$$\left|\frac{z-a}{1-\overline{a}z}g(z)\right| = |g(z)| \text{ for } |z| = 1.$$

This property of the functions φ_a will be applied several times in this section and the problems following it.

We turn now to what is often called Pick's generalization of Schwarz's lemma.

4.6.3 Theorem

Let $f : D \to D$ be analytic. then for any $a \in D$ and any $z \in D$,

$$\left|\frac{f(z)-f(a)}{1-\overline{f(a)}f(z)}\right| \le \left|\frac{z-a}{1-\overline{a}z}\right| \tag{i}$$

and

$$|f'(a)| \le \frac{1-|f(a)|^2}{1-|a|^2}. \tag{ii}$$

Furthermore, if equality holds in (i) for some $z \neq a$, or if equality holds in (ii), then f is a linear fractional transformation. In fact, there is a unimodular complex number λ such that with $b = f(a)$, f is the composition $\varphi_{-b} \circ \lambda\varphi_a = \varphi_b^{-1} \circ \lambda\varphi_a$. That is,

$$f(z) = \frac{\lambda\varphi_a(z) + b}{1 + \overline{b}\lambda\varphi_a(z)}, \quad |z| < 1.$$

Proof. Let $a \in D$ and set $b = f(a)$. We are going to apply Schwarz's lemma (2.4.16) to the function $g = \varphi_b \circ f \circ \varphi_{-a}$. First, since f maps D into D, so does g. Also,

$$g(0) = \varphi_b(f(\varphi_{-a}(0))) = \varphi_b(f(a)) = \varphi_b(b) = 0.$$

By Schwarz's lemma, $|g(w)| \leq |w|$ for $|w| < 1$, and replacing w by $\varphi_a(z)$ and noting that $g(\varphi_a(z)) = \varphi_b(f(z))$, we obtain (i). Also by (2.4.16), we have $|g'(0) \leq 1$. But by (4.6.1),

$$\begin{aligned} g'(0) &= \varphi_b'(f(\varphi_{-a}(0)))f'(\varphi_{-a}(0))\varphi_{-a}'(0) \\ &= \varphi_b'(f(a))f'(a)(1-|a|^2) \\ &= \frac{1}{1-|f(a)|^2)} f'(a)(1-|a|^2). \end{aligned}$$

Thus the condition $|g'(0)| \leq 1$ implies (ii).

Now if equality holds in (i) for some $z \neq a$, then $|g(\varphi_a(z))| = |\varphi_a(z)|$ for some $z \neq a$, hence $|g(w)| = |w|$ for some $w \neq 0$. If equality holds in (ii), then $|g'(0)| = 1$. In either case, (2.4.16) yields a unimodular complex number λ such that $g(w) = \lambda w$ for $|w| < 1$. Set $w = \varphi_a(z)$ to obtain $\varphi_b(f(z)) = \lambda\varphi_a(z)$, that is, $f(z) = \varphi_{-b}(\lambda\varphi_a(z))$ for $|z| < 1$. ♣

An important application of Theorem 4.6.3 is in characterizing the one-to-one analytic maps of D onto itself as having the form $\lambda\varphi_a$ where $|\lambda| = 1$ and $a \in D$.

4.6.4 Theorem

Suppose f is a one-to-one analytic map of D onto D. then $f = \lambda\varphi_a$ for some unimodular λ and $a \in D$.

Proof. Let $a \in D$ be such that $f(a) = 0$ and let $g = f^{-1}$, so $g(0) = a$. Now since $g(f(z)) = z$, we have $1 = g'(f(z))f'(z)$, in particular, $1 = g'(f(a))f'(a) = g'(0)f'(a)$. Next, (4.6.3ii) implies that $|g'(0)| \leq 1 - |a|^2$ and $|f'(a)| \leq 1/(1-|a|^2)$. Thus

$$1 = |g'(0)||f'(a)| \leq \frac{1-|a|^2}{1-|a|^2} = 1.$$

Necessarily then, $|f'(a)| = 1/(1-|a|^2)$ (and $|g'(0)| = 1-|a|^2$). Consequently, by the condition for equality in (4.6.3ii), $f = \lambda\varphi_a$, as required. ♣

4.6.5 Remark

One implication of the previous theorem is that any one-to-one analytic map of D onto D actually extends to a homeomorphism of $\overline{D}$ onto $\overline{D}$. We will see when we study the Riemann mapping theorem in the next chapter that more generally, if f maps D onto a special type of region Ω, then f again extends to a homeomorphism of $\overline{D}$ onto $\overline{\Omega}$.

Our final result is a characterization of those continuous functions on $\overline{D}$ which are analytic on D and have constant modulus on the boundary $|z| = 1$. The technique mentioned in (4.6.2) will be used.

4.6.6 Theorem

Suppose f is continuous on $\overline{D}$, analytic on D, and $|f(z)| = 1$ for $|z| = 1$. Then there is a unimodular λ, finitely many points $a_1, \dots, a_n$ in D, and positive integers $k_1, \dots, k_n$, such that

$$f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}.$$

In other words, f is, to within a multiplicative constant, a finite product of functions of the type φ_a. (If f is constant on $\overline{D}$, the product is empty and we agree that it is identically 1 in this case.)

Proof. First note that $|f(z)| = 1$ for $|z| = 1$ implies that f has at most finitely many zeros in D. If f has no zeros in D, then by the maximum and minimum principles, f is constant on $\overline{D}$. Suppose then that f has its zeros at the points $a_1, \dots, a_n$ with orders $k_1, \dots, k_n$ respectively. Put

$$g(z) = \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}, \quad z \in D.$$

Then f/g has only removable singularities in D, the analytic extension of f/g has no zeros in D, and $|f/g| = 1$ on ∂D. Again by the maximum and minimum principles, f/g is constant on $D \setminus \{a_1, \dots, a_n\}$. Thus $f = \lambda g$ with $|\lambda| = 1$. ♣

Problems

1. Derive the inequality (4.6.3ii) *directly* from (4.6.3i).
2. Let f be an analytic map of $D(0, 1)$ into the right half plane $\{z : \operatorname{Re} z > 0\}$. Show that

$$\frac{1 - |z|}{1 + |z|} |f(0)| \le |f(z)| \le \frac{1 + |z|}{1 - |z|} |f(0)|, \quad z \in D(0, 1),$$

   and

$$|f'(0)| \le 2 |\operatorname{Re} f(0)|.$$

   Hint: Apply Schwarz's lemma to $T \circ f$, where $T(w) = (w - f(0))/(w + \overline{f(0)})$.
3. Show that if f is an analytic map of $D(0, 1)$ into itself, and f has two or more fixed points, then $f(z) = z$ for all $z \in D(0, 1)$.
4. (a) Characterize the entire functions f such that $|f(z)| = 1$ for $|z| = 1$ [see (4.6.6)].
   (b) Characterize the meromorphic functions f on $\mathbb{C}$ such that $|f(z)| = 1$ for $|z| = 1$. (Hint: If f has a pole of order k at $a \in D(0, 1)$, then $[(z - a)/(1 - \overline{a}z)]^k f(z)$ has a removable singularity at a.)
5. Suppose that in Theorem 4.6.3, the unit disk D is replaced by $D(0, R)$ and $D(0, M)$. That is, suppose $f : D(0, R) \to D(0, M)$. How are the conclusions (i) and (ii) modified in this case? (Hint: Consider $g(z) = f(Rz)/M$.)
6. Suppose $f : \overline{D}(0, 1) \to \overline{D}(0, 1)$ is continuous and f is analytic on $D(0, 1)$. Assume that f has zeros at $z_1, \dots, z_n$ of orders $k_1, \dots, k_n$ respectively. Show that

$$|f(z)| \le \prod_{j=1}^{n} \left| \frac{z - z_j}{1 - \overline{z}_j z} \right|^{k_j}.$$

Suppose equality holds for some $z \in D(0,1)$ with $z \neq z_j, j = 1, \dots, n$. Find a formula for $f(z)$.

4.7 The Poisson Integral Formula and its Applications

Our aim in this section is to solve the Dirichlet problem for a disk, that is, to construct a solution of Laplace's equation in the disk subject to prescribed boundary values. The basic tool is the Poisson integral formula, which may be regarded as an analog of the Cauchy integral formula for harmonic functions. We will begin by extending Cauchy's theorem and the Cauchy integral formula to functions continuous on a disk and analytic on its interior.

4.7.1 Theorem

Suppose f is continuous on $\overline{D}(0,1)$ and analytic on $D(0,1)$. Then
(i) $\int_{C(0,1)} f(w)\,dw = 0$ and
(ii) $f(z) = \frac{1}{2\pi i}\int_{C(0,1)} \frac{f(w)}{w-z}\,dw$ for all $z \in D(0,1)$.

Proof. For $0 < r < 1$, $\int_{C(0,r)} f(w)\,dw = 0$ by Cauchy's theorem. For $n = 1, 2, \dots$, put $f_n(z) = f(\frac{n}{n+1}z)$. Then f_n is analytic on $D(0, \frac{n+1}{n})$ and the sequence $\{f_n\}$ converges to f uniformly on $C(0,1)$ [by continuity of f on $\overline{D}(0,1)$]. Hence $\int_{C(0,1)} f_n(w)\,dw \to \int_{C(0,1)} f(w)\,dw$. Since $\int_{C(0,1)} f_n(w)\,dw = \frac{n+1}{n}\int_{C(0,\frac{n}{n+1})} f(w)\,dw = 0$, we have (i). To prove (ii), we apply (i) to the function g, where

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(z), & w = z. \end{cases} \clubsuit$$

Note that the same proof works with only minor modifications if $D(0,1)$ is replaced by an arbitrary disk $D(z_0, R)$.

4.7.2 Definition

For $z \in D(0,1)$, define functions P_z and Q_z on the real line $\mathbb{R}$ by

$$P_z(t) = \frac{1-|z|^2}{|e^{it}-z|^2} \quad \text{and} \quad Q_z(t) = \frac{e^{it}+z}{e^{it}-z};$$

$P_z(t)$ is called the *Poisson kernel* and $Q_z(t)$ the *Cauchy kernel*. We have

$$\mathrm{Re}[Q_z(t)] = \mathrm{Re}\left[\frac{(e^{it}+z)(e^{-it}-\overline{z})}{|e^{it}-z|^2}\right] = \mathrm{Re}\left[\frac{1-|z|^2+ze^{-it}-\overline{z}e^{it}}{|e^{it}-z|^2}\right] = P_z(t).$$

Note also that if $z = re^{i\theta}$, then

$$P_z(t) = \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} = \frac{1-r^2}{|e^{i(t-\theta)}-r|^2} = P_r(t-\theta).$$

Since $|e^{i(t-\theta)} - r|^2 = 1 - 2r\cos(t-\theta) + r^2$, we see that

$$P_r(t-\theta) = \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} = P_r(\theta - t).$$

Thus for $0 \le r < 1$, $P_r(x)$ is an even function of x. Note also that $P_r(x)$ is positive and decreasing on $[0, \pi]$.

After these preliminaries, we can establish the Poisson integral formula for the unit disk, which states that the value of an analytic function at a point inside the disk is a weighted average of its values on the boundary, the weights being given by the Poisson kernel. The precise statement is as follows.

4.7.3 Poisson Integral Formula

Suppose f is continuous on $\overline{D}(0,1)$ and analytic on $D(0,1)$. then for $z \in D(0,1)$ we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t) f(e^{it})\,dt$$

and therefore

$$\operatorname{Re} f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t) \operatorname{Re} f(e^{it})\,dt.$$

Proof. By Theorem 4.7.1(ii),

$$f(0) = \frac{1}{2\pi i}\int_{C(0,1)} \frac{f(w)}{w}\,dw = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,dt,$$

hence $f(0) = (1/2\pi)\int_0^{2\pi} P_0(t) f(e^{it})\,dt$ because $P_0(t) \equiv 1$. This takes care of the case $z = 0$. If $z \neq 0$, then again by (4.7.1) we have

$$f(z) = \frac{1}{2\pi i}\int_{C(0,1)} \frac{f(w)}{w-z}\,dw \quad \text{and} \quad 0 = \frac{1}{2\pi i}\int_{C(0,1)} \frac{f(w)}{w - 1/\overline{z}}\,dw,$$

the second equation holding because $1/\overline{z} \notin D(0,1)$. Subtracting the second equation from the first, we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C(0,1)} [\frac{1}{w-z} - \frac{1}{w-1/\overline{z}}] f(w)\,dw \\
&= \frac{1}{2\pi}\int_0^{2\pi} [\frac{1}{e^{it}-z} - \frac{1}{e^{it}-1/\overline{z}}] e^{it} f(e^{it})\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} [\frac{e^{it}}{e^{it}-z} + \frac{\overline{z}e^{it}}{1-\overline{z}e^{it}}] f(e^{it})\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} [\frac{e^{it}}{e^{it}-z} + \frac{\overline{z}}{e^{-it}-\overline{z}}] f(e^{it})\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it})\,dt
\end{aligned}$$

which proves the first formula. Taking real parts, we obtain the second. ♣

4.7.4 Corollary

For $|z| < 1$, $\frac{1}{2\pi}\int_0^{2\pi} P_z(t)\,dt = 1$.

Proof. Take $f \equiv 1$ in (4.7.3). ♣

Using the formulas just derived for the unit disk $D(0,1)$, we can obtain formulas for functions defined on arbitrary disks.

4.7.5 Poisson Integral Formula for Arbitrary Disks

Let f be continuous on $\overline{D}(z_0,R)$ and analytic on $D(z_0,R)$. Then for $z \in D(z_0,R)$,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{(z-z_0)/R}(t) f(z_0 + Re^{it})\,dt.$$

In polar form, if $z = z_0 + re^{i\theta}$, then

$$f(z_0 + re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\theta - t) f(z_0 + Re^{it})\,dt.$$

Proof. Define g on $\overline{D}(0,1)$ by $g(w) = f(z_0 + Rw)$. Then (4.7.3) applies to g, and we obtain

$$g(w) = \frac{1}{2\pi}\int_0^{2\pi} P_w(t) g(e^{it})\,dt, \ \ |w| < 1.$$

If $z \in D(z_0,R)$, then $w = (z - z_0)/R \in D(0,1)$ and

$$f(z) = g(\frac{z-z_0}{R}) = \frac{1}{2\pi}\int_0^{2\pi} P_{(z-z_0)/R}(t) f(z_0 + Re^{it})\,dt$$

which establishes the first formula. For the second, apply (4.7.2). [See the discussion beginning with "Note also that ... ".] ♣

We now have the necessary machinery available to solve the Dirichlet problem for disks. Again, for notational reasons we will solve the problem for the unit disk $D(0,1)$. If desired, the statement and proof for an arbitrary disk can be obtained by the same technique we used to derive (4.7.5) from (4.7.3).

4.7.6 The Dirichlet Problem

Suppose u_0 is a real-valued continuous function on $C(0,1)$. Define a function u on $\overline{D}(0,1)$ by

$$u(z) = \begin{cases} u_0(z) & \text{for } |z| = 1, \\ \frac{1}{2\pi}\int_0^{2\pi} P_z(t) u_0(e^{it})\,dt & \text{for } |z| < 1. \end{cases}$$

Then u is continuous on $\overline{D}(0,1)$ and harmonic on $D(0,1)$. Furthermore (since P_z is the real part of Q_z), for $z \in D(0,1)$,

$$u(z) = \text{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} Q_z(t)u_0(e^{it})\,dt\right].$$

In particular, the given continuous function u_0 on $C(0,1)$ has a continuous extension to $\overline{D}(0,1)$ which is harmonic on the interior $D(0,1)$.

Proof. The function $z \to \frac{1}{2\pi}\int_0^{2\pi} Q_z(t)u_0(e^{it})\,dt$ is analytic on $D(0,1)$ by (3.3.3), and therefore u is harmonic, hence continuous, on $D(0,1)$. All that remains is to show that u is continuous at points of the boundary $C(0,1)$.

We will actually show that $u(re^{i\theta}) \to u_0(e^{i\theta})$ uniformly in θ as $r \to 1$. Since u_0 is continuous on $C(0,1)$, this will prove that u is continuous at each of point of $C(0,1)$, by the triangle inequality. Thus let θ and r be real numbers with $0 < r < 1$. Then by (4.7.2), (4.7.4) and the definition of $u(z)$,

$$u(re^{i\theta}) - u_0(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)[u_0(e^{it}) - u_0(e^{i\theta})]\,dt.$$

Make the change of variable $x = t - \theta$ and recall that P_r is an even function. The above integral becomes

$$\frac{1}{2\pi}\int_{-\theta}^{2\pi-\theta} P_r(x)[u_0(e^{i(\theta+x)}) - u_0(e^{i\theta})]\,dx,$$

and the limits of integration can be changed to $-\pi$ and π, because the integrand has 2π as a period. Now fix δ with $0 < \delta < \pi$ and write the last integral above as the sum,

$$\frac{1}{2\pi}\int_{-\pi}^{-\delta} + \frac{1}{2\pi}\int_{-\delta}^{\delta} + \frac{1}{2\pi}\int_{\delta}^{\pi}.$$

We can estimate each of these integrals. The first and third have absolute value at most $2\sup\{|u_0(e^{it})| : -\pi \le t \le \pi\}P_r(\delta)$, because $P_r(x)$ is a positive and decreasing function on $[0,\pi]$ and $P_r(-x) = P_r(x)$. The middle integral has absolute value at most $\sup\{|u_0(e^{i(\theta+x)}) - u_0(e^{i\theta})| : -\delta \le x \le \delta\}$, by (4.7.4).

But for fixed $\delta > 0$, $P_r(\delta) \to 0$ as $r \to 1$, while $\sup\{|u_0(e^{i(\theta+x)}) - u_0(e^{i\theta})| : -\delta \le x \le \delta\}$ approaches 0 as $\delta \to 0$, uniformly in θ because u_0 is uniformly continuous on $C(0,1)$. Putting this all together, we see that given $\epsilon > 0$ there is an $r_0, 0 < r_0 < 1$, such that for $r_0 < r < 1$ and all θ, we have $|u(re^{i\theta}) - u_0(e^{i\theta})| < \epsilon$. This, along with the continuity of u_0 on $C(0,1)$, shows that u is continuous at each point of $C(0,1)$. ♣

4.7.7 Uniqueness of Solutions to the Dirichlet Problem

We saw in (2.4.15) that harmonic functions satisfy the maximum and minimum principles. Specifically, if u is continuous on $\overline{D}(0,1)$ and harmonic on $D(0,1)$, then

$$\max_{z\in\overline{D}(0,1)} u(z) = \max_{z\in C(0,1)} u(z) \quad \text{and} \quad \min_{z\in\overline{D}(0,1)} u(z) = \min_{z\in C(0,1)} u(z).$$

Thus if $u \equiv 0$ on $C(0,1)$, then $u \equiv 0$ on $\overline{D}(0,1)$.

Now suppose that u_1 and u_2 are solutions to a Dirichlet problem on $\overline{D}(0,1)$ with boundary function u_0. Then $u_1 - u_2$ is continuous on $\overline{D}(0,1)$, harmonic on $D(0,1)$, and identically 0 on $C(0,1)$, hence identically 0 on $\overline{D}(0,1)$. Therefore $u_1 \equiv u_2$, so *the solution to any given Dirichlet problem is unique.*

Here is a consequence of the uniqueness result.

4.7.8 Poisson Integral Formula for Harmonic Functions

Suppose u is continuous on $\overline{D}(0,1)$ and harmonic on $D(0,1)$. Then for $z \in D(0,1)$, we have

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)u(e^{it})\,dt.$$

More generally, if $D(0,1)$ is replaced by $D(z_0,R)$, then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{(z-z_0)/R}(t)u(z_0+Re^{it})\,dt;$$

equivalently,

$$u(z_0+re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\theta-t)u(z_0+Re^{it})\,dt$$

for $0 \le r < R$ and all θ.

Proof. The result for $D(0,1)$ follows from (4.7.6) and (4.7.7). To prove the result for $D(z_0,R)$, we apply (4.7.6) and (4.7.7) to $u^*(w) = u(z_0+Rw), w \in D(0,1)$. If $z = z_0+re^{i\theta}, 0 \le r < R$, then $u(z) = u^*((z-z_0)/R)$, hence

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{(z-z_0)/R}(t)u^*(e^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\theta-t)u(z_0+Re^{it})\,dt$$

as in (4.7.5). ♣

The Poisson integral formula allows us to derive a mean value property for harmonic functions.

4.7.9 Corollary

Suppose u is harmonic on an open set Ω. If $z_0 \in \Omega$ and $\overline{D}(z_0,R) \subseteq \Omega$, then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+Re^{it})\,dt.$$

That is, $u(z_0)$ is the average of its values on circles with center at z_0.

Proof. Apply (4.7.8) with $r = 0$. ♣

It is interesting that the mean value property characterizes harmonic functions.

4.7.10 Theorem

Suppose φ is a continuous, real-valued function on Ω such that whenever $\overline{D}(z_0, R) \subseteq \Omega$, it is true that $\varphi(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(z_0 + Re^{it})\,dt$. Then φ is harmonic on Ω.

Proof. Let $D(z_0, R)$ be any disk such that $\overline{D}(z_0, R) \subseteq \Omega$. Let u_0 be the restriction of φ to the circle $C(z_0, R)$ and apply (4.7.6) [for the disk $D(z_0, R)$] to produce a continuous function u on $\overline{D}(z_0, R)$ such that $u = u_0 = \varphi$ on $C(z_0, R)$. We will show that $\varphi = u$ on $D(z_0, R)$, thereby proving that φ is harmonic on $D(z_0, R)$. Since $D(z_0, R)$ is an arbitrary subdisk, this will prove that φ is harmonic on Ω.

The function $\varphi - u$ is continuous on $\overline{D}(z_0, R)$, and hence assumes its maximum and minimum at some points z_1 and z_2 respectively. If both z_1 and z_2 belong to $C(z_0, R)$, then since $u = \varphi$ on $C(z_0, R)$, the maximum and minimum values of $\varphi - u$ are both 0. It follows that $\varphi - u \equiv 0$ on $\overline{D}(z_0, R)$ and we are finished. On the other hand, suppose that (say) z_1 belongs to the open disk $D(z_0, R)$. Define a set A by

$$A = \{z \in D(z_0, R) : (\varphi - u)(z) = (\varphi - u)(z_1)\}.$$

Then A is closed in $D(z_0, R)$ by continuity of $\varphi - u$. We will also show that A is open, and thus conclude by connectedness that $A = D(z_0, R)$. For suppose that $a \in A$ and $r > 0$ is chosen so that $\overline{D}(a, r) \subseteq D(z_0, R)$. Then for $0 < \rho \leq r$ we have

$$\varphi(a) - u(a) = \frac{1}{2\pi}\int_0^{2\pi} [\varphi(a + \rho e^{it}) - u(a + \rho e^{it})]\,dt.$$

Since $\varphi(a + \rho e^{it}) - u(a + \rho e^{it}) \leq \varphi(a) - u(a)$, it follows from Lemma 2.4.11 that $\varphi - u$ is constant on $D(a, r)$. Thus $D(a, r) \subseteq A$, so A is open. A similar argument is used if $z_2 \in D(z_0, R)$. ♣

Remark

The above proof shows that a continuous function with the mean value property that has an absolute maximum or minimum in a region Ω is constant.

Problems

1. Let $Q_z(t)$ be as in (4.7.2). Prove that $\frac{1}{2\pi}\int_0^{2\pi} Q_z(t)\,dt = 1$.
2. Use (4.7.8) to prove *Harnack's inequality*: Suppose u satisfies the hypothesis of (4.7.8), and in addition $u \geq 0$. Then for $0 \leq r < 1$ and all θ,

$$\frac{1-r}{1+r}u(0) \leq u(re^{i\theta}) \leq \frac{1+r}{1-r}u(0).$$

3. Prove the following analog (for harmonic functions) of Theorem 2.2.17. Let $\{u_n\}$ be a sequence of harmonic functions on Ω such that $u_n \to u$ uniformly on compact subsets of Ω. Then u is harmonic on Ω. (Hint: If $\overline{D}(z_0, R) \subseteq \Omega$, the Poisson integral formula holds for u on $D(z_0, R)$.)

4. In Theorem 1.6.2 we showed that every harmonic function is locally the real part of an analytic function. Using results of this section, give a new proof of this fact.

5. Let Ω be a bounded open set and γ a closed path such that the following conditions are satisfied:
(a) $\gamma^* = \partial\Omega$, the boundary of Ω.
(b) There exists z_0 such that for every $\delta, 0 \le \delta < 1$, the path $\gamma_\delta = z_0 + \delta(\gamma - z_0)$ has its range in Ω (see Figure 4.7.1).

If f is continuous on $\overline{\Omega}$ and analytic on Ω, show that

$$\int_\gamma f(w)\,dw = 0 \quad \text{and} \quad n(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw,\ z \in \Omega.$$

Outline:
(i) First show that Ω must be starlike with star center z_0 by showing that if $z \in \Omega$, then the ray $[z_0, z, \infty)$ meets $\partial\Omega$ at some point β. By (a) and (b), $[z_0, \beta) \subseteq \Omega$. Next show that $z \in [z_0, \beta)$, hence $[z_0, z] \subseteq \Omega$.
(ii) The desired conclusions hold with γ replaced by γ_δ; let $\delta \to 1$ to complete the proof.

6. (Poisson integral formula for a half plane). Let f be analytic on $\{z : \operatorname{Im} z > 0\}$ and continuous on $\{z : \operatorname{Im} z \ge 0\}$. If $u = \operatorname{Re} f$, establish the formula

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yu(t, 0)}{(t-x)^2 + y^2}\,dt,\ \operatorname{Im} z > 0$$

under an appropriate hypothesis on the growth of f as $z \to \infty$. (Consider the path γ indicated back in Figure 4.2.6. Write, for $\operatorname{Im} z > 0$, $f(z) = (2\pi i)^{-1}\int_\gamma [f(w)/(w-z)]\,dw$ and $0 = (2\pi i)^{-1}\int_\gamma [f(w)/(w-\overline{z})]\,dw$ by using either Problem 5 or a technique similar to that given in the proof of (4.7.1). Then subtract the second equation from the first.)

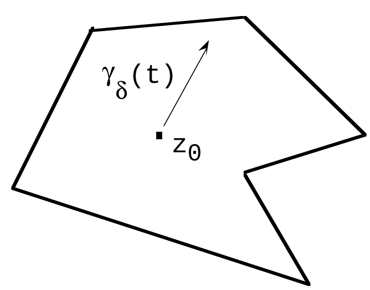

Figure 4.7.1

4.8 The Jensen and Poisson-Jensen Formulas

Suppose f is continuous on $\overline{D}(0, R)$, analytic on $D(0, R)$ and f has no zeros in $\overline{D}(0, R)$. Then we know that f has an analytic logarithm on $D(0, R)$ whose real part $\ln|f|$ is continuous on $\overline{D}(0,R)$ and harmonic on $D(0,R)$. Thus by (4.7.8), the Poisson integral formula for harmonic functions, we have

$$\ln|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t)\ln|f(Re^{it})|\,dt$$

or in polar form,

$$\ln|f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\theta - t)\ln|f(Re^{it})|\,dt.$$

If f has zeros in $\overline{D}(0,R)$, then this derivation fails. However, the above formula can be modified to take the zeros of f into account.

4.8.1 Poisson-Jensen Formula

Suppose that f is continuous on $\overline{D}(0,R)$, analytic on $D(0,R)$ and that f has no zeros on $C(0,R)$. Let $a_1,\dots,a_n$ be the distinct zeros of f in $D(0,R)$ with multiplicities $k_1,\dots,k_n$ respectively. Then for $z \in D(0,R)$, z unequal to any of the a_j, we have

$$\ln|f(z)| = \sum_{j=1}^n k_j \ln\left|\frac{R(z-a_j)}{R^2 - \overline{a}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t)\ln|f(Re^{it}|\,dt.$$

Proof. We first give a proof for the case $R = 1$. By (4.6.2), there is a continuous function g on $\overline{D}(0,1)$, analytic on $D(0,1)$, such that g has no zeros in $\overline{D}(0,1)$ and such that

$$f(z) = \left[\prod_{j=1}^n \left(\frac{z-a_j}{1-\overline{a}_j z}\right)^{k_j}\right] g(z).$$

Since the product has modulus one when $|z| = 1$ we have $|f(z)| = |g(z)|$ for $|z| = 1$. Thus if $f(z) \neq 0$, then

$$\ln|f(z)| = \sum_{j=1}^n k_j \ln\left|\frac{z-a_j}{1-\overline{a}_j z}\right| + \ln|g(z)|.$$

But g has no zeros in $\overline{D}(0,1)$, so by the discussion in the opening paragraph of this section,

$$\ln|g(z)| = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)\ln|g(e^{it})|\,dt = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)\ln|f(e^{it})|\,dt.$$

This gives the result for $R = 1$. To obtain the formula for arbitrary R, we apply what was just proved to $F(w) = f(Rw), |w| \le 1$. Thus

$$\ln|F(w)| = \sum_{j=1}^n k_j \ln\left|\frac{w-(a_j/R)}{1-(\overline{a}_j w/R)}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_w(t)\ln|F(e^{it})|\,dt.$$

If we let $z = Rw$ and observe that

$$\frac{w - (a_j/R)}{1 - (\overline{a}_j w/R)} = \frac{R(z - a_j)}{R^2 - \overline{a}_j z},$$

we have the desired result. ♣

The Poisson-Jensen formula has several direct consequences.

4.8.2 Corollary

Assume that f satisfies the hypothesis of (4.8.1). Then

(a) $\ln|f(z)| \le \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t)\ln|f(Re^{it})|\,dt$.

If in addition, $f(0) \ne 0$, then

(b) $\ln|f(0)| = \sum_{j=1}^n k_j \ln|a_j/R| + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{it})|\,dt$, hence

(c) $\ln|f(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{it})|\,dt$.

Part (b) is known as *Jensen's formula*.

Proof. It follows from (4.6.1) and the proof of (4.8.1) that

$$\left|\frac{R(z - a_j)}{R^2 - \overline{a}_j z}\right| < 1, \quad \text{hence} \quad k_j \ln\left|\frac{R(z - a_j)}{R^2 - \overline{a}_j z}\right| < 0,$$

proving (a). Part (b) follows from (4.8.1) with $z = 0$, and (c) follows from (b). ♣

Jensen's formula (4.8.2b) does not apply when $f(0) = 0$, and the Poisson-Jensen formula (4.8.1) requires that f have no zeros on $C(0, R)$. It is natural to ask whether any modifications of our formulas are available so that these situations are covered.

First, if f has a zero of order k at 0, with $f(z) \ne 0$ for $|z| = R$, then the left side of Jensen's formula is modified to $k \ln R + \ln|f^{(k)}(0)/k!|$ rather than $\ln|f(0)|$. This can be verified by considering $f(z)/z^k$ and is left as Problem 1 at the end of the section.

However, if $f(z) = 0$ for some $z \in C(0, R)$, then the situation is complicated for several reasons. For example, it is possible that $f(z) = 0$ for infinitely many points on $C(0, R)$ without being identically zero on $\overline{D}(0, R)$ if f is merely assumed continuous on $\overline{D}(0, R)$ and analytic on $D(0, R)$. Thus $\ln|f(z)| = -\infty$ at infinitely many points in $C(0, R)$ and so the Poisson integral of $\ln|f|$ does not à priori exist. It turns out that the integral does exist in the sense of Lebesgue, but Lebesgue integration is beyond the scope of this text. Thus we will be content with a version of the Poisson-Jensen formula requiring analyticity on $\overline{D}(0, R)$, but allowing zeros on the boundary.

4.8.3 Poisson-Jensen Formula, Second Version

Let f be analytic and not identically zero on $\overline{D}(0, R)$. Let $a_1, \dots, a_n$ be the zeros of f in $D(0, R)$, with multiplicities $k_1, \dots, k_n$ respectively. Then for $z \in D(0, R) \setminus Z(f)$,

$$\ln|f(z)| = \sum_{j=1}^n k_j \ln\left|\frac{R(z - a_j)}{R^2 - \overline{a}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t)\ln|f(Re^{it})|\,dt$$

where the integral exists as an improper Riemann integral.

Proof. Suppose that in addition to $a_1, \dots, a_n$, f has zeros on $C(0,R)$ at $a_{n+1}, \dots, a_m$ with multiplicities $k_{n+1}, \dots, k_m$. There is an analytic function g on $\overline{D}(0,R)$ with no zeros on $C(0,R)$ such that

$$f(z) = (z - a_{n+1})^{k_{n+1}} \cdots (z - a_m)^{k_m} g(z). \tag{1}$$

The function g satisfies the hypothesis of (4.8.1) and has the same zeros as f in $D(0,R)$. Now if $z \in D(0,R) \setminus Z(f)$, then

$$\ln|f(z)| = \sum_{j=1}^{n} k_j \ln|z - a_j| + \ln|g(z)|.$$

But by applying (4.8.1) to g we get

$$\ln|g(z)| = \sum_{j=1}^{n} \ln\left|\frac{R(z - a_j)}{R^2 - \overline{a}_j z}\right| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t) \ln|g(Re^{it})|\, dt,$$

so the problem reduces to showing that

$$\sum_{j=n+1}^{m} k_j \ln|z - a_j| + \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t) \ln|g(Re^{it})|\, dt = \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t) \ln|f(Re^{it})|\, dt.$$

Since by (1), $f(Re^{it}) = [\prod_{j=1}^{n}(Re^{it} - a_j)^{k_j}]g(Re^{it}), 0 \le t \le 2\pi$, we see that it is sufficient to show that

$$\ln|z - a_j| = \frac{1}{2\pi}\int_0^{2\pi} P_{z/R}(t) \ln|Re^{it} - a_j|\, dt$$

for $j = n+1, \dots, m$. In other words, the Poisson integral formula (4.7.8) holds for the functions $u(z) = \ln|z - a|$ when $|a| = R$ (as well as for $|a| < R$. This is essentially the content of the following lemma, where to simplify the notation we have taken $R = 1$ and $a = 1$

4.8.4 Lemma

For $|z| < 1$,

$$\ln|z - 1| = \frac{1}{2\pi}\int_0^{2\pi} P_z(t) \ln|e^{it} - 1|\, dt,$$

where the integral is to be understood as an improper Riemann integral at 0 and 2π. In particular,

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|e^{it} - 1|\, dt = 0.$$

Proof. We note first that the above improper integral exists, because if $0 \le t \le \pi$, then $|e^{it} - 1| = \sqrt{2(1 - \cos t)} = 2 \sin(t/2) \ge 2t/\pi$. Therefore

$$P_z(t) \ln |e^{it} - 1| \ge P_z(t) \ln(2t/\pi) = P_z(t)[\ln(2/\pi) + \ln t].$$

Since the improper integral $\int_0^{\pi/2} \ln t \, dt$ exists by elementary calculus and $P_z(t)$ is continuous, the above inequalities imply that

$$\lim_{\delta \to 0^+} \frac{1}{2\pi} \int_\delta^\pi P_z(t) \ln |e^{it} - 1| \, dt > -\infty \text{ and } \lim_{\delta \to 0^+} \frac{1}{2\pi} \int_\pi^{2\pi - \delta} P_z(t) \ln |e^{it} - 1| \, dt > -\infty.$$

Thus it remains to show that the *value* of the improper Riemann integral in the statement of the lemma is $\ln |z - 1|$. We will use a limit argument to evaluate the integral

$$I = \frac{1}{2\pi} \int_0^{2\pi} P_z(t) \ln |e^{it} - 1| \, dt.$$

For $r > 1$, define

$$I_r = \frac{1}{2\pi} \int_0^{2\pi} P_z(t) \ln |e^{it} - r| \, dt.$$

We will show that $I_r \to I$ as $r \to 1^+$. Now for any fixed $r > 1$, the function $z \to \ln |z - r|$ is continuous on $\overline{D}(0,1)$ and harmonic in $D(0,1)$, hence by (4.7.8), $\ln |z - r| = I_r$. Since $\ln |z - r| \to \ln |z - 1|$ as $r \to 1^+$, this will show that $I = \ln |z - 1|$, completing the proof. So consider, for $r > 1$,

$$|I_r - I| = \left| \frac{1}{2\pi} \int_0^{2\pi} P_z(t) \ln \left| \frac{e^{it} - r}{e^{it} - 1} \right| \, dt \right| = \frac{1}{2\pi} \int_0^{2\pi} P_z(t) \ln \left| \frac{e^{it} - r}{e^{it} - 1} \right| \, dt.$$

(The outer absolute values may be removed because $|e^{it} - r| > |e^{it} - 1|$ and therefore the integrand is positive.) Using the 2π-periodicity of the integrand, we may write

$$\int_0^{2\pi} = \int_{-\pi}^{\pi} = \int_{-\pi}^{0} + \int_0^{\pi}$$

and since $P_z(-t) = P_z(t)$, this becomes

$$\int_0^\pi + \int_0^\pi = 2 \int_0^\pi .$$

Now if $0 \le t \le \pi$, then

$$\frac{e^{it} - r}{e^{it} - 1} = \frac{e^{it} - 1 + 1 - r}{e^{it} - 1} = 1 + \frac{1 - r}{e^{it} - 1}.$$

But as we noted at the beginning of the proof, $|e^{it} - 1| \ge 2t/\pi$, so the above expression is bounded in absolute value by $1 + [\pi(r - 1)/2t]$. Thus

$$0 < \ln \left| \frac{e^{it} - r}{e^{it} - 1} \right| \le \ln \left[1 + \frac{\pi(r - 1)}{2t} \right].$$

Also, the Poisson kernel satisfies

$$P_z(t) = \frac{1-|z|^2}{|e^{it}-z|^2} \le \frac{(1-|z|)(1+|z|)}{(1-|z|)^2} = \frac{1+|z|}{1-|z|},$$

an estimate that was used to establish Harnack's inequality. (See the solution to Section 4.7, Problem 2. Thus we now have

$$0 < |I_r - I| \le \frac{1+|z|}{1-|z|} \cdot \frac{1}{\pi} \int_0^\pi \ln\left[1 + \frac{\pi(r-1)}{2t}\right] dt.$$

Now fix $\delta, 0 < \delta < \pi$, and write

$$\int_0^\pi \ln\left[1 + \frac{\pi(r-1)}{2t}\right] dt = \int_0^\delta \ln\left[1 + \frac{\pi(r-1)}{2t}\right] dt + \int_\delta^\pi \ln\left[1 + \frac{\pi(r-1)}{2t}\right] dt.$$

Since the integral on the left side is finite (this is essentially the same as saying that $\int_0^\pi \ln t\, dt > -\infty$), and the integrand increases as r increases ($r > 1$), the first integral on the right side approaches 0 as $\delta \to 0^+$, uniformly in r. On the other hand, the second integral on the right side is bounded by $(\pi - \delta)\ln(1 + [\pi(r-1)/2\delta])$, which for fixed $\delta > 0$, approaches 0 as $r \to 1^+$. This completes the proof of the lemma, and as we noted earlier, finishes the proof of (4.8.3). ♣

The Poisson-Jensen formula has a number of interesting corollaries, some of which will be stated below. The proof of the next result (4.8.5), as well as other consequences, will be left for the problems.

4.8.5 Jensen's Formula, General Case

Let f be analytic on an open disc $D(0,R)$ and assume that $f \not\equiv 0$. Assume that f has a zero of order $k \ge 0$ at 0 and $a_1, a_2, \dots$ are the zeros of f in $D(0,R) \setminus \{0\}$, each appearing as often as its multiplicity and arranged so that $0 < |a_1| \le |a_2| \le \cdots$. Then for $0 < r < R$ we have

$$k \ln r + \ln\left|\frac{f^{(k)}(0)}{k!}\right| = \sum_{j=1}^{n(r)} \ln\left|\frac{a_j}{r}\right| + \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{it})|\, dt$$

where $n(r)$ is the number of terms of the sequence $a_1, a_2, \dots$ that are in the disk $D(0,r)$.

Problems

1. Prove (4.8.5).
2. Let f be as in (4.8.2), except that instead of being analytic on all of $D(0,R)$, f has poles at $b_1, \dots, b_m$ in $D(0,R) \setminus \{0\}$, of orders $l_1, \dots, l_m$ respectively. State and prove an appropriate version of Jensen's formula in this case.
3. Let $n(r)$ be as in (4.8.5). Show that

$$\int_0^r \frac{n(t)}{t}\, dt = \sum_{j=1}^{n(r)} \ln \frac{r}{|a_j|}.$$

4. With f as in (4.8.5) and $M(r) = \max\{|f(z)| : |z| = r\}$, show that for $0 < r < R$,

$$\int_0^r \frac{n(t)}{t}\,dt \le \ln\left[\frac{M(r)}{|f^{(k)}(0)|r^k/k!}\right].$$

5. Let f be as in (4.8.5). Show that the function

$$r \to \frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{it})|\,dt$$

is increasing, and discuss the nature of its graph on the interval $(0, R)$.

4.9 Analytic Continuation

In this section we examine the problem of extending an analytic function to a larger domain. An example of this has already been encountered in the Schwarz reflection principle (2.2.15). We first consider a function defined by a power series.

4.9.1 Definition

Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ have radius of convergence $r, 0 < r < \infty$. Let z^* be a point such that $|z^* - z_0| = r$ and let $r(t)$ be the radius of convergence of the expansion of f about the point $z_1 = (1 - t)z_0 + tz^*, 0 < t < 1$. Then $r(t) \ge (1 - t)r$ (Figure 4.9.1). If

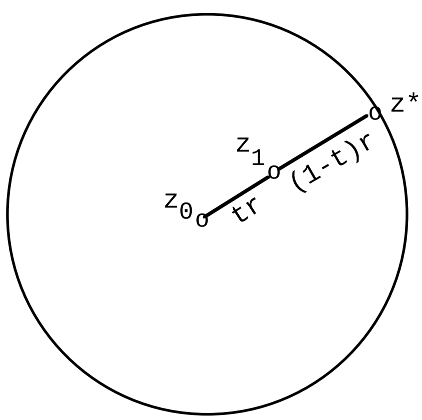

Figure 4.9.1

$r(t) = (1 - t)r$ for some (hence for all) $t \in (0, 1)$, so that there is *no* function g analytic on an open set containing $D(z_0, r) \cup \{z^*\}$ and such that $g = f$ on $D(z_0, r)$, then z^* is said to be a *singular point* of f. Equivalently, $z^* \in D(0, r)$ is *not* a singular point of f iff f has an analytic extension to an open set containing $D(z_0, r) \cup \{z^*\}$.

We are going to show that there is always at least one singular point on the circle of convergence, although in general, its exact location will not be known. Before doing this, we consider a special case in which it *is* possible to locate a singular point.

4.9.2 Theorem

In (4.9.1), if a_n is real and nonnegative for all n, then $z_0 + r$ is a singular point.

Proof. Fix z_1 between z_0 and $z_0 + r$. Note that since $a_n \geq 0$ for all n and $z_1 - z_0$ is a positive real number, f and its derivatives are nonnegative at z_1. Now assume, to the contrary, that the Taylor series expansion of f about z_1 *does* converge for some z_2 to the right of $z_0 + r$. Then we have

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!}(z_2 - z_1)^k < +\infty.$$

But by the remark after (2.2.18),

$$f^{(k)}(z_1) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z_1 - z_0)^{n-k}$$

for $k = 0, 1, 2, \ldots$. Substituting this for $f^{(k)}(z_1)$ in the Taylor expansion of f about z_1 and using the fact that the order of summation in a double series with nonnegative terms can always be reversed, we get

$$\begin{aligned} +\infty > & \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z_1 - z_0)^{n-k} \right] \frac{(z_2 - z_1)^k}{k!} \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} \binom{n}{k} a_n (z_1 - z_0)^{n-k} \right] (z_2 - z_1)^k \\ &= \sum_{n=0}^{\infty} a_n \left[\sum_{k=0}^{n} \binom{n}{k} (z_1 - z_0)^{n-k}(z_2 - z_1)^k \right] \\ &= \sum_{n=0}^{\infty} a_n (z_2 - z_0)^n \end{aligned}$$

by the binomial theorem. But this implies that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence greater than r, a contradiction. ♣

The preceding theorem is illustrated by the geometric series $1 + z + z^2 + \cdots$, which has radius of convergence equal to 1 and which converges to $1/(1-z)$ for $|z| < 1$. In this case, $z^* = 1$ is the only singular point, but as we will see later, the other extreme is also possible, namely that *every* point on the circle of convergence is a singular point.

4.9.3 Theorem

In (4.9.1), let $\Gamma = \{z : |z - z_0| = r\}$ be the circle of convergence. Then there is at least one singular point on Γ.

Proof. If $z \in \Gamma$ is not a singular point, then there is a function f_z analytic on a disk $D(z, \epsilon_z)$ such that $f_z = f$ on $D(z_0, r) \cap D(z, \epsilon_z)$. Say there are no singular points on Γ.

By compactness, Γ is covered by finitely many such disks, say by $D(z_j, \epsilon_j), j = 1, \dots, n$. Define

$$g(z) = \begin{cases} f(z), & z \in D(z_0, r) \\ f_{z_j}(z), & z \in D(z_j, \epsilon_j), j = 1, \dots, n. \end{cases}$$

We show that g is well defined. If $D(z_j, \epsilon_j) \cap D(z_k, \epsilon_k) \neq \emptyset$, then also $D(z_j, \epsilon_j) \cap D(z_k, \epsilon_k) \cap D(z_0, r) \neq \emptyset$, as is verified by drawing a picture. Now $f_{z_j} - f_{z_k} = f - f = 0$ on $D(z_j, \epsilon_j) \cap D(z_k, \epsilon_k)$ by the identity theorem (2.4.8), proving that g is well defined. Thus g is analytic on $D(z_0, s)$ for some $s > r$, and the Taylor expansion of g about z_0 coincides with that of f since $g = f$ on $D(z_0, r)$. This means that the expansion of f converges in a disk of radius greater than r, a contradiction. ♣

We are now going to construct examples of power series for which the circle of convergence is a *natural boundary*, that is, every point on the circle of convergence is a singular point. The following result will be needed.

4.9.4 Lemma

Let $f_1(w) = (w^p + w^{p+1})/2$, p a positive integer. Then $|w| < 1$ implies $|f_1(w)| < 1$, and if $\Omega = D(0, 1) \cup D(1, \epsilon), \epsilon > 0$, then $f_1(D(0, r)) \subseteq \Omega$ for some $r > 1$.

Proof. If $|w| \leq 1$, then $|f_1(w)| = |w|^p |1 + w|/2 \leq |1 + w|/2$, which is less than 1 unless $w = 1$, in which case $f_1(w) = 1$. Thus $|w| < 1$ implies $|f_1(w)| < 1$, and $f_1(\overline{D}(0, 1)) \subseteq \Omega$. Hence $f_1^{-1}(\Omega)$ is an open set containing $\overline{D}(0, 1)$. Consequently, there exists $r > 1$ such that $D(0, r) \subseteq f_1^{-1}(\Omega)$, from which it follows that $f_1(D(0, r) \subseteq f_1(f_1^{-1}(\Omega)) \subseteq \Omega$. ♣

The construction of natural boundaries is now possible.

4.9.5 Hadamard Gap Theorem

Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ and, for some $s > 1$, $n_{k+1}/n_k \geq s$ for all k. (We say that $\sum_k a_k z^{n_k}$ is a *gap series*.) If the radius of convergence of the series is 1, then every point on the circle of convergence is a singular point.

Proof. We will show that 1 is a singular point, from which it will follow (under these hypotheses) that every point on the unit circle is a singular point. Thus assume, to the contrary, that 1 is not a singular point. Then, for some $\epsilon > 0$, f has an analytic extension g to $D(0, 1) \cup D(1, \epsilon)$. Let p be a positive integer such that $s > (p + 1)/p$, and let f_1 and $r > 1$ be as in Lemma 4.9.4. Then $h(w) = g(f_1(w))$ is analytic on $D(0, r)$, and for

$|w| < 1$,

$$\begin{aligned} g(f_1(w)) = f(f_1(w)) &= \sum_{k=1}^{\infty} a_k (f_1(w))^{n_k} \\ &= \sum_{k=1}^{\infty} a_k 2^{-n_k} (w^p + w^{p+1})^{n_k} \\ &= \sum_{k=1}^{\infty} a_k 2^{-n_k} \sum_{n=0}^{n_k} \binom{n_k}{n} w^{p(n_k-n)} w^{(p+1)n} \\ &= \sum_{k=1}^{\infty} a_k 2^{-n_k} \sum_{n=0}^{n_k} \binom{n_k}{n} w^{pn_k+n}. \end{aligned}$$

Now for each k we have $n_{k+1}/n_k \geq s > (p+1)/p$, so $pn_k + n_k < pn_{k+1}$. Therefore, the highest power of w that appears in $\sum_{n=0}^{n_k} \binom{n_k}{n} w^{pn_k+n}$, namely $w^{pn_k+n_k}$, is less than the lowest power $w^{pn_{k+1}}$ that appears in $\sum_{n=0}^{n_{k+1}} \binom{n_{k+1}}{n} w^{pn_{k+1}+n}$. This means that the series

$$\sum_{k=1}^{\infty} a_k 2^{-n_k} \sum_{n=0}^{n_k} \binom{n_k}{n} w^{pn_k+n}$$

is (with a grouping of terms) precisely the Taylor expansion of h about $w = 0$. But since h is analytic on $D(0, r)$, this expansion converges absolutely on $D(0, r)$, hence (as there are no repetition of powers of w),

$$\sum_{k=1}^{\infty} |a_k| 2^{-n_k} \sum_{n=0}^{n_k} \binom{n_k}{n} |w|^{pn_k+n} < \infty,$$

that is [as in the above computation of $g(f_1(w))$],

$$\sum_{k=1}^{\infty} |a_k| 2^{-n_k} (|w|^p + |w|^{p+1})^{n_k} < \infty$$

for $|w| < r$. But if $1 < |w| < r$, then $2^{-n_k}(|w|^p + |w|^{p+1})^{n_k} = \left[|w|^p(\frac{1+|w|}{2})\right]^{n_k} > 1$. Consequently, $\sum_{k=1}^{\infty} a_k z^{n_k}$ converges for some z with $|z| > 1$, contradicting the assumption that the series defining f has radius of convergence 1.

Finally, if $z^* = e^{i\theta}$ is not a singular point, let $q(z) = f(e^{i\theta} z) = \sum_{k=1}^{\infty} a_k e^{i\theta n_k} z^{n_k}$ (with radius of convergence 1, as before, because $|e^{i\theta n_k}| = 1$). Now f extends to a function analytic on $D(0, 1) \cup D(z^*, \epsilon)$ for some $\epsilon > 0$, and thus q extends to a function analytic on $D(0, 1) \cup D(1, \epsilon)$, contradicting the above argument. ♣

Some typical examples of gap series are $\sum_{k=1}^{\infty} z^{2^k}$ and $\sum_{k=1}^{\infty} z^{k!}$.

Remarks

The series $\sum_{n=0}^{\infty} z^n$ diverges at every point of the circle of convergence since $|z|^n$ does not approach 0 when $|z| = 1$. However, $z = 1$ is the only singular point since $(1 - z)^{-1}$

is analytic except at $z = 1$. On the other hand, $\sum_{n=1}^{\infty} \frac{1}{n!} z^{2^n}$ has radius of convergence 1, for if $a_k = 0, k \neq 2^n; a_{2^n} = 1/n!$, then

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_{2^n}|^{1/2^n} = \limsup_{n\to\infty} (1/n!)^{1/2^n} = 1$$

because

$$\ln[(n!)^{1/2^n}] = 2^{-n} \ln(n!) = 2^{-n} \sum_{k=1}^{n} \ln k \leq 2^{-n} n \ln n \to 0.$$

The series converges (as does every series obtained from it by termwise differentiation) at each point of the circle of convergence, and yet by (4.9.5), each such point is singular.

The conclusion of Theorem 4.9.5 holds for any (finite) radius of convergence. For if $\sum a_k z^{n_k}$ has radius of convergence r, then $\sum a_k (rz)^{n_k}$ has radius of convergence 1.

We now consider chains of functions defined by power series.

4.9.6 Definitions

A *function element* in Ω is a pair (f, D), where D is a disk contained in Ω and f is analytic on D. (The convention $D = D(0, 1)$ is no longer in effect.) If z is an element of D, then (f, D) is said to be a *function element at z*. Two function elements (f_1, D_1) and (f_2, D_2) in Ω are *direct analytic continuations* of each other (relative to Ω) if $D_1 \cap D_2 \neq \emptyset$ and $f_1 = f_2$ on $D_1 \cap D_2$. Note that in this case, $f_1 \cup f_2$ is an extension of f_1 (respectively f_2) from D_1 (respectively D_2) to $D_1 \cup D_2$. If there is a *chain* $(f_1, D_1), (f_2, D_2), \ldots, (f_n, D_n)$ of function elements in Ω, with (f_i, D_i) and (f_{i+1}, D_{i+1}) direct analytic continuations of each other for $i = 1, 2, \ldots, n-1$, then (f_1, D_1) and (f_n, D_n) are said to be *analytic continuations* of each other relative to Ω.

Suppose that γ is a curve in Ω, with γ defined on the interval $[a, b]$. If there is a partition $a = t_0 < t_1 < \cdots < t_n = b$, and a chain$(f_1, D_1), \ldots, (f_n, D_n)$ of function elements in Ω such that (f_{i+1}, D_{i+1}) is a direct analytic continuation of (f_i, D_i) for $i = 1, 2, \ldots, n-1$, and $\gamma(t) \in D_i$ for $t_{i-1} \leq t \leq t_i, i = 1, 2, \ldots, n$, then (f_n, D_n) is said to be an *analytic continuation* of (f_1, D_1) *along the curve* γ.

4.9.7 Theorem

Analytic continuation of a *given* function element along a *given* curve is unique, that is, if (f_n, D_n) and (g_m, E_m) are two continuations of (f_1, D_1) along the same curve γ, then $f_n = g_m$ on $D_n \cap E_m$.

Proof. Let the first continuation be $(f_1, D_1), \ldots, (f_n, D_n)$, and let the second continuation be $(g_1, E_1), \ldots, (g_m, E_m)$, where $g_1 = f_1, E_1 = D_1$. There are partitions $a = t_0 < t_1 < \cdots < t_n = b, a = s_0 < s_1 < \cdots < s_m = b$ such that $\gamma(t) \in D_i$ for $t_{i-1} \leq t \leq t_i, i = 1, 2, \ldots, n, \gamma(t) \in E_j$ for $s_{j-1} \leq t \leq s_j, j = 1, 2, \ldots, m$.

We claim that if $1 \leq i \leq n, 1 \leq j \leq m$, and $[t_{i-1}, t_i] \cap [s_{j-1}, s_j] \neq \emptyset$, then (f_i, D_i) and (g_j, E_j) are direct analytic continuations of each other. This is true when $i = j = 1$, since $g_1 = f_1$ and $E_1 = D_1$. If it is not true for all i and j, pick from all (i, j) for which the statement is false a pair such that $i+j$ is minimal. Say $t_{i-1} \geq s_{j-1}$ [then $i \geq 2$, for if $i = 1$, then $s_{j-1} = t_0 = a$, hence $j = 1$, and we know that the result holds for the pair (1,1)]. We have $t_{i-1} \leq s_j$ since $[t_{i-1}, t_i] \cap [s_{j-1}, s_j] \neq \emptyset$, hence $s_{j-1} \leq t_{i-1} \leq s_j$. Therefore $\gamma(t_{i-1}) \in D_{i-1} \cap D_i \cap E_j$, in particular, this intersection is not empty. Now (f_i, D_i) is a direct analytic continuation of (f_{i-1}, D_{i-1}), and furthermore (f_{i-1}, D_{i-1}) is a direct analytic continuation of (g_j, E_j) by minimality of $i + j$ (note that $t_{i-1} \in [t_{i-2}, t_{i-1}] \cap [s_{j-1}, s_j]$, so the hypothesis of the claim is satisfied). Since $D_{i-1} \cap D_i \cap E_j \neq \emptyset, (f_i, D_i)$ must be a direct continuation of $g_j, E_j)$, a contradiction. Thus the claim holds for all i and j, in particular for $i = n$ and $j = m$. The result follows. ♣

4.9.8 Definition

Let Ω be an open connected subset of $\mathbb{C}$. The function elements (f_1, D_1) and (f_2, D_2) in Ω are said to be *equivalent* if they are analytic continuations of each other relative to Ω. (It is immediate that this is an equivalence relation.) An equivalence class Φ of function elements in Ω such that for every $z \in \Omega$ there is an element $(f, D) \in \Phi$ with $z \in D$ is called a *generalized analytic function* on Ω.

Note that connectedness of Ω is necessary in this definition if there are to be any generalized analytic functions on Ω at all. For if $z_1, z_2 \in \Omega$, there must exist equivalent function elements (f_1, D_1) and (f_2, D_2) at z_1 and z_2 respectively. This implies that there is a curve in Ω joining z_1 to z_2.

Note also that if g is analytic on Ω, then g determines a generalized analytic function Φ on Ω in the following sense. Take

$$\Phi = \{(f, D) : D \subseteq \Omega \text{ and } f = g|_D\}.$$

However, not every generalized analytic function arises from a single analytic function in this way (see Problem 2). The main result of this section, the monodromy theorem (4.9.11), addresses the question of when a generalized analytic function *is* determined by a single analytic function.

4.9.9 Definition

Let γ_0 and γ_1 be curves in a set $S \subseteq \mathbb{C}$ (for convenience, let γ_0 and γ_1 have common domain $[a, b]$). Assume $\gamma_0(a) = \gamma_1(a) = z_0, \gamma_0(b) = \gamma_1(b) = z_1$, that is, the curves have the same endpoints. Then γ_0 and γ_1 are said to be *homotopic* (in S) if there is a continuous map $H : [a, b] \times [0, 1] \to S$ (called a *homotopy* of γ_0 and γ_1) such that $H(t, 0) = \gamma_0(t), H(t, 1) = \gamma_1(t), a \leq t \leq t; H(a, s) = z_0, H(b, s) = z_1, 0 \leq s \leq 1$. Intuitively, H deforms γ_0 into γ_1 without moving the endpoints or leaving the set S. For $0 \leq s \leq 1$, the curve $t \to H(t, s)$ represents the state of the deformation at "time s".

4.9.10 Theorem

Let Ω be an open connected subset of $\mathbb{C}$, and let γ_0, γ_1 be curves in Ω that are homotopic in Ω. Let (f, D) be a function element at z_0, the initial point of γ_0 and γ_1. Assume that

(f, D) can be continued along all possible curves in Ω, that is, if γ is a curve in Ω joining z_0 to another point z_n, there is an analytic continuation (f_n, D_n) of (f, D) along γ.

If (g_0, D_0) is a continuation of (f, D) along γ_0 and (g_1, D_1) is a continuation of (f, D) along γ_1, then $g_0 = g_1$ on $D_0 \cap D_1$. (Note that $D_0 \cap D_1 \neq \emptyset$ since the terminal point z_1 of γ_0 and γ_1 belongs to $D_0 \cap D_1$.) Thus (g_0, D_0) and (g_1, D_1) are direct analytic continuations of each other.

Proof. Let H be a homotopy of γ_0 and γ_1. By hypothesis, if $0 \leq s \leq 1$, then (f, D) can be continued along the curve $\gamma_s = H(\cdot\ , s)$, say to (g_s, D_s). Fix s and pick one such continuation, say $(h_1, E_1), \ldots, (h_n, E_n)$ [with $(h_1, E_1) = (f, D), (h_n, E_n) = (g_s, D_s)$]. There is a partition $a = t_0 < t_1 < \cdots t_n = b$ such that $\gamma_s(t) \in E_i$ for $t_{i-1} \leq t \leq t_i, i = 1, \ldots, n$. Let K_i be the compact set $\gamma_s([t_{i-1}, t_i]) \subseteq E_i$, and let

$$\epsilon = \min_{1 \leq i \leq n} \{\operatorname{dist}(K_i, \mathbb{C} \setminus E_i\} > 0.$$

Since H is uniformly continuous, there exists $\delta > 0$ such that if $|s - s_1| < \delta$, then $|\gamma_s(t) - \gamma_{s_1}(t)| < \epsilon$ for all $t \in [a, b]$. In particular, if $t_{i-1} \leq t \leq t_i$, then since $\gamma_s(t) \in K_i$ and $|\gamma_s(t) - \gamma_{s_1}(t)| < \epsilon$, we have $\gamma_{s_1}(t) \in E_i$.

Thus by definition of continuation along a curve, $(h_1, E_1), \ldots, (h_n, E_n)$ is also a continuation of (f, D) along γ_{s_1}. But we specified at the beginning of the proof that (f, D) is continued along γ_{s_1} to (g_{s_1}, D_{s_1}). By (4.9.7), $g_s = g_{s_1}$ on $D_s \cap D_{s_1}$. Thus for each $s \in [0, 1]$ there is an open interval I_s such that $g_s = g_{s_1}$ on $D_s \cap D_{s_1}$ whenever $s_1 \in I_s$. Since $[0, 1]$ can be covered by finitely many such intervals, it follows that $g_0 = g_1$ on $D_0 \cap D_1$. ♣

4.9.11 Monodromy Theorem

Let Ω be an open connected subset of $\mathbb{C}$ with the property that every closed curve γ in Ω is homotopic to a point, that is, homotopic (in Ω) to $\gamma_0 \equiv z$, where z is the initial and terminal point of γ. Let Φ be a generalized analytic function on Ω, and assume that each element of Φ can be continued along all possible curves in Ω. Then there is a function g analytic on Ω such that whenever $(f, D) \in \Phi$ we have $g = f$ on D. Thus Φ is determined by a single analytic function.

Proof. If $z \in \Omega$ there is a function element $(f, D) \in \Phi$ such that $z \in D$. Define $g(z) = f(z)$. We must show that g is well defined. If $(f^*, D^*) \in \Phi$ and $z \in D^*$, we have to show that $f(z) = f^*(z)$. But since $(f, D), (f^*, D^*) \in \Phi$, there is a continuation in Ω from (f, D) to (f^*, D^*); since $z \in D \cap D^*$, we can find a curve γ (in fact a polygonal path) in Ω with initial and terminal point z such that the continuation is along γ. But by hypothesis, γ is homotopic to the curve $\gamma_0 \equiv z$. Since (f, D) is a continuation of (f, D) along γ_0, it follows from (4.9.10) that $f = f^*$ on $D \cap D^*$, in particular, $f(z) = f^*(z)$. Since $g = f$ on D, g is analytic on Ω. ♣

Remarks

Some authors refer to (4.9.10), rather than (4.9.11), as the monodromy theorem. Still others attach this title to our next result (4.9.13), which is a corollary of (4.9.11). It is appropriate at this point to assign a name to the topological property of Ω that appears in the hypothesis of (4.9.11).

4.9.12 Definition

Let Ω be a plane region, that is, an open connected subset of $\mathbb{C}$. We say that Ω is (homotopically) *simply connected* if every closed curve in Ω is homotopic (in Ω) to a point. In the next chapter, we will show that the homotopic and homological versions of simple connectedness are equivalent.

Using this terminology, we have the following corollary to the monodromy theorem (4.9.11).

4.9.13 Theorem

Let Ω be simply connected region and let (f, D) be a function element in Ω such that (f, D) can be continued along all curves in Ω whose initial points are in D. Then there is an analytic function g on Ω such that $g = f$ on D.

Proof (outline). Let Φ be the collection of all function elements (h, E) such that (h, E) is a continuation of (f, D). One can then verify that Φ satisfies the hypothesis of (4.9.11). Since $(f, D) \in \Phi$, the result follows. ♣

Alternatively, we need not introduce Φ at all, but instead imitate the proof of (4.9.11).

We conclude this section with an important and interesting application of analytic continuation in simply connected regions.

4.9.14 Theorem

If Ω is a (homotopically) simply connected region, then every harmonic function on Ω has a harmonic conjugate.

Proof. If u is harmonic on Ω, we must produce an analytic function g on Ω such that $u = \operatorname{Re} g$. We make use of previous results for disks; if D is a disk contained in Ω, then by (1.6.2), there is an analytic function f on D such that $\operatorname{Re} f = u$. That is, (f, D) is a function element in Ω with $\operatorname{Re} f = u$ on D.

If $\gamma : [a, b] \to \Omega$ is any curve in Ω such that $\gamma(a) \in D$, we need to show that (f, D) can be continued along γ. As in the proof of (3.1.7), there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ and disks $D_1, \dots, D_n$ with centers at $\gamma(t_1), \dots, \gamma_(t_n)$ respectively, such that if $t_{j-1} \le t \le t_j$, then $\gamma(t) \in D_j$. Now $D \cap D_1 \ne \emptyset$, and by repeating the above argument we see that there exists f_1 analytic on D_1 such that $\operatorname{Re} f_1 = u$ on D_1. Since $f - f_1$ is pure imaginary on $D \cap D_1$, it follows (from the open mapping theorem (4.3.1), for example) that $f - f_1$ is a purely imaginary constant on $D \cap D_1$. By adding this constant to f_1 on D_1, we obtain a new f_1 on D_1 such that (f_1, D_1) is a direct continuation of (f, D). Repeating this process with (f_1, D_1) and (f_2, D_2), and so on, we obtain a continuation (f_n, D_n) of (f, D) along γ. Thus by (4.9.13), there is an analytic function g on Ω such that $g = f$ on D. Then $\operatorname{Re} g = u$ on D, and hence by the identity theorem for harmonic functions (2.4.14), $\operatorname{Re} g = u$ on Ω. ♣

In the next chapter we will show that the converse of (4.9.14) holds. However, this will require a closer examination of the connection between homology and homotopy. Also, we can give an alternative (but less constructive) proof of (4.9.14) after proving the Riemann mapping theorem.

Problems

1. Let $f(z) = \sum_{n=0}^{\infty} z^{n!}, z \in D(0,1)$. Show directly that f has $C(0,1)$ as its natural boundary without appealing to the Hadamard gap theorem. (Hint: Look at f on radii which terminate at points of the form $e^{i2\pi p/q}$ where p and q are integers.)

2. Let $f(z) = \operatorname{Log} z = \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^n/n,\ z \in D = D(1,1)$. Let $\Omega = \mathbb{C} \setminus \{0\}$ and let Φ be the equivalence class determined by (f, D).
(a) Show that Φ is actually a generalized analytic function on Ω, that is, if $z \in \Omega$ then there is an element $(g, E) \in \Phi$ with $z \in E$.
(b) Show that there is no function h analytic on Ω such that for every $(g, E) \in \Phi$ we have $h = g$ on E.

3. Criticize the following argument: Let $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ have radius of convergence r. If $z_1 \in D(z_0, r)$, then by the rearrangement procedure of (4.9.2) we can find the Taylor expansion of f about z_1, namely
$$f(z) = \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} \binom{n}{k} a_n (z_1 - z_0)^{n-k} \right] (z - z_1)^k.$$
If the expansion about z_1 converges at some point $z \notin \overline{D}(z_0, r)$, then since power series converge absolutely inside the circle of convergence, we may rearrange the expansion about z_1 to show that the original expansion about z_0 converges at z, a contradiction. Consequently, for any function defined by a power series, the circle of convergence is a natural boundary.

4. (Law of permanence of functional equations). Let $F : \mathbb{C}^{k+1} \to \mathbb{C}$ be such that F and all its first order partial derivatives are continuous. Let $f_1, \dots, f_k$ be analytic on a disk D, and assume that $F(z, f_1(z), \dots, f_k(z)) = 0$ for all $z \in D$. Let $(f_{i1}, D_1), (f_{i2}, D_2), \dots, (f_{in}, D_n)$, with $f_{i1} = f_i, D_1 = D$, form a continuation of $(f_i, D), i = 1, \dots, k$. Show that $F(z, f_{1n}(z), \dots, f_{kn}(z)) = 0$ for all $z \in D_n$. An example: If $e^g = f$ on D and the continuation carries f into f^* and g into g^*, then $e^{g^*} = f^*$ on D_n (take $F(z, w_1, w_2) = w_1 - e^{w_2}, f_1 = f, f_2 = g$).

5. Let (f^*, D^*) be a continuation of (f, D). Show that $(f^{*\prime}, D^*)$ is a continuation of (f', D). ("The derivative of the continuation, that is, $f^{*\prime}$, is the continuation of the derivative.")

Reference

W. Rudin, "Real and Complex Analysis," 3rd ed., McGraw Hill Series in Higher Mathematics, New York, 1987.