# Chapter 3

# The General Cauchy Theorem

In this chapter, we consider two basic questions. First, for a given open set $\Omega$, we try to determine which closed paths $\gamma$ in $\Omega$ have the property that $\int_\gamma f(z)\,dz = 0$ for *every* analytic function $f$ on $\Omega$. Then second, we try to characterize those open sets $\Omega$ having the property that $\int_\gamma f(z)\,dz = 0$ for all closed paths $\gamma$ in $\Omega$ and all analytic functions $f$ on $\Omega$. The results, which may be grouped under the name "Cauchy's theorem", form the cornerstone of analytic function theory.

A basic concept in the general Cauchy theory is that of *winding number* or *index* of a point with respect to a closed curve not containing the point. In order to make this precise, we need several preliminary results on logarithm and argument functions.

## 3.1 Logarithms and Arguments

In (2.3.1), property (j), we saw that given a real number $\alpha$, the exponential function when restricted to the strip $\{x+iy : \alpha \le y < \alpha + 2\pi\}$ is a one-to-one analytic map of this strip onto the nonzero complex numbers. With this in mind, we make the following definition.

### 3.1.1 Definition

We take $\log_\alpha$ to be the inverse of the exponential function restricted to the strip $S_\alpha = \{x+iy : \alpha \le y < \alpha + 2\pi\}$. We define $\arg_\alpha$ to the the imaginary part of $\log_\alpha$.

Consequently, $\log_\alpha(\exp z) = z$ for each $z \in S_\alpha$, and $\exp(\log_\alpha z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$.

Several important properties of $\log_\alpha$ and $\arg_\alpha$ follow readily from Definition 3.1.1 and the basic properties of exp.

### 3.1.2 Theorem

(a) If $z \ne 0$, then $\log_\alpha(z) = \ln|z| + i\arg_\alpha(z)$, and $\arg_\alpha(z)$ is the unique number in $[\alpha, \alpha+2\pi)$ such that $z/|z| = e^{i\arg_\alpha(z)}$, in other words, the unique argument of $z$ in $[\alpha, \alpha+2\pi)$.

(b) Let $R_\alpha$ be the ray $[0, e^{i\alpha}, \infty) = \{re^{i\alpha} : r \geq 0\}$. The functions $\log_\alpha$ and $\arg_\alpha$ are continuous at each point of the "slit" complex plane $\mathbb{C} \setminus R_\alpha$, and discontinuous at each point of $R_\alpha$.

(c) The function $\log_\alpha$ is analytic on $\mathbb{C} \setminus R_\alpha$, and its derivative is given by $\log'_\alpha(z) = 1/z$.

*Proof.*

(a) If $w = \log_\alpha(z), z \neq 0$, then $e^w = z$, hence $|z| = e^{\operatorname{Re} w}$ and $z/|z| = e^{i \operatorname{Im} w}$. Thus $\operatorname{Re} w = \ln |z|$, and $\operatorname{Im} w$ is an argument of $z/|z|$. Since $\operatorname{Im} w$ is restricted to $[\alpha, \alpha + 2\pi)$ by definition of $\log_\alpha$, it follows that $\operatorname{Im} w$ is the unique argument for $z$ that lies in the interval $[\alpha, \alpha + 2\pi)$.

(b) By (a), it suffices to consider $\arg_\alpha$. If $z_0 \in \mathbb{C} \setminus R_\alpha$ and $\{z_n\}$ is a sequence converging to $z_0$, then $\arg_\alpha(z_n)$ must converge to $\arg_\alpha(z_0)$. (Draw a picture.) On the other hand, if $z_0 \in R_\alpha \setminus \{0\}$, there is a sequence $\{z_n\}$ converging to $z_0$ such that $\arg_\alpha(z_n) \to \alpha + 2\pi \neq \arg_\alpha(z_0) = \alpha$.

(c) This follows from Theorem 1.3.2 (with $g = \exp, \Omega_1 = \mathbb{C}, f = \log_\alpha$, and $\Omega = \mathbb{C} \setminus R_\alpha$) and the fact that exp is its own derivative. ♣

### 3.1.3 Definition

The *principal branches* of the logarithm and argument functions, to be denoted by Log and Arg, are obtained by taking $\alpha = -\pi$. Thus, $\operatorname{Log} = \log_{-\pi}$ and $\operatorname{Arg} = \arg_{-\pi}$.

### Remark

The definition of principal branch is not standardized; an equally common choice for $\alpha$ is $\alpha = 0$. Also, having made a choice of principal branch, one can define $w^z = \exp(z \operatorname{Log} w)$ for $z \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$. We will not need this concept, however.

### 3.1.4 Definition

Let $S$ be a subset of $\mathbb{C}$ (or more generally any metric space), and let $f : S \to \mathbb{C} \setminus \{0\}$ be continuous. A function $g : S \to \mathbb{C}$ is a *continuous logarithm* of $f$ if $g$ is continuous on $S$ and $f(s) = e^{g(s)}$ for all $s \in S$. A function $\theta : S \to \mathbb{R}$ is a *continuous argument* of $f$ if $\theta$ is continuous on $S$ and $f(s) = |f(s)|e^{i\theta(s)}$ for all $s \in S$.

### 3.1.5 Examples

(a) If $S = [0, 2\pi]$ and $f(s) = e^{is}$, then $f$ has a continuous argument on $S$, namely $\theta(s) = s + 2k\pi$ for any fixed integer $k$.

(b) If for some $\alpha$, $f$ is a continuous mapping of $S$ into $\mathbb{C} \setminus R_\alpha$, then $f$ has a continuous argument, namely $\theta(s) = \arg_\alpha(f(s))$.

(c) If $S = \{z : |z| = 1\}$ and $f(z) = z$, then $f$ does not have a continuous argument on $S$.

Part (a) is a consequence of Definition 3.1.4, and (b) follows from (3.1.4) and (3.1.2b). The intuition underlying (c) is that if we walk entirely around the unit circle, a continuous argument of $z$ must change by $2\pi$. Thus the argument of $z$ must abruptly jump by $2\pi$

at the end of the trip, which contradicts continuity. A formal proof will be easier after further properties of continuous arguments are developed (see Problem 3.2.5).

Continuous logarithms and continuous arguments are closely related, as follows.

## 3.1.6 Theorem

Let $f : S \to \mathbb{C}$ be continuous.

(a) If $g$ is a continuous logarithm of $f$, then $\operatorname{Im} g$ is a continuous argument of $f$.

(b) If $\theta$ is a continuous argument of $f$, then $\ln|f| + i\theta$ is a continuous logarithm of $f$.

Thus $f$ has a continuous logarithm iff $f$ has a continuous argument.

(c) Assume that $S$ is connected, and $f$ has continuous logarithms $g_1$ and $g_2$, and continuous arguments $\theta_1$ and $\theta_2$. Then there are integers $k$ and $l$ such that $g_1(s) - g_2(s) = 2\pi ik$ and $\theta_1(s) - \theta_2(s) = 2\pi l$ for all $s \in S$. Thus $g_1 - g_2$ and $\theta_1 - \theta_2$ are constant on $S$.

(d) If $S$ is connected and $s, t \in S$, then $g(s) - g(t) = \ln|f(s)| - \ln|f(t)| + i(\theta(s) - \theta(t))$ for all continuous logarithms $g$ and all continuous arguments $\theta$ of $f$.

*Proof.*

(a) If $f(s) = e^{g(s)}$, then $|f(s)| = e^{\operatorname{Re} g(s)}$, hence $f(s)/|f(s)| = e^{i \operatorname{Im} g(s)}$ as required.

(b) If $f(s) = |f(s)|e^{i\theta(s)}$, then $f(s) = e^{\ln|f(s)| + i\theta(s)}$, so $\ln|f| + i\theta$ is a continuous logarithm.

(c) We have $f(s) = e^{g_1(s)} = e^{g_2(s)}$, hence $e^{g_1(s) - g_2(s)} = 1$, for all $s \in S$. By (2.3.1f), $g_1(s) - g_2(s) = 2\pi ik(s)$ for some integer-valued function $k$. Since $g_1$ and $g_2$ are continuous on $S$, so is $k$. But $S$ is connected, so $k$ is a constant function. A similar proof applies to any pair of continuous arguments of $f$.

(d) If $\theta$ is a continuous argument of $f$, then $\ln|f| + i\theta$ is a continuous logarithm of $f$ by part (b). Thus if $g$ is any continuous logarithm of $f$, then $g = \ln|f| + i\theta + 2\pi ik$ by (c). The result follows. ♣

As Example 3.1.5(c) indicates, a given zero-free continuous function on a set $S$ need not have a continuous argument. However, a continuous argument must exist when $S$ is an *interval*, as we now show.

## 3.1.7 Theorem

Let $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ be continuous, that is, $\gamma$ is a curve and $0 \notin \gamma^*$. Then $\gamma$ has a continuous argument, hence by (3.1.6), a continuous logarithm.

*Proof.* Let $\epsilon$ be the distance from 0 to $\gamma^*$, that is, $\epsilon = \min\{|\gamma(t)| : t \in [a, b]\}$. Then $\epsilon > 0$ because $0 \notin \gamma^*$ and $\gamma^*$ is a closed set. By the uniform continuity of $\gamma$ on $[a, b]$, there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ such that if $1 \le j \le n$ and $t \in [t_{j-1}, t_j]$, then $\gamma(t) \in D(\gamma(t_j), \epsilon)$. By (3.1.5b), the function $\gamma$, restricted to the interval $[t_0, t_1]$, has a continuous argument $\theta_1$, and $\gamma$ restricted to $[t_1, t_2]$ has a continuous argument $\theta_2$. Since $\theta_1(t_1)$ and $\theta_2(t_1)$ differ by an integer multiple of $2\pi$, we may (if necessary) redefine $\theta_2$ on $[t_1, t_2]$ so that the relation $\theta_1 \cup \theta_2$ is a continuous argument of $\gamma$ on $[t_0, t_2]$. Proceeding in this manner, we obtain a continuous argument of $\gamma$ on the entire interval $[a, b]$. ♣

For a generalization to other subsets $S$, see Problem 3.2.6.

### 3.1.8 Definition

Let $f$ be analytic on $\Omega$. We say that $g$ is an *analytic logarithm* of $f$ if $g$ is analytic on $\Omega$ and $e^g = f$.

Our next goal is to show that if $\Omega$ satisfies certain conditions, in particular, if $\Omega$ is a starlike region, then every zero-free analytic function $f$ on $\Omega$ has an analytic logarithm on $\Omega$. First, we give necessary and sufficient conditions for $f$ to have an analytic logarithm.

### 3.1.9 Theorem

Let $f$ be analytic and never zero on the open set $\Omega$. Then $f$ has an analytic logarithm on $\Omega$ iff the "logarithmic derivative" $f'/f$ has a primitive on $\Omega$. Equivalently, by (2.1.6) and (2.1.10), $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$ for every closed path $\gamma$ in $\Omega$.

*Proof.* If $g$ is an analytic logarithm of $f$, then $e^g = f$, hence $f'/f = g'$. Conversely, if $f'/f$ has a primitive $g$, then $f'/f = g'$, and therefore

$$(fe^{-g})' = -fe^{-g}g' + f'e^{-g} = e^{-g}(f' - fg')$$

which is identically zero on $\Omega$. Thus $fe^{-g}$ is constant on each component of $\Omega$. If $fe^{-g} = k_A$ on the component $A$, then $k_A$ cannot be zero, so we can write $k_A = e^{l_A}$ for some constant $l_A$. We then have $f = e^{g+l_A}$, so that $g + l_A$ is an analytic logarithm of $f$ on $A$. Finally, $\cup_A(g + l_A)$ is an analytic logarithm of $f$ on $\Omega$. ♣

We may now give a basic sufficient condition on $\Omega$ under which every zero-free analytic function on $\Omega$ has an analytic logarithm.

### 3.1.10 Theorem

If $\Omega$ is an open set such that $\int_\gamma h(z)\,dz = 0$ for every analytic function $h$ on $\Omega$ and every closed path $\gamma$ in $\Omega$, in particular if $\Omega$ is a starlike region, then every zero-free analytic function $f$ on $\Omega$ has an analytic logarithm.

*Proof.* The result is a consequence of (3.1.9). If $\Omega$ is starlike, then $\int_\gamma h(z)\,dz = 0$ by Cauchy's theorem for starlike regions (2.1.9). ♣

### 3.1.11 Remark

If $g$ is an analytic logarithm of $f$ on $\Omega$, then $f$ has an analytic $n$-th root, namely $f^{1/n} = \exp(g/n)$. If $f(z) = z$ and $g = \log_\alpha$, we obtain

$$z^{1/n} = \exp\left(\frac{1}{n}\ln|z| + i\frac{1}{n}\arg_\alpha z\right) = |z|^{1/n}\exp\left(\frac{i}{n}\arg_\alpha z\right).$$

More generally, we may define an analytic version of $f^w$ for any complex number $w$, via $f^w = e^{wg}$.

## 3.2 The Index of a Point with Respect to a Closed Curve

In the introduction to this chapter we raised the question of which closed paths $\gamma$ in an open set $\Omega$ have the property that $\int_\gamma f(z)\,dz = 0$ for every analytic function $f$ on $\Omega$. As we will see later, a necessary and sufficient condition on $\gamma$ is that "$\gamma$ not wind around any points outside of $\Omega$." That is to say, if $z_0 \notin \Omega$ and $\gamma$ is defined on $[a,b]$, there is "no net change in the argument of $\gamma(t) - z_0$" as $t$ increases from $a$ to $b$. To make this precise, we define the notion of the *index* (or *winding number*) of a point with respect to a closed curve. The following observation will be crucial in showing that the index is well-defined.

### 3.2.1 Theorem

Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve. Fix $z_0 \notin \gamma^*$, and let $\theta$ be a continuous argument of $\gamma - z_0$ [$\theta$ exists by (3.1.7)]. Then $\theta(b) - \theta(a)$ is an integer multiple of $2\pi$. Furthermore, if $\theta_1$ is another continuous argument of $\gamma - z_0$, then $\theta_1(b) - \theta_1(a) = \theta(b) - \theta(a)$.

*Proof.* By (3.1.4), we have $(\gamma(t) - z_0)/|\gamma(t) - z_0| = e^{i\theta(t)}, a \le t \le b$. Since $\gamma$ is a closed curve, $\gamma(a) = \gamma(b)$, hence

$$1 = \frac{\gamma(b) - z_0}{|\gamma(b) - z_0|} \cdot \frac{|\gamma(a) - z_0|}{\gamma(a) - z_0} = e^{i(\theta(b) - \theta(a))}.$$

Consequently, $\theta(b) - \theta(a)$ is an integer multiple of $2\pi$. If $\theta_1$ is another continuous argument of $\gamma - z_0$, then by (3.1.6c), $\theta_1 - \theta = 2\pi l$ for some integer $l$. Thus $\theta_1(b) = \theta(b) + 2\pi l$ and $\theta_1(a) = \theta(a) + 2\pi l$, so $\theta_1(b) - \theta_1(a) = \theta(b) - \theta(a)$. ♣

It is now possible to define the index of a point with respect to a closed curve.

### 3.2.2 Definition

Let $\gamma : [a,b] \to \mathbb{C}$ be a closed curve. If $z_0 \notin \gamma^*$, let $\theta_{z_0}$ be a continuous argument of $\gamma - z_0$. The *index of* $z_0$ *with respect to* $\gamma$, denoted by $n(\gamma, z_0)$, is

$$n(\gamma, z_0) = \frac{\theta_{z_0}(b) - \theta_{z_0}(a)}{2\pi}.$$

By (3.2.1), $n(\gamma, z_0)$ is well-defined, that is, $n(\gamma, z_0)$ does not depend on the particular continuous argument chosen. Intuitively, $n(\gamma, z_0)$ is the net number of revolutions of $\gamma(t), a \le t \le b$, about the point $z_0$. This is why the term *winding number* is often used for the index. Note that by the above definition, for any complex number $w$ we have $n(\gamma, z_0) = n(\gamma + w, z_0 + w)$.

If $\gamma$ is sufficiently smooth, an integral representation of the index is available.

### 3.2.3 Theorem

Let $\gamma$ be a closed *path*, and $z_0$ a point not belonging to $\gamma^*$. Then

$$n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\,dz.$$

More generally, if $f$ is analytic on an open set $\Omega$ containing $\gamma^*$, and $z_0 \notin (f \circ \gamma)^*$, then

$$n(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - z_0}\, dz.$$

*Proof.* Let $\epsilon$ be the distance from $z_0$ to $\gamma^*$. As in the proof of (3.1.7), there is a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $t_{j-1} \le t \le t_j$ implies $\gamma(t) \in D(\gamma(t_j), \epsilon)$. For each $j$, $z_0 \notin D(\gamma(t_j), \epsilon)$ by definition of $\epsilon$. Consequently, by (3.1.10), the analytic function $z \to z - z_0$, when restricted to $D(\gamma(t_j), \epsilon)$ has an analytic logarithm $g_j$. Now if $g$ is an analytic logarithm of $f$, then $g' = f'/f$ [see (3.1.9)]. Therefore $g_j'(z) = 1/(z - z_0)$ for all $z \in D(\gamma(t_j), \epsilon)$. The path $\gamma$ restricted to $[t_{j-1}, t_j]$ lies in the disk $D(\gamma(t_j), \epsilon)$, and hence by (2.1.6),

$$\int_{\gamma|_{[t_{j-1},t_j]}} \frac{1}{z - z_0}\, dz = g_j(\gamma(t_j)) - g_j(\gamma(t_{j-1})).$$

Thus

$$\int_\gamma \frac{1}{z - z_0}\, dz = \sum_{j=1}^n [g_j(\gamma(t_j)) - g_j(\gamma(t_{j-1}))].$$

If $\theta_j = \operatorname{Im} g_j$, then by (3.1.6a), $\theta_j$ is a continuous argument of $z \to z - z_0$ on $D(\gamma(t_j), \epsilon)$. By (3.1.6d), then,

$$\int_\gamma \frac{1}{z - z_0}\, dz = \sum_{j=1}^n [\theta_j(\gamma(t_j)) - \theta_j(\gamma(t_{j-1}))].$$

If $\theta$ is any continuous argument of $\gamma - z_0$, then $\theta|_{[t_{j-1},t_j]}$ is a continuous argument of $(\gamma - z_0)|_{[t_{j-1},t_j]}$. But so is $\theta_j \circ \gamma|_{[t_{j-1},t_j]}$, hence by (3.1.6c),

$$\theta_j(\gamma(t_j)) - \theta_j(\gamma(t_{j-1})) = \theta(t_j) - \theta(t_{j-1}).$$

Therefore,

$$\int_\gamma \frac{1}{z - z_0}\, dz = \sum_{j=1}^n [\theta(t_j) - \theta(t_{j-1})] = \theta(b) - \theta(a) = 2\pi n(\gamma, z_0)$$

completing the proof of the first part of the theorem. Applying this result to the path $f \circ \gamma$, we get the second statement. Specifically, if $z_0 \notin (f \circ \gamma)^*$, then

$$n(f \circ \gamma, z_0) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z - z_0}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - z_0}\, dz. \quad ♣$$

The next result contains additional properties of winding numbers that will be useful later, and which are also interesting (and amusing, in the case of (d)) in their own right.

### 3.2.4 Theorem

Let $\gamma, \gamma_1, \gamma_2 : [a, b] \to \mathbb{C}$ be closed curves.
(a) If $z \notin \gamma^*$, then $n(\gamma, z) = n(\gamma - z, 0)$.
(b) If $0 \notin \gamma_1^* \cup \gamma_2^*$, then $n(\gamma_1\gamma_2, 0) = n(\gamma_1, 0) + n(\gamma_2, 0)$ and $n(\gamma_1/\gamma_2, 0) = n(\gamma_1, 0) - n(\gamma_2, 0)$.
(c) If $\gamma^* \subseteq D(z_0, r)$ and $z \notin D(z_0, r)$, then $n(\gamma, z) = 0$.
(d) If $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|, a \le t \le b$, then $0 \notin \gamma_1^* \cup \gamma_2^*$ and $n(\gamma_1, 0) = n(\gamma_2, 0)$.
*Proof.*
(a) This follows from Definition 3.2.2.
(b) Since $0 \notin \gamma_1^* \cup \gamma_2^*$, both $n(\gamma_1, 0)$ and $\gamma_2, 0)$ are defined. If $\theta_1$ and $\theta_2$ are continuous arguments of $\gamma_1$ and $\gamma_2$ respectively, then $\gamma_j(t) = |\gamma_j(t)|e^{i\theta_j(t)}, j = 1, 2$, so

$$\gamma_1(t)\gamma_2(t) = |\gamma_1(t)\gamma_2(t)|e^{i(\theta_1(t)+\theta_2(t))}, \quad \gamma_1(t)/\gamma_2(t) = |\gamma_1(t)/\gamma_2(t)|e^{i(\theta_1(t)-\theta_2(t))}.$$

Thus

$$\begin{aligned} n(\gamma_1\gamma_2, 0) &= (\theta_1(b) + \theta_2(b)) - (\theta_1(a) + \theta_2(a)) = (\theta_1(b) - \theta_1(a)) + (\theta_2(b) - \theta_2(a)) \\ &= n(\gamma_1, 0) + n(\gamma_2, 0). \end{aligned}$$

Similarly, $n(\gamma_1/\gamma_2, 0) = n(\gamma_1, 0) - n(\gamma_2, 0)$.
(c) If $z \notin D(z_0, r)$, then by (3.1.10), the function $f$ defined by $f(w) = w - z, w \in D(z_0, r)$, has an analytic logarithm $g$. If $\theta$ is the imaginary part of $g$, then by (3.1.6a), $\theta \circ \gamma$ is a continuous argument of $\gamma - z$. Consequently, $n(\gamma, z) = (2\pi)^{-1}[\theta(\gamma(b)) - \theta(\gamma(a))] = 0$ since $\gamma(b) = \gamma(a)$.
(d) First note that if $\gamma_1(t) = 0$ or $\gamma_2(t) = 0$, then $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ is false; therefore, $0 \notin \gamma_1^* \cup \gamma_2^*$. Let $\gamma$ be the closed curve defined by $\gamma(t) = \gamma_2(t)/\gamma_1(t)$. By the hypothesis, we have $|1 - \gamma(t)| < 1$ on $[a, b]$, hence $\gamma^* \subseteq D(1, 1)$. But by (c) and (b), $0 = n(\gamma, 0) = n(\gamma_2, 0) - n(\gamma_1, 0)$. ♣

Part (d) of (3.2.4) is sometimes called the "dog-walking theorem". (See the text by W. Veech,*A Second Course in Complex Analysis*, page 30.) For if $\gamma_1(t)$ and $\gamma_2(t)$ are respectively the positions of a man and a dog on a variable length leash, and a tree is located at the origin, then the hypothesis states that the length of the leash is always less than the distance from the man to the tree. The conclusion states that the man and the dog walk around the tree exactly the same number of times. See Problem 4 for a generalization of (d).

The final theorem of this section deals with $n(\gamma, z_0)$ when viewed as a function of $z_0$.

### 3.2.5 Theorem

If $\gamma$ is a closed curve, then the function $z \to n(\gamma, z), z \notin \gamma^*$, is constant on each component of $\mathbb{C} \setminus \gamma^*$, and is 0 on the unbounded component of $\mathbb{C} \setminus \gamma^*$.
*Proof.* Let $z_0 \in \mathbb{C} \setminus \gamma^*$, and choose $r > 0$ such that $D(z_0, r) \subseteq \mathbb{C} \setminus \gamma^*$. If $z \in D(z_0, r)$, then by parts (a) and (b) of (3.2.4),

$$n(\gamma, z) - n(\gamma, z_0) = n(\gamma - z, 0) - n(\gamma - z_0, 0) = n\left(\frac{\gamma - z}{\gamma - z_0}, 0\right) = n\left(1 + \frac{z_0 - z}{\gamma - z_0}, 0\right).$$

But for each $t$,

$$\left|\frac{z_0 - z}{\gamma(t) - z_0}\right| < \frac{r}{|\gamma(t) - z_0|} \leq 1$$

since $D(z_0, r) \subseteq \mathbb{C}\setminus\gamma^*$. Therefore the curve $1+(z_0-z)/(\gamma-z_0)$ lies in $D(1,1)$. By part (c) of (3.2.4), $n(1+(z_0-z)/(\gamma-z_0), 0) = 0$, so $n(\gamma, z) = n(\gamma, z_0)$. This proves that the function $z \to n(\gamma, z)$ is continuous on the open set $\mathbb{C}\setminus\gamma^*$ and locally constant. By an argument that we have seen several times, the function is constant on components of $\mathbb{C}\setminus\gamma^*$. (If $z_0 \in \mathbb{C}\setminus\gamma^*$ and $\Omega$ is that component of $\mathbb{C} \setminus \gamma^*$ containing $z_0$, let $A = \{z \in \Omega : n(\gamma, z) = n(\gamma, z_0)\}$. Then $A$ is a nonempty subset of $\Omega$ and $A$ is both open and closed in $\Omega$, so $A = \Omega$.) To see that $n(\gamma, z) = 0$ on the unbounded component of $\mathbb{C} \setminus \gamma^*$, note that $\gamma^* \subseteq D(0, R)$ for $R$ sufficiently large. By (3.2.4c), $n(\gamma, z) = 0$ for $z \notin D(0, R)$. Since all $z$ outside of $D(0, R)$ belong to the unbounded component of $\mathbb{C} \setminus \gamma^*$, we are finished. ♣

## Problems

1. Suppose $\Omega$ is a region in $\mathbb{C} \setminus \{0\}$ such that every ray from 0 meets $\Omega$.
   (a) Show that for any $\alpha \in \mathbb{R}$, $\log_\alpha$ is not analytic on $\Omega$.
   (b) Show, on the other hand, that there exist regions of this type such that $z$ *does* have an analytic logarithm on $\Omega$.
2. Let $f(z) = (z - a)(z - b)$ for $z$ in the region $\Omega = \mathbb{C} \setminus [a, b]$, where $a$ and $b$ are distinct complex numbers. Show that $f$ has an analytic square root, but not an analytic logarithm, on $\Omega$.
3. Let $f$ be an analytic zero-free function on $\Omega$. Show that the following are equivalent.
   (a) $f$ has an analytic logarithm on $\Omega$.
   (b) $f$ has an analytic $k$-th root on $\Omega$ (that is, an analytic function $h$ such that $h^k = f$) for every positive integer $k$.
   (c) $f$ has an analytic $k$-th root on $\Omega$ for infinitely many positive integers $k$.
4. Prove the following extension of (3.2.4d), the "generalized dog-walking theorem". Let $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C}$ be closed curves such that $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| + |\gamma_2(t)|$ for all $t \in [a, b]$. Prove that $n(\gamma_1, 0) = n(\gamma_2, 0)$. (Hint: Define $\gamma$ as in the proof of (3.2.4d), and investigate the location of $\gamma^*$.) Also, what does the hypothesis imply about the dog and the man in this case?
5. Prove the result given in Example 3.1.5(c).
6. Let $f$ be a continuous mapping of the rectangle $S = \{x + iy : a \leq x \leq b, c \leq y \leq d\}$ into $\mathbb{C} \setminus \{0\}$. Show that $f$ has a continuous logarithm. This can be viewed as a generalization of Theorem 3.1.7; to obtain (3.1.7) (essentially), take $c = d$.
7. Let $f$ be analytic and zero-free on $\Omega$, and suppose that $g$ is a continuous logarithm of $f$ on $\Omega$. Show that $g$ is actually analytic on $\Omega$.
8. Characterize the entire functions $f, g$ such that $f^2 + g^2 = 1$. (Hint: $1 = f^2 + g^2 = (f + ig)(f - ig)$, so $f + ig$ is never 0.)
9. Let $f$ and $g$ be continuous mappings of the connected set $S$ into $\mathbb{C} \setminus \{0\}$.
   (a) If $f^n = g^n$ for some positive integer $n$, show that $f = g\exp(i2\pi k/n)$ for some

$k = 0, 1, \dots, n-1$.Hence if $f(s_0) = g(s_0)$ for some $s_0 \in S$, then $f \equiv g$.
(b) Show that $\mathbb{C} \setminus \{0\}$ cannot be replaced by $\mathbb{C}$ in the hypothesis.

## 3.3 Cauchy's Theorem

This section is devoted to a discussion of the global (or homology) version of Cauchy's theorem. The elementary proof to be presented below is due to John Dixon, and appeared in Proc. Amer. Math. Soc. 29 (1971), pp. 625-626, but the theorem as stated is originally due to E.Artin.

### 3.3.1 Cauchy's Theorem

Let $\gamma$ be closed path in $\Omega$ such that $n(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$.

(i) For all analytic functions $f$ on $\Omega$, $\int_\gamma f(w)\, dw = 0$;

(ii) If $z \in \Omega \setminus \gamma^*$, then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw.$$

A path $\gamma$ in $\Omega$ with $n(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$ is said to be $\Omega$-*homologous to zero*. Dixon's proof requires two preliminary lemmas.

### 3.3.2 Lemma

Let $f$ be analytic on $\Omega$, and define $g$ on $\Omega \times \Omega$ by

$$g(w, z) = \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z \\ f'(z), & w = z. \end{cases}$$

Then $g$ is continuous, and for each fixed $w \in \Omega$, the function given by $z \to g(w, z)$ is analytic on $\Omega$.

*Proof.* Let $\{(w_n, z_n), n = 1, 2, \dots\}$ be any sequence in $\Omega \times \Omega$ converging to $(w, z) \in \Omega \times \Omega$. If $w \neq z$, then eventually $w_n \neq z_n$, and by continuity of $f$, $g(w_n, z_n) = \frac{f(w_n)-f(z_n)}{w_n - z_n} \to \frac{f(w)-f(z)}{w-z} = g(w, z)$. However, if $w = z$, then

$$g(w_n, z_n) = \begin{cases} \frac{1}{w_n - z_n} \int_{[z_n, w_n]} f'(\tau)\, d\tau & \text{if } w_n \neq z_n \\ \\ f'(z_n) & \text{if } w_n = z_n. \end{cases}$$

In either case, the continuity of $f'$ at $z$ implies that $g(w_n, z_n) \to f'(z) = g(z, z)$.

Finally, the function $z \to g(w, z)$ is continuous on $\Omega$ and analytic on $\Omega \setminus \{w\}$ (because $f$ is analytic on $\Omega$). Consequently, $z \to g(w, z)$ is analytic on $\Omega$ by (2.2.13). ♣

### 3.3.3 Lemma

Suppose $[a, b] \subseteq \mathbb{R}$, and let $\varphi$ be a continuous complex-valued function on the product space $\Omega \times [a, b]$. Assume that for each $t \in [a, b]$, the function $z \to \varphi(z, t)$ is analytic on $\Omega$. Define $F$ on $\Omega$ by $F(z) = \int_a^b \varphi(z, t)\, dt$, $z \in \Omega$. Then $F$ is analytic on $\Omega$ and

$$F'(z) = \int_a^b \frac{\partial \varphi}{\partial z}(z, t)\, dt, z \in \Omega.$$

Note that Theorem 2.2.10 on integrals of the Cauchy type is special case of this result. However, (2.2.10) will itself play a part in the proof of (3.3.3).

*Proof.* Fix any disk $D(z_0, r)$ such that $\overline{D}(z_0, r) \subseteq \Omega$. Then for each $z \in D(z_0, r)$,

$$\begin{aligned} F(z) &= \int_a^b \varphi(z, t)\, dt \\ &= \frac{1}{2\pi i} \int_a^b \left( \int_{C(z_0, r)} \frac{\varphi(w, t)}{w - z}\, dw \right) dt \qquad \text{(by 2.2.9)} \\ &= \frac{1}{2\pi i} \int_{C(z_0, r)} \left( \int_a^b \varphi(w, t)\, dt \right) \frac{1}{w - z}\, dw \end{aligned}$$

(Write the path integral as an ordinary definite integral and observe that the interchange in the order of integration is justified by the result that applies to continuous functions on rectangles.) Now $\int_a^b \varphi(w, t)\, dt$ is a continuous function of $w$ (to see this use the continuity of $\varphi$ on $\Omega \times [a, b]$), hence by (2.2.10), $F$ is analytic on $D(z_0, r)$ and for each $z \in D(z_0, r)$,

$$\begin{aligned} F'(z) &= \frac{1}{2\pi i} \int_{C(z_0, r)} \left[ \int_a^b \varphi(w, t)\, dt \right] \frac{1}{(w - z)^2}\, dw \\ &= \int_a^b \left[ \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{\varphi(w, t)}{(w - z)^2}\, dw \right] dt \\ &= \int_a^b \frac{\partial \varphi}{\partial z}(z, t)\, dt \end{aligned}$$

by (2.2.10) again. ♣

*Proof of Cauchy's Theorem.*

Let $\gamma$ be a closed path in the open set $\Omega$ such that $n(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$, and let $f$ be an analytic function on $\Omega$. Define $\Omega' = \{z \in \mathbb{C} \setminus \gamma^* : n(\gamma, z) = 0\}$. Then $\mathbb{C} \setminus \Omega \subseteq \Omega'$, so $\Omega \cup \Omega' = \mathbb{C}$; furthermore, $\Omega'$ is open by (3.2.5). If $z \in \Omega \cap \Omega'$ and $g$ is defined as in (3.3.2), then $g(w, z) = (f(w) - f(z))/(w - z)$ since $z \notin \gamma^*$. Thus

$$\int_\gamma g(w, z)\, dw = \int_\gamma \frac{f(w)}{w - z}\, dw - 2\pi i n(\gamma, z) f(z) = \int_\gamma \frac{f(w)}{w - z}\, dw$$

since $n(\gamma, z) = 0$ for $z \in \Omega'$. The above computation shows that we can define a function $h$ on $\mathbb{C}$ by

$$h(z) = \begin{cases} \int_\gamma g(w,z)\,dw & \text{if } z \in \Omega \\ \\ \int_\gamma \frac{f(w)}{w-z}\,dw & \text{if } z \in \Omega'. \end{cases}$$

By (2.2.10), $h$ is analytic on $\Omega'$, and by (3.3.2) and (3.3.3), $h$ is analytic on $\Omega$. Thus $h$ is an entire function. But for $|z|$ sufficiently large, $n(\gamma, z) = 0$ by (3.2.5), hence $z \in \Omega'$. Consequently, $h(z) = \int_\gamma \frac{f(w)}{w-z}\,dw \to 0$ as $|z| \to \infty$. By Liouville's theorem (2.4.2), $h \equiv 0$. Now if $z \in \Omega \setminus \gamma^*$ we have, as at the beginning of the proof,

$$0 = h(z) = \int_\gamma g(w,z)\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw - 2\pi i n(\gamma, z) f(z)$$

proving (ii). To obtain (i), choose any $z \in \Omega \setminus \gamma^*$ and apply (ii) to the function $w \to (w-z)f(w), w \in \Omega$. ♣

### 3.3.4 Remarks

Part (i) of (3.3.1) is usually referred to as Cauchy's *theorem*, and part (ii) as Cauchy's *integral formula*. In the above proof we derived (i) from (ii); see Problem 1 for the reverse implication.

Also, there is a converse to part (i): If $\gamma$ is a closed path in $\Omega$ such that $\int_\gamma f(w)\,dw = 0$ for every $f$ analytic on $\Omega$, then $n(\gamma, z) = 0$ for every $z \notin \Omega$. To prove this, take $f(w) = 1/(w-z)$ and apply (3.2.3).

It is sometimes convenient to integrate over objects slightly more general than closed paths.

### 3.3.5 Definitions

Let $\gamma_1, \gamma_2, \dots, \gamma_m$ be closed paths. If $k_1, k_2, \dots, k_m$ are integers, then the formal sum $\gamma = k_1\gamma_1 + \cdots + k_m\gamma_m$ is called a *cycle*. We define $\gamma^* = \cup_{j=1}^m \gamma_j^*$, and for any continuous function $f$ on $\gamma^*$,

$$\int_\gamma f(w)\,dw = \sum_{j=1}^m k_j \int_{\gamma_j} f(w)\,dw.$$

Finally, for $z \notin \gamma^*$, define

$$n(\gamma, z) = \sum_{j=1}^m k_j n(\gamma_j, z).$$

It follows directly from the above definitions that the integral representation (3.2.3) for winding numbers extends to cover cycles as well. Also, the proof of Cauchy's theorem (3.3.1) may be repeated almost verbatim for cycles (Problem 2).

Cauchy's theorem, along with the remarks and definitions following it combine to yield the following equivalence.

### 3.3.6 Theorem

Let $\gamma$ be a closed path (or cycle) in the open set $\Omega$. Then $\int_\gamma f(z)\,dz = 0$ for every analytic function $f$ on $\Omega$ iff $n(\gamma, z) = 0$ for every $z \notin \Omega$.

*Proof.* Apply (3.3.1), (3.3.4) and (3.3.5). Note that the proof of the converse of (i) of (3.3.1) given in (3.3.4) works for cycles, because the integral representation (3.2.3) still holds. ♣

### 3.3.7 Corollary

Let $\gamma_1$ and $\gamma_2$ be closed paths (or cycles) in the open set $\Omega$. Then $\int_{\gamma_1} f(w)\,dw = \int_{\gamma_2} f(w)\,dw$ for every analytic function $f$ on $\Omega$ iff $n(\gamma_1, z) = n(\gamma_2, z)$ for every $z \notin \Omega$.

*Proof.* Apply (3.3.6) to the cycle $\gamma_1 - \gamma_2$. ♣

Note that Theorem 3.3.6 now provides a solution of the first problem posed at the beginning of the chapter, namely, a characterization of those closed paths $\gamma$ in $\Omega$ such that $\int_\gamma f(z)\,dz = 0$ for every analytic function $f$ on $\Omega$.

## Problems

1. Show that (i) implies (ii) in (3.3.1).
2. Explain briefly how the proof of (3.3.1) is carried out for cycles.
3. Let $\Omega, \gamma$ and $f$ be as in (3.3.1). Show that for each $k = 0, 1, 2, \dots$ and $z \in \Omega \setminus \gamma^*$, we have

$$n(\gamma, z) f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{k+1}}\,dw.$$

4. Compute $\int_{C(0,2)} \frac{1}{z^2-1}\,dz$.
5. Use Problem 3 to calculate each of the integrals $\int_{\gamma_j} \frac{e^z + \cos z}{z^4}\,dz, j = 1, 2$, where the $\gamma_j$ are the paths indicated in Figure 3.3.1.
6. Consider $\gamma : [0.2\pi] \to \mathbb{C}$ given by $\gamma(t) = a\cos t + ib\sin t$, where $a$ and $b$ are nonzero real numbers. Evaluate $\int_\gamma dz/z$, and using this result, deduce that

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}.$$

## 3.4 Another Version of Cauchy's Theorem

In this section we consider the second question formulated at the beginning of the chapter: Which open sets $\Omega$ have the property that $\int_\gamma f(z)\,dz = 0$ for all analytic functions $f$ on $\Omega$ and all closed paths (or cycles) $\gamma$ in $\Omega$? A concise answer is given by Theorem 3.4.6, but several preliminaries are needed.

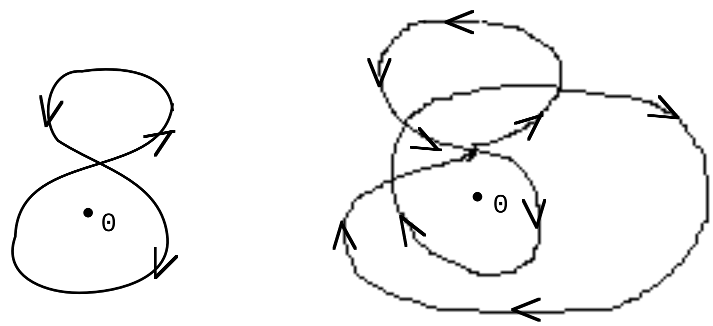


Figure 3.3.1

### 3.4.1 The Extended Complex Plane

Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - 1/2)^2 = 1/4\}$. Thus $S$ is the sphere in $\mathbb{R}^3$ (called the *Riemann sphere*) with center at (0,0,1/2) and radius 1/2 (Figure 3.4.1). The line segment joining (0,0,1), the north pole of $S$, to a point $(x, y, 0)$ is $\{(tx, ty, 1-t) : 0 \le t \le 1\}$, and this segment meets $S$ when and only when

$$t^2(x^2 + y^2) + (\frac{1}{2} - t)^2 = \frac{1}{4}, \quad \text{or} \quad t = \frac{1}{1 + x^2 + y^2}.$$

Therefore the intersection point is $(x_1, x_2, x_3)$, where

$$x_1 = \frac{x}{1 + x^2 + y^2},\ x_2 = \frac{y}{1 + x^2 + y^2},\ x_3 = \frac{x^2 + y^2}{1 + x^2 + y^2}. \tag{1}$$

Since $1 - x_3 = 1/(1 + x^2 + y^2)$, it follows from (1) that

$$x = \frac{x_1}{1 - x_3},\ y = \frac{x_2}{1 - x_3}. \tag{2}$$

Let $h$ be the mapping that takes $(x, y, 0)$ to the point $(x_1, x_2, x_3)$ of $S$. Then $h$ maps $\mathbb{R}^2 \times \{0\}$, which can be identified with $\mathbb{C}$, one-to-one onto $S \setminus \{(0, 0, 1)\}$. Also, by (2), $h^{-1}(x_1, x_2, x_3) = (\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0)$. Consequently, $h$ is a homeomorphism, that is, $h$ and $h^{-1}$ are continuous.

We can identify $\mathbb{C}$ and $S \setminus \{(0, 0, 1)\}$ formally as follows. Define $k : \mathbb{C} \to \mathbb{R}^2 \times \{0\}$. by $k(x + iy) = (x, y.0)$. Then $k$ is an isometry (a one-to-one, onto, distance-preserving map), hence $h \circ k$ is a homeomorphism of $\mathbb{C}$ onto $S \setminus \{(0, 0, 1)\}$. Next let $\infty$ denote a point not belonging to $\mathbb{C}$, and take $\hat{\mathbb{C}}$ to be $\mathbb{C} \cup \{\infty\}$. Define $g : \hat{\mathbb{C}} \to S$ by

$$g(z) = \begin{cases} h(k(z)), & z \in \mathbb{C} \\ (0, 0, 1), & z = \infty. \end{cases}$$

Then $g$ maps $\mathbb{C}$ one-to-one onto $S$. If $\rho$ is the usual Euclidean metric of $\mathbb{R}^3$ and $\hat{d}$ is defined on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ by

$$\hat{d}(z, w) = \rho(g(z), g(w)),$$

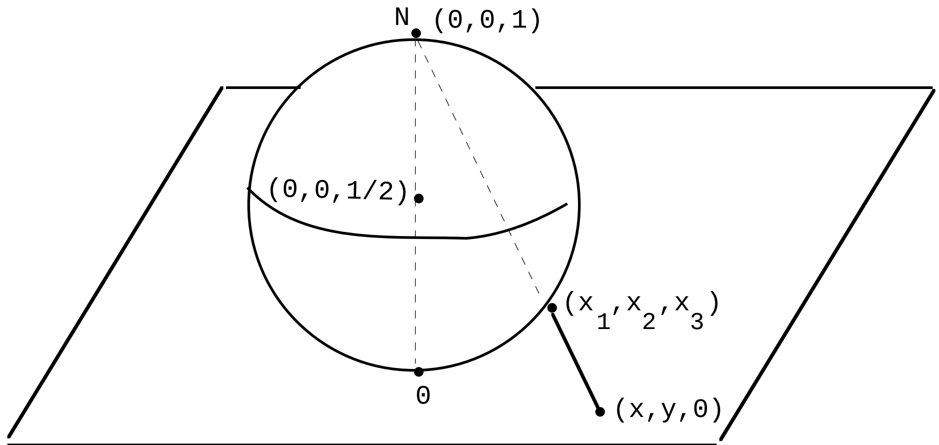


Figure 3.4.1

then $\hat{d}$ is a metric on $\hat{\mathbb{C}}$. (The $\hat{d}$-distance between points of $\hat{\mathbb{C}}$ is the Euclidean distance between the corresponding points on the Riemann sphere.) The metric space $(\hat{\mathbb{C}}, \hat{d})$ is called the *extended plane*, and $\hat{d}$ is called the *chordal metric* on $\hat{\mathbb{C}}$. It is a consequence of the definition of $\hat{d}$ that $(\hat{\mathbb{C}}, \hat{d})$ and $(S, \rho)$ are isometric spaces. The following formulas for $\hat{d}$ hold.

### 3.4.2 Lemma

$$\hat{d}(z,w) = \begin{cases} \frac{|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}}, & z, w \in \mathbb{C} \\ \\ \frac{1}{(1+|z|^2)^{1/2}}, & z \in \mathbb{C}, w = \infty. \end{cases}$$

*Proof.* Suppose $z = x + iy, w = u + iv$. Then by (1) of (3.4.1),

$$\begin{aligned} [\hat{d}(z,w)]^2 &= \left[\frac{x}{1+|z|^2} - \frac{u}{1+|w|^2}\right]^2 + \left[\frac{y}{1+|z|^2} - \frac{v}{1+|w|^2}\right]^2 + \left[\frac{|z|^2}{1+|z|^2} - \frac{|w|^2}{1+|w|^2}\right]^2 \\ &= \frac{x^2+y^2+|z|^4}{(1+|z|^2)^2} + \frac{u^2+v^2+|w|^4}{(1+|w|^2)^2} - 2\frac{xu+yv+|z|^2|w|^2}{(1+|z|^2)(1+|w|^2)} \\ &= \frac{|z|^2}{1+|z|^2} + \frac{|w|^2}{1+|w|^2} - \frac{|z|^2+|w|^2-|z-w|^2+2|z|^2|w|^2}{(1+|z|^2)(1+|w|^2)} \\ &= \frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)} \end{aligned}$$

as desired. Also,

$$\begin{aligned}[\hat{d}(z,\infty)]^2 &= [\rho(g(z),(0,0,1))]^2\\ &= \frac{x^2+y^2+1}{(1+x^2+y^2)^2} \text{ by (1) of (3.4.1)}\\ &= \frac{1}{1+|z|^2}. \ ♣\end{aligned}$$

Here is a list of the most basic properties of $\hat{\mathbb{C}}$.

## 3.4.3 Theorem

(a) The metric space $(\hat{\mathbb{C}},\hat{d})$ is compact, and the identity function on $\mathbb{C}$ is a homeomorphism of $\mathbb{C}$ (with the usual metric) onto $(\mathbb{C},\hat{d})$.

(b) The complex plane is a dense subspace of $\hat{\mathbb{C}}$. In fact, a sequence $\{z_n\}$ in $\mathbb{C}$ converges to $\infty$ iff $\{|z_n|\}$ converges to $+\infty$.

(c) The metric space $(\hat{\mathbb{C}},\hat{d})$ is connected and complete.

(d) Let $\gamma$ be a closed curve in $\mathbb{C}$, and define $n(\gamma,\infty)=0$. Then the function $n(\gamma,\cdot)$ is continuous on $\hat{\mathbb{C}}\setminus\gamma^*$.

(e) The identity map on $\hat{\mathbb{C}}$ is a homeomorphism of $(\hat{\mathbb{C}},\hat{d})$ with the one-point compactification $(\mathbb{C}_\infty,\mathcal{T})$ of $\mathbb{C}$. (Readers unfamiliar with the one-point compactification of a locally compact space may simply ignore this part of the theorem, as it will not be used later.)

*Proof.* Since the Riemann sphere is compact, connected and complete, so is $(\hat{\mathbb{C}},\hat{d})$. The formula for $\hat{d}$ in (3.4.2) shows that the identity map on $\mathbb{C}$ is a homeomorphism of $\mathbb{C}$ into $\hat{\mathbb{C}}$, and that $z_n\to\infty$ iff $|z_n|\to+\infty$. This proves (a), (b) and (c). Part (d) follows from (3.2.5). For (e), see Problem 4. ♣

We are now going to make precise, in two equivalent ways, the notion that an open set has no holes.

## 3.4.4 Theorem

Let $\Omega$ be open in $\mathbb{C}$. Then $\hat{\mathbb{C}}\setminus\Omega$ is connected iff each closed curve (and each cycle) $\gamma$ in $\Omega$ is $\Omega$-homologous to 0, that is, $n(\gamma,z)=0$ for all $z\notin\Omega$.

*Proof.* Suppose first that $\hat{\mathbb{C}}\setminus\Omega$ is connected, and let $\gamma$ be a closed curve in $\Omega$. Since $z\to n(\gamma,z)$ is a continuous integer-valued function on $\mathbb{C}\setminus\gamma^*$ [by (3.2.5) and (3.4.3d)], it must be constant on the connected set $\hat{\mathbb{C}}\setminus\Omega$. But $n(\gamma,\infty)=0$, hence $n(\gamma,z)=0$ for all $z\in\hat{\mathbb{C}}\setminus\Omega$. The statement for cycles now follows from the result for closed curves.

The converse is considerably more difficult, and is a consequence of what we will call the *hexagon lemma*. As we will see, this lemma has several applications in addition to its use in the proof of the converse.

### 3.4.5 The Hexagon Lemma

Let $\Omega$ be an open subset of $\mathbb{C}$, and let $K$ be a nonempty compact subset of $\Omega$. Then there are closed polygonal paths $\gamma_1, \gamma_2, \dots, \gamma_m$ in $\Omega \setminus K$ such that

$$\sum_{j=1}^{m} n(\gamma_j, z) = \begin{cases} 1 & \text{if } z \in K \\ 0 & \text{if } z \notin \Omega. \end{cases}$$

The lemma may be expressed by saying there is a (polygonal) cycle in $\Omega \setminus K$ which winds around each point of $K$ exactly once, but does not wind around any point of $\mathbb{C} \setminus \Omega$.

*Proof.* For each positive integer $n$, let $\mathcal{P}_n$ be the hexagonal partition of $\mathbb{C}$ determined by the hexagon with base $[0, 1/n]$; see Figure 3.4.2. Since $K$ is a compact subset of the open

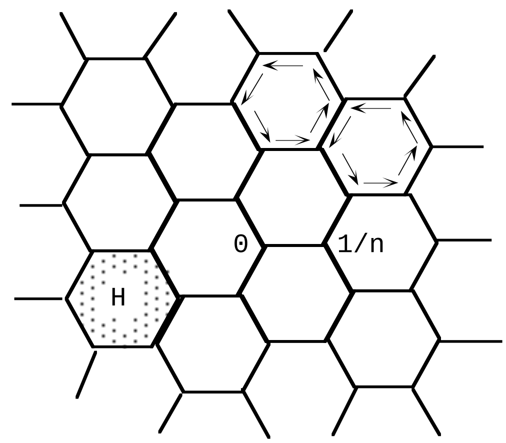


Figure 3.4.2. A Hexagonal Partition of $\mathbb{C}$.

set $\Omega$, we have $\operatorname{dist}(K, \mathbb{C} \setminus \Omega) > 0$, and therefore we can choose $n$ large enough so that if $H \in \mathcal{P}_n$ and $H \cap K \neq \emptyset$, then $H \subseteq \Omega$. Define $\mathcal{K} = \{H \in \mathcal{P}_n : H \cap K \neq \emptyset\}$. Since $K$ is nonempty and bounded, $\mathcal{K}$ is a nonempty finite collection and

$$K \subseteq \cup\{H : H \in \mathcal{K}\} \subseteq \Omega.$$

Now assign a positive (that is, counterclockwise) orientation to the sides of each hexagon (see Figure 3.4.2). Let $S$ denote the collection of all oriented sides of hexagons in $\mathcal{K}$ that are sides of exactly one member of $\mathcal{K}$. Observe that given an oriented side $\vec{ab} \in S$, there are *unique* oriented sides $\vec{ca}$ and $\vec{bd}$ in $S$. (This uniqueness property is the motivation for tiling with hexagons instead of squares. If we used squares instead, as in Figure 3.4.3, we have $\vec{ab}, \vec{bc}$ and $\vec{bd} \in S$, so $\vec{ab}$ does not have a unique successor, thus complicating the argument that follows.)

By the above observations, and the fact that $S$ is a finite collection, it follows that given $\vec{a_1 a_2} \in S$, there is a *uniquely* defined closed polygonal path $\gamma_1 = [a_1, a_2, \dots, a_k, a_1]$ with all sides in $S$. If $S_1$ consists of the edges of $\gamma_1$ and $S \setminus S_1 \neq \emptyset$, repeat the above

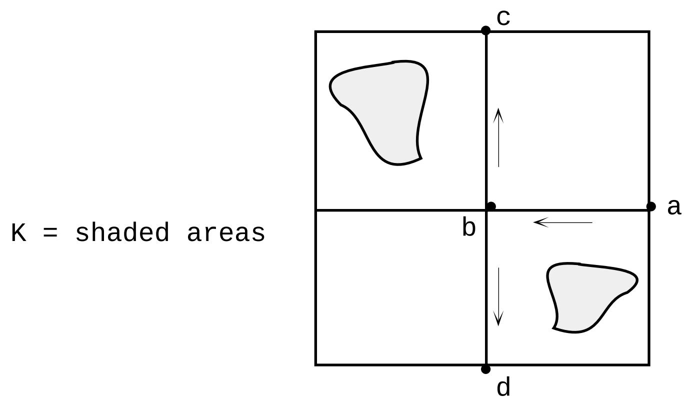


Figure 3.4.3 Nonuniqueness when squares are used.

construction with $S$ replaced by $S \setminus S_1$. Continuing in this manner, we obtain pairwise disjoint collections $S_1, S_2, \ldots, S_n$ such that $S = \cup_{j=1}^{m} S_j$, and corresponding closed polygonal paths $\gamma_1, \gamma_2, \ldots, \gamma_m$ (Figure 3.4.4).

Suppose now that the hexagons in $\mathcal{K}$ are $H_1, H_2, \ldots, H_p$, and let $\sigma_j$ denote the boundary of $H_j$, oriented positively. If $z$ belongs to the interior of some $H_r$, then $n(\sigma_r, z) = 1$ and $n(\sigma_j, z) = 0, j \neq r$. Consequently, $n(\sigma_1 + \sigma_2 + \cdots + \sigma_p, z) = 1$ by (3.3.5). But by construction, $n(\gamma_1 + \cdots + \gamma_m, z) = n(\sigma_1 + \cdots + \sigma_p, z)$. (The key point is that if both hexagons containing a particular side $[a, b]$ belong to $\mathcal{K}$, then $\vec{ab} \notin S$ and $\vec{ba} \notin S$. Thus $[a, b]$ will not contribute to either $n(\gamma_1 + \cdots + \gamma_m, z)$ or to $n(\sigma_1 + \cdots + \sigma_p, z)$. If only one hexagon containing $[a, b]$ belongs to $\mathcal{K}$, then $\vec{ab}$ (or $\vec{ba}$) appears in both cycles.) Therefore $n(\gamma_1 + \cdots + \gamma_m, z) = 1$. Similarly, if $z \notin \Omega$, then $n(\gamma_1 + \cdots + \gamma_m, z) = n(\sigma_1 + \cdots + \sigma_p, z) = 0$.

Finally, assume $z \in K$ and $z$ belongs to a side $s$ of some $H_r$. Then $s$ cannot be in $S$, so $z \notin (\gamma_1 + \cdots + \gamma_m)^*$. Let $\{w_k\}$ be a sequence of interior points of $H_r$ with $w_k$ converging to $z$. We have shown that $n(\gamma_1 + \cdots + \gamma_m, w_k) = 1$ for all $k$, so by (3.2.5), $n(\gamma_1 + \cdots + \gamma_m, z) = 1$. ♣

## Completion of the Proof of (3.4.4)

If $\hat{\mathbb{C}} \setminus \Omega$ is not connected, we must exhibit a cycle in $\Omega$ that is not $\Omega$-homologous to 0. Now since $\hat{\mathbb{C}}$ is closed and not connected, it can be expressed as the union of two nonempty disjoint closed sets $K$ and $L$. One of these two sets must contain $\infty$; assume that $\infty \in L$. Then $K$ must be a compact subset of the complex plane $\mathbb{C}$, and $K$ is contained in the plane open set $\Omega_1 = \mathbb{C} \setminus L$. apply the hexagon lemma (3.4.5) to $\Omega_1$ and $K$ to obtain a cycle $\sigma$ in $\Omega_1 \setminus K = \mathbb{C} \setminus (K \cup L) = \Omega$ such that $n(\sigma, z) = 1$ for each $z \in K$ (and $n(\sigma, z) = 0$ for $z \notin \Omega_1$). Pick any point $z$ in the nonempty set $K \subseteq \mathbb{C} \setminus \Omega$. Then $z \notin \Omega$ and $n(\sigma, z) = 1 \neq 0$. ♣

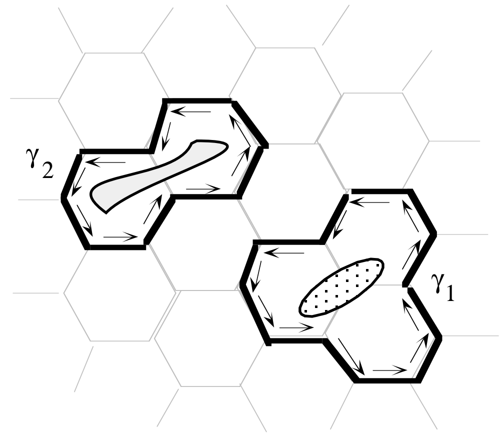


Figure 3.4.4. Construction of the closed paths.

## Remark

As a consequence of the definition (3.3.5) of the index of a cycle, if $\hat{\mathbb{C}} \setminus \Omega$ is not connected, there must actually be a closed *path* $\gamma$ in $\Omega$ such that $n(\gamma, z) \neq 0$ for some $z \notin \Omega$.

The list of equivalences below is essentially a compilation of results that have already been established.

### 3.4.6 Second Cauchy Theorem

Let $\Omega$ be an open subset of $\mathbb{C}$. The following are equivalent.

(1) $\hat{\mathbb{C}} \setminus \Omega$ is connected.

(2) $n(\gamma, z) = 0$ for each closed path (or cycle) $\gamma$ in $\Omega$ and each point $z \in \mathbb{C} \setminus \Omega$.

(3) $\int_\gamma f(z)\,dz = 0$ for every closed path (or cycle) $\gamma$ in $\Omega$ and every analytic function $f$ on $\Omega$.

(4) Every analytic function on $\Omega$ has a primitive on $\Omega$.

(5) Every zero-free analytic function on $\Omega$ has an analytic logarithm.

(6) Every zero-free analytic function on $\Omega$ has an analytic $n$-th root for $n = 1, 2, \ldots$.

*Proof.*

(1) is equivalent to (2) by Theorem 3.4.4.

(2) is equivalent to (3) by Theorem 3.3.6.

(3) is equivalent to (4) by Theorems 2.1.6 and 2.1.10.

(3) implies (5) by Theorem 3.1.10.

(5) is equivalent to (6) by Problem 3.2.3.

(5) implies (2): If $z_0 \notin \Omega$, let $f(z) = z - z_0, z \in \Omega$. Then $f$ has an analytic logarithm on $\Omega$, and hence for each closed path (or cycle) $\gamma$ in $\Omega$ we have, by (3.2.3) and (3.1.9),

$$n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - z_0}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0. \;\clubsuit$$

We will be adding to the above list in later chapters. An open subset of $\mathbb{C}$ satisfying any (and hence all) of the conditions of (3.4.6) is said to be *(homologically) simply connected*.

It is true that in complex analysis, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are used almost exclusively. The rather tedious hexagon lemma was required to establish the reverse implication (2) $\Rightarrow$ (1). Thus one might wonder why we have gone to the trouble of obtaining the hexagon lemma at all. One answer is that it has other applications, including the following global integral representation formula. This formula should be compared with Cauchy's integral formula for a circle (2.2.9). It will also be used later in the proof of Runge's theorem on rational approximation.

### 3.4.7 Theorem

Let $K$ be a compact subset of the open set $\Omega$. Then there is a cycle $\gamma$ in $\Omega \setminus K$ such that $\gamma$ is a formal sum of closed polygonal paths, and for every analytic function $f$ on $\Omega$,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = 0 \quad \text{for all } z \in K.$$

*Proof.* Apply the hexagon lemma and part (ii) of (3.3.1). ♣

## Problems

1. (a) Give an example of an open connected set that is not simply connected. For this set, describe explicitly an analytic function $f$ and a closed path $\gamma$ such that $\int_\gamma f(z)\,dz \neq 0$.
   (b) Give an example of an open, simply connected set that is not connected.
2. Suppose that in the hexagon lemma, $\Omega$ is assumed to be connected. Can a *cycle* that satisfies the conclusion be taken to be a closed *path*?
3. Let $\Gamma_1$ be the ray $[1, i/2, \infty) = \{1 - t + ti/2 : 0 \le t < \infty\}$ and let $\Gamma_2$ be the ray $[1, 2, \infty)$.
   (a) Show that $1 - z$ has analytic square roots $f$ and $g$ on $\mathbb{C} \setminus \Gamma_1$ and $\mathbb{C} \setminus \Gamma_2$ respectively, such that $f(0) = g(0) = 1$.
   (b) Show that $f = g$ below $\Gamma = \Gamma_1 \cup \Gamma_2$ and $f = -g$ above $\Gamma$. (Compare Problem 3.2.9.)
   (c) Let $h(z)$ be given by the binomial expansion of $(1-z)^{1/2}$, that is,

$$h(z) = \sum_{n=0}^{\infty} \binom{1/2}{n}(-z)^n, \quad |z| < 1,$$

   where $\binom{w}{n} = \frac{w(w-1)\cdots(w-n+1)}{n!}$. What is the relationship between $h$ and $f$?
4. Prove Theorem 3.4.3(e).