Chapter 1

Introduction

The reader is assumed to be familiar with the complex plane $\mathbb{C}$ to the extent found in most college algebra texts, and to have had the equivalent of a standard introductory course in real analysis (advanced calculus). Such a course normally includes a discussion of continuity, differentiation, and Riemann-Stieltjes integration of functions from the real line to itself. In addition, there is usually an introductory study of metric spaces and the associated ideas of open and closed sets, connectedness, convergence, compactness, and continuity of functions from one metric space to another. For the purpose of review and to establish notation, some of these concepts are discussed in the following sections.

1.1 Basic Definitions

The complex plane $\mathbb{C}$ is the set of all ordered pairs (a, b) of real numbers, with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{and} \qquad (a, b)(c, d) = (ac - bd, ad + bc).$$

If $i = (0, 1)$ and the real number a is identified with $(a, 0)$, then $(a, b) = a + bi$. The expression $a + bi$ can be manipulated as if it were an ordinary binomial expression of real numbers, subject to the relation $i^2 = -1$. With the above definitions of addition and multiplication, $\mathbb{C}$ is a field.

If $z = a + bi$, then a is called the *real part* of z, written $a = \operatorname{Re} z$, and b is called the *imaginary part* of z, written $b = \operatorname{Im} z$. The *absolute value* or *magnitude* or *modulus* of z is defined as $(a^2 + b^2)^{1/2}$. A complex number with magnitude 1 is said to be *unimodular.* An *argument* of z (written $\arg z$) is defined as the angle which the line segment from $(0, 0)$ to (a, b) makes with the positive real axis. The argument is not unique, but is determined up to a multiple of 2π.

If r is the magnitude of z and θ is an argument of z, we may write

$$z = r(\cos\theta + i\sin\theta)$$

and it follows from trigonometric identities that

$$|z_1 z_2| = |z_1||z_2| \qquad \text{and} \qquad \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

(that is, if θ_k is an argument of $z_k, k = 1, 2$, then $\theta_1 + \theta_2$ is an argument of $z_1 z_2$). If $z_2 \neq 0$, then $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$. If $z = a + bi$, the *conjugate* of z is defined as $\overline{z} = a - bi$, and we have the following properties:

$$|\overline{z}| = |z|, \qquad \arg \overline{z} = -\arg z, \qquad \overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2, \qquad \overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2,$$

$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2, \qquad \operatorname{Re} z = (z + \overline{z})/2, \qquad \operatorname{Im} z = (z - \overline{z})/2i, \qquad z\overline{z} = |z|^2.$$

The *distance* between two complex numbers z_1 and z_2 is defined as $d(z_1, z_2) = |z_1 - z_2|$. So $d(z_1, z_2)$ is simply the Euclidean distance between z_1 and z_2 regarded as points in the plane. Thus d defines a metric on $\mathbb{C}$, and furthermore, d is complete, that is, every Cauchy sequence converges. If $z_1, z_2, \ldots$ is sequence of complex numbers, then $z_n \to z$ if and only if $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z$. We say that $z_n \to \infty$ if the sequence of real numbers $|z_n|$ approaches $+\infty$.

Many of the above results are illustrated in the following analytical proof of the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

The geometric interpretation is that the length of a side of a triangle cannot exceed the sum of the lengths of the other two sides. See Figure 1.1.1, which illustrates the familiar representation of complex numbers as vectors in the plane.

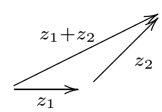

Figure 1.1.1

The proof is as follows:

$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) = |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + \overline{z}_1 z_2\\ &= |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + \overline{z_1\overline{z}_2} = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\overline{z}_2)\\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1\overline{z}_2| = (|z_1| + |z_2|)^2.\end{aligned}$$

The proof is completed by taking the square root of both sides.

If a and b are complex numbers, $[a, b]$ denotes the closed line segment with endpoints a and b. If t_1 and t_2 are arbitrary real numbers with $t_1 < t_2$, then we may write

$$[a, b] = \{a + \frac{t - t_1}{t_2 - t_1}(b - a) : t_1 \leq t \leq t_2\}.$$

The notation is extended as follows. If $a_1, a_2, \ldots, a_{n+1}$ are points in $\mathbb{C}$, a *polygon* from a_1 to a_{n+1} (or a polygon joining a_1 to a_{n+1}) is defined as

$$\bigcup_{j=1}^{n} [a_j, a_{j+1}],$$

often abbreviated as $[a_1, \ldots, a_{n+1}]$.

1.2 Further Topology of the Plane

Recall that two subsets S_1 and S_2 of a metric space are *separated* if there are open sets $G_1 \supseteq S_1$ and $G_2 \supseteq S_2$ such that $G_1 \cap S_2 = G_2 \cap S_1 = \emptyset$, the empty set. A set is *connected* iff it cannot be written as the union of two nonempty separated sets. An open (respectively closed) set is connected iff it is not the union of two nonempty disjoint open (respectively closed) sets.

1.2.1 Definition

A set $S \subseteq \mathbb{C}$ is said to be *polygonally connected* if each pair of points in S can be joined by a polygon that lies in S.

Polygonal connectedness is a special case of path (or arcwise) connectedness, and it follows that a polygonally connected set, in particular a polygon itself, is connected. We will prove in Theorem 1.2.3 that any *open* connected set is polygonally connected.

1.2.2 Definitions

If $a \in \mathbb{C}$ and $r > 0$, then $D(a, r)$ is the open disk with center a and radius r; thus $D(a, r) = \{z : |z - a| < r\}$. The closed disk $\{z : |z - a| \leq r\}$ is denoted by $\overline{D}(a, r)$, and $C(a, r)$ is the circle with center a and radius r.

1.2.3 Theorem

If Ω is an open subset of $\mathbb{C}$, then Ω is connected iff Ω is polygonally connected.

Proof. If Ω is connected and $a \in \Omega$, let Ω_1 be the set of all z in Ω such that there is a polygon in Ω from a to z, and let $\Omega_2 = \Omega \setminus \Omega_1$. If $z \in \Omega_1$, there is an open disk $D(z, r) \subseteq \Omega$ (because Ω is open). If $w \in D(z, r)$, a polygon from a to z can be extended to w, and it follows that $D(z, r) \subseteq \Omega_1$, proving that Ω_1 is open. Similarly, Ω_2 is open. (Suppose $z \in \Omega_2$, and choose $D(z, r) \subseteq \Omega$. Then $D(z, r) \subseteq \Omega_2$ as before.)

Thus Ω_1 and Ω_2 are disjoint open sets, and $\Omega_1 \neq \emptyset$ because $a \in \Omega_1$. Since Ω is connected we must have $\Omega_2 = \emptyset$, so that $\Omega_1 = \Omega$. Therefore Ω is polygonally connected. The converse assertion follows because *any* polygonally connected set is connected. ♣

1.2.4 Definitions

A *region* in $\mathbb{C}$ is an open connected subset of $\mathbb{C}$. A set $E \subseteq \mathbb{C}$ is *convex* if for each pair of points $a, b \in E$, we have $[a, b] \subseteq E$; E is *starlike* if there is a point $a \in E$ (called a *star center*) such that $[a, z] \subseteq E$ for each $z \in E$. Note that any nonempty convex set is starlike and that starlike sets are polygonally connected.

1.3 Analytic Functions

1.3.1 Definition

Let $f : \Omega \to \mathbb{C}$, where Ω is a subset of $\mathbb{C}$. We say that f is *complex-differentiable* at the point $z_0 \in \Omega$ if for some $\lambda \in \mathbb{C}$ we have

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lambda \tag{1}$$

or equivalently,

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lambda. \tag{2}$$

Conditions (3), (4) and (5) below are also equivalent to (1), and are sometimes easier to apply.

$$\lim_{n\to \infty} \frac{f(z_0+h_n)-f(z_0)}{h_n} = \lambda \tag{3}$$

for each sequence $\{h_n\}$ such that $z_0 + h_n \in \Omega \setminus \{z_0\}$ and $h_n \to 0$ as $n \to \infty$.

$$\lim_{n\to \infty} \frac{f(z_n)-f(z_0)}{z_n-z_0} = \lambda \tag{4}$$

for each sequence $\{z_n\}$ such that $z_n \in \Omega \setminus \{z_0\}$ and $z_n \to z_0$ as $n \to \infty$.

$$f(z) = f(z_0) + (z-z_0)(\lambda + \epsilon(z)) \tag{5}$$

for all $z \in \Omega$, where $\epsilon : \Omega \to \mathbb{C}$ is continuous at z_0 and $\epsilon(z_0) = 0$.

To show that (1) and (5) are equivalent, just note that ϵ may be written in terms of f as follows:

$$\epsilon(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} - \lambda & \text{if } z \neq z_0 \\ 0 & \text{if } z = z_0. \end{cases}$$

The number λ is unique. It is usually written as $f'(z_0)$, and is called the *derivative* of f at z_0.

If f is complex-differentiable at every point of Ω, f is said to be *analytic* or *holomorphic* on Ω. Analytic functions are the basic objects of study in complex variables.

Analyticity on a nonopen set $S \subseteq \mathbb{C}$ means analyticity on an open set $\Omega \supseteq S$. In particular, f is analytic at a point z_0 iff f is analytic on an open set Ω with $z_0 \in \Omega$. If f_1 and f_2 are analytic on Ω, so are $f_1 + f_2, f_1 - f_2, kf_1$ for $k \in \mathbb{C}, f_1 f_2$, and f_1/f_2 (provided that f_2 is never 0 on Ω). Furthermore,

$$(f_1+f_2)' = f_1' + f_2', \quad (f_1-f_2)' = f_1' - f_2', \quad (kf_1)' = kf_1'$$

$$(f_1 f_2)' = f_1 f_2' + f_1' f_2, \quad \left(\frac{f_1}{f_2}\right)' = \frac{f_2 f_1' - f_1 f_2'}{f_2^2}.$$

The proofs are identical to the corresponding proofs for functions from $\mathbb{R}$ to $\mathbb{R}$.

Since $\frac{d}{dz}(z) = 1$ by direct computation, we may use the rule for differentiating a product (just as in the real case) to obtain

$$\frac{d}{dz}(z^n) = nz^{n-1},\ n = 0, 1, \ldots$$

This extends to $n = -1, -2, \ldots$ using the quotient rule.

If f is analytic on Ω and g is analytic on $f(\Omega) = \{f(z) : z \in \Omega\}$, then the composition $g \circ f$ is analytic on Ω and

$$\frac{d}{dz} g(f(z)) = g'(f(z)f'(z)$$

just as in the real variable case.

As an example of the use of Condition (4) of (1.3.1), we now prove a result that will be useful later in studying certain inverse functions.

1.3.2 Theorem

Let g be analytic on the open set Ω_1, and let f be a continuous complex-valued function on the open set Ω. Assume
(i) $f(\Omega) \subseteq \Omega_1$,
(ii) g' is never 0,
(iii) $g(f(z)) = z$ for all $z \in \Omega$ (thus f is 1-1).
Then f is analytic on Ω and $f' = 1/(g' \circ f)$.

Proof. Let $z_0 \in \Omega$, and let $\{z_n\}$ be a sequence in $\Omega \setminus \{z_0\}$ with $z_n \to z_0$. Then

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} = \frac{f(z_n) - f(z_0)}{g(f(z_n)) - g(f(z_0))} = \left[\frac{g(f(z_n)) - g(f(z_0))}{f(z_n) - f(z_0)}\right]^{-1}.$$

(Note that $f(z_n) \neq f(z_0)$ since f is 1-1 and $z_n \neq z_0$.) By continuity of f at z_0, the expression in brackets approaches $g'(f(z_0))$ as $n \to \infty$. Since $g'(f(z_0)) \neq 0$, the result follows. ♣

1.4 Real-Differentiability and the Cauchy-Riemann Equations

Let $f : \Omega \to \mathbb{C}$, and set $u = \operatorname{Re} f, v = \operatorname{Im} f$. Then u and v are real-valued functions on Ω and $f = u + iv$. In this section we are interested in the relation between f and its real and imaginary parts u and v. For example, f is continuous at a point z_0 iff both u and v are continuous at z_0. Relations involving derivatives will be more significant for us, and for this it is convenient to be able to express the idea of differentiability of real-valued function of two real variables by means of a single formula, without having to consider partial derivatives separately. We do this by means of a condition analogous to (5) of (1.3.1).

Convention

From now on, Ω will denote an open subset of $\mathbb{C}$, unless otherwise specified.

1.4.1 Definition

Let $g : \Omega \to \mathbb{R}$. We say that g is *real-differentiable* at $z_0 = x_0 + iy_0 \in \Omega$ if there exist real numbers A and B, and real functions ϵ_1 and ϵ_2 defined on a neighborhood of (x_0, y_0), such that ϵ_1 and ϵ_2 are continuous at (x_0, y_0), $\epsilon_1(x_0, y_0) = \epsilon_2(x_0, y_0) = 0$, and

$$g(x, y) = g(x_0, y_0) + (x - x_0)[A + \epsilon_1(x, y)] + (y - y_0)[B + \epsilon_2(x, y)]$$

for all (x, y) in the above neighborhood of (x_0, y_0).

It follows from the definition that if g is real-differentiable at (x_0, y_0), then the partial derivatives of g exist at (x_0, y_0) and

$$\frac{\partial g}{\partial x}(x_0, y_0) = A, \quad \frac{\partial g}{\partial y}(x_0, y_0) = B.$$

If, on the other hand, $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ exist at (x_0, y_0) and one of these exists in a neighborhood of (x_0, y_0) and is continuous at (x_0, y_0), then g is real-differentiable at (x_0, y_0). To verify this, assume that $\frac{\partial g}{\partial x}$ is continuous at (x_0, y_0), and write

$$g(x, y) - g(x_0, y_0) = g(x, y) - g(x_0, y) + g(x_0, y) - g(x_0, y_0).$$

Now apply the mean value theorem and the definition of partial derivative respectively (Problem 4).

1.4.2 Theorem

Let $f : \Omega \to \mathbb{C}, u = \operatorname{Re} f, v = \operatorname{Im} f$. Then f is complex-differentiable at (x_0, y_0) iff u and v are real-differentiable at (x_0, y_0) and the *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied at (x_0, y_0). Furthermore, if $z_0 = x_0 + iy_0$, we have

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Proof. Assume f complex-differentiable at z_0, and let ϵ be the function supplied by (5) of (1.3.1). Define $\epsilon_1(x, , y) = \operatorname{Re} \epsilon(x, y), \epsilon_2(x, y) = \operatorname{Im} \epsilon(x, y)$. If we take real parts of both sides of the equation

$$f(x) = f(z_0) + (z - z_0)(f'(z_0) + \epsilon(z)) \tag{1}$$

we obtain

$$\begin{aligned} u(x, y) = u(x_0, y_0) &+ (x - x_0)[\operatorname{Re} f'(z_0) + \epsilon_1(x, y)] \\ &+ (y - y_0)[- \operatorname{Im} f'(z_0) - \epsilon_2(x, y)]. \end{aligned}$$

It follows that u is real-differentiable at (x_0, y_0) with

$$\frac{\partial u}{\partial x}(x_0, y_0) = \operatorname{Re} f'(z_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\operatorname{Im} f'(z_0). \tag{2}$$

Similarly, take imaginary parts of both sides of (1) to obtain

$$\begin{aligned} v(x, y) = v(x_0, y_0) &+ (x - x_0)[\operatorname{Im} f'(z_0) + \epsilon_2(x, y)] \\ &+ (y - y_0)[\operatorname{Re} f'(z_0) + \epsilon_1(x, y)] \end{aligned}$$

and conclude that

$$\frac{\partial v}{\partial x}(x_0, y_0) = \operatorname{Im} f'(z_0), \quad \frac{\partial v}{\partial y}(x_0, y_0) = \operatorname{Re} f'(z_0). \tag{3}$$

The Cauchy-Riemann equations and the desired formulas for $f'(z_0)$ follow from (2) and (3).

Conversely, suppose that u and v are real-differentiable at (x_0, y_0) and satisfy the Cauchy-Riemann equations there. Then we may write equations of the form

$$\begin{aligned} u(x, y) = u(x_0, y_0) &+ (x - x_0)[\frac{\partial u}{\partial x}(x_0, y_0) + \epsilon_1(x, y)] \\ &+ (y - y_0)[\frac{\partial u}{\partial y}(x_0, y_0) + \epsilon_2(x, y)], \qquad (4) \\ v(x, y) = v(x_0, y_0) &+ (x - x_0)[\frac{\partial v}{\partial x}(x_0, y_0) + \epsilon_3(x, y)] \\ &+ (y - y_0)[\frac{\partial v}{\partial y}(x_0, y_0) + \epsilon_4(x, y)]. \qquad (5) \end{aligned}$$

Since $f = u + iv$, (4) and (5) along with the Cauchy-Riemann equations yield

$$f(z) = f(z_0) + (z - z_0)[\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) + \epsilon(z)]$$

where, at least in a neighborhood of z_0,

$$\epsilon(z) = \left[\frac{x - x_0}{z - z_0}\right][\epsilon_1(x, y) + i\epsilon_3(x, y)] + \left[\frac{y - y_0}{z - z_0}\right][\epsilon_2(x, y) + i\epsilon_4(x, y)] \text{ if } z \neq z_0; \ \epsilon(z_0) = 0.$$

It follows that f is complex-differentiable at z_0. ♣

1.5 The Exponential Function

In this section we extend the domain of definition of the exponential function (as normally encountered in calculus) from the real line to the entire complex plane. If we require that the basic rules for manipulating exponentials carry over to the extended function, there is only one possible way to define $\exp(z)$ for $z = x+iy \in \mathbb{C}$. Consider the following sequence of "equations" that exp should satisfy:

$$\begin{aligned}
\exp(z) &= \exp(x+iy) \\
&\text{“}=\text{”}\ \exp(x)\exp(iy) \\
&\text{“}=\text{”}\ e^x\left(1+iy+\frac{(iy)^2}{2!}+\cdots\right) \\
&\text{“}=\text{”}\ e^x\left[\left(1-\frac{y^2}{2!}+\frac{y^4}{4!}-\cdots\right)+i\left(y-\frac{y^3}{3!}+\frac{y^5}{5!}-\cdots\right)\right] \\
&\text{“}=\text{”}\ e^x(\cos y+i\sin y).
\end{aligned}$$

Thus we have only one candidate for the role of exp on $\mathbb{C}$.

1.5.1 Definition

If $z = x+iy \in \mathbb{C}$, let $\exp(z) = e^x(\cos y + i\sin y)$. Note that if $z = x \in \mathbb{R}$, then $\exp(z) = e^x$ so exp is indeed a extension of the real exponential function.

1.5.2 Theorem

The exponential function is analytic on $\mathbb{C}$ and $\frac{d}{dz}\exp(z) = \exp(z)$ for all z.

Proof. The real and imaginary parts of $\exp(x+iy)$ are, respectively, $u(x,y) = e^x\cos y$ and $v(x,y) = e^x \sin y$. At any point (x_0, y_0), u and v are real-differentiable (see Problem 4) and satisfy the Cauchy-Riemann equations there. The result follows from (1.4.2). ♣

Functions such as exp and the polynomials that are analytic on $\mathbb{C}$ are called *entire functions*.

The exponential function is of fundamental importance in mathematics, and the investigation of its properties will be continued in Section 2.3.

1.6 Harmonic Functions

1.6.1 Definition

A function $g : \Omega \to \mathbb{R}$ is said to be *harmonic* on Ω if g has continuous first and second order partial derivatives on Ω and satisfies *Laplace's equation*

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$$

on all of Ω.

After some additional properties of analytic functions have been developed, we will be able to prove that the real and imaginary parts of an analytic function on Ω are harmonic on Ω. The following theorem is a partial converse to that result, namely that a harmonic on Ω is locally the real part of an analytic function.

1.6.2 Theorem

Suppose $u : \Omega \to \mathbb{R}$ is harmonic on Ω, and D is any open disk contained in Ω. Then there exists a function $v : D \to \mathbb{R}$ such that $u + iv$ is analytic on D.

The function v is called a *harmonic conjugate* of u.

Proof. Consider the differential $Pdx + Qdy$, where $P = -\frac{\partial u}{\partial y}$, $Q = \frac{\partial u}{\partial x}$. Since u is harmonic, P and Q have continuous partial derivatives on Ω and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. It follows (from calculus) that $Pdx + Qdy$ is a locally exact differential. In other words, there is a function $v : D \to \mathbb{R}$ such that $dv = Pdx + Qdy$. But this just means that on D we have

$$\frac{\partial v}{\partial x} = P = -\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = Q = \frac{\partial u}{\partial x}.$$

Hence by (1.4.2) (and Problem 4), $u + iv$ is analytic on D.

Problems

1. Prove the parallelogram law $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$ and give a geometric interpretation.
2. Show that $|z_1 + z_2| = |z_1| + |z_2|$ iff z_1 and z_2 lie on a common ray from 0 iff one of z_1 or z_2 is a nonnegative multiple of the other.
3. Let z_1 and z_2 be nonzero complex numbers, and let $\theta, 0 \leq \theta \leq \pi$, be the angle between them. Show that
   (a) $\operatorname{Re} z_1\overline{z}_2 = |z_1||z_2|\cos\theta$, $\operatorname{Im} z_1\overline{z}_2 = \pm|z_1||z_2|\sin\theta$, and consequently
   (b) The area of the triangle formed by z_1, z_2 and $z_2 - z_1$ is $|\operatorname{Im} z_1\overline{z}_2|/2$.
4. Let $g : \Omega \to \mathbb{R}$ be such that $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ exist at $(x_0, y_0) \in \Omega$, and suppose that one of these partials exists in a neighborhood of (x_0, y_0) and is continuous at (x_0, y_0). Show that g is real-differentiable at (x_0, y_0).
5. Let $f(x) = \overline{z}, z \in \mathbb{C}$. Show that although f is continuous everywhere, it is nowhere differentiable.
6. Let $f(z) = |z|^2, z \in \mathbb{C}$. Show that f is complex-differentiable at $z = 0$, but nowhere else.
7. Let $u(x, y) = \sqrt{|xy|}, (x, y) \in \mathbb{C}$. Show that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ both exist at (0,0), but u is not real-differentiable at (0,0).
8. Show that the field of complex numbers is isomorphic to the set of matrices of the form

   $$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

   with $a, b \in \mathbb{R}$.
9. Show that the complex field cannot be ordered. That is, there is no subset $P \subseteq \mathbb{C}$ of "positive elements" such that
   (a) P is closed under addition and multiplication.
   (b) If $z \in P$, then exactly one of the relations $z \in P,\ z = 0,\ -z \in P$ holds.

10. (A characterization of absolute value) Show that there is a unique function $\alpha : \mathbb{C} \to \mathbb{R}$ such that
(i) $\alpha(x) = x$ for all real $x \geq 0$;
(ii) $\alpha(zw) = \alpha(z)\alpha(w), z, w \in \mathbb{C}$;
(iii) α is bounded on the unit circle $C(0, 1)$.

    Hint: First show that $\alpha(z) = 1$ for $|z| = 1$.

11. (Another characterization of absolute value) Show that there is a unique function $\alpha : \mathbb{C} \to \mathbb{R}$ such that
(i) $\alpha(x) = x$ for all real $x \geq 0$;
(ii) $\alpha(zw) = \alpha(z)\alpha(w), z, w \in \mathbb{C}$;
(iii) $\alpha(z + w) \leq \alpha(z) + \alpha(w), z, w \in \mathbb{C}$.

12. Let α be a complex number with $|\alpha| < 1$. Prove that

$$\left| \frac{z - \alpha}{1 - \overline{\alpha} z} \right| = 1 \quad \text{iff} \quad |z| = 1.$$

13. Suppose $z \in \mathbb{C}, z \neq 0$. Show that $z + \frac{1}{z}$ is real iff $\operatorname{Im} z = 0$ or $|z| = 1$.

14. In each case show that u is harmonic and find the harmonic conjugate v such that $v(0, 0) = 0$.
(i) $u(x, y) = e^y \cos x$;
(ii) $u(x, y) = 2x - x^3 + 3xy^2$.

15. Let $a, b \in \mathbb{C}$ with $a \neq 0$, and let $T(z) = az + b, z \in \mathbb{C}$.
(i) Show that T maps the circle $C(z_0, r)$ onto the circle $C(T(z_0), r|a|)$.
(ii) For which choices of a and b will T map $C(0, 1)$ onto $C(1 + i, 2)$?
(iii) In (ii), is it possible to choose a and b so that $T(1) = -1 + 3i$?

16. Show that $f(z) = e^{\operatorname{Re} z}$ is nowhere complex-differentiable.

17. Let f be a complex-valued function defined on an open set Ω that is symmetric with respect to the real line, that is, $z \in \Omega$ implies $\overline{z} \in \Omega$. (Examples are $\mathbb{C}$ and $D(x, r)$ where $x \in \mathbb{R}$.) Set $g(z) = \overline{f(\overline{z})}$, and show that g is analytic on Ω if and only if f is analytic on Ω.

18. Show that an equation for the circle $C(z_0, r)$ is $z\overline{z} - \overline{z}_0 z - z_0 \overline{z} + z_0 \overline{z}_0 = r^2$.

19. (Enestrom's theorem) Suppose that $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $n \geq 1$ and $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$. Prove that the zeros of the polynomial $P(z)$ all lie outside the open unit disk $D(0, 1)$.

    Hint: Look at $(1 - z)P(z)$, and show that $(1 - z)P(z) = 0$ implies that $a_0 = (a_0 - a_1)z + (a_1 - a_2)z^2 + \cdots + (a_{n-1} - a_n)z^n + a_n z^{n+1}$, which is impossible for $|z| < 1$.

20. Continuing Problem 19, show that if $a_{j-1} > a_j$ for all j, then all the zeros of $P(z)$ must be outside the *closed* unit disk $\overline{D}(0, 1)$.

    Hint: If the last equation of Problem 19 holds for some z with $|z| \leq 1$, then $z = 1$.