Solutions to Problems

CHAPTER 1

Section 1.2

1. \( A \cap B \cap C = \{4\}, A \cup (B \cap C^c) = \{0, 1, 2, 3, 4, 5, 7\}, \)
   \( (A \cup B) \cap C^c = \{0, 1, 3, 5, 7\}, (A \cap B) \cap [(A \cup C)^c] = \emptyset \)

3. \( W \) = registered Whigs
   \( F = \) those who approve of Fillmore
   \( E = \) those who favor the electoral college
   \[ x + y + z + 100 = 550 \]
   \[ x + y + 25 = 325 \]
   Thus \( z + 100 - 25 = 550 - 325 = 225 \), so \( z = 150 \)

\[ \begin{array}{c}
\text{F} & 25 \\
\text{W} \\
\text{E}_{(550)}
\end{array} \]

Problem 1.2.3
5. If \( a < x < b \) then \( x \leq b - 1/n \) for some \( n \), hence \( x \in \bigcup_{n=1}^{\infty} (a, b - 1/n] \). Conversely, if \( x \in \bigcup_{n=1}^{\infty} (a, b - 1/n] \), then \( a < x \leq b - 1/n \) for some \( n \), hence \( x \in (a, b) \). The other arguments are similar.

9. \( x \in A \cap (\bigcup_{i} B_i) \) iff \( x \in A \) and \( x \in B_i \) for at least one \( i \)
   \( \quad \) iff \( x \in A \cap B_i \) for at least one \( i \)
   \( \quad \) iff \( x \in \bigcup_{i} (A \cap B_i) \).

Section 1.3

1. (a) Let \( A \subset \Omega \), and let \( \mathcal{F} \) consist of \( \varnothing \), \( \Omega \), \( A \) and \( A^c \).
   (b) Let \( A_1, A_2, \ldots \) be disjoint sets whose union is \( \Omega \). Let \( \mathcal{F} \) consist of all finite or countable unions of the \( A_i \) (including \( \varnothing \) and \( \Omega \)).

Section 1.4

1. 432

2. (a) The sequence of face values may be chosen in 9 ways (the high card may be anything from an ace to a six); the suit of each card in the straight may be chosen in 4 ways. Thus the probability of a straight is \( 9(\binom{4}{3})(\binom{3}{3})(\binom{3}{3})(\binom{3}{3})(\binom{3}{3})(\binom{3}{3}) \).
   (b) Select the face value to appear three times (13 choices); select the two other face values [\( \binom{13}{2} \) choices]. Select three suits out of four for the face value appearing three times, and select one suit for each of the two odd cards. Thus \( p = 13(\binom{13}{2})(\binom{4}{3})(16)/(\binom{52}{3}) \).
   (c) Select two face values for the pairs [\( \binom{13}{2} \) possibilities]; then choose two suits out of four for each of the pairs. Finally, choose the odd card from the 44 cards that remain when the two face values are removed from the deck. Thus \( p = (\binom{13}{2})(\binom{4}{3})(44)/(\binom{52}{3}) \).

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4. (a) \( (52)(48) \cdots (20)(16)/52^{10} \)
   (b) \[ 4(\binom{4}{3})(\binom{13}{2}) + 4(\binom{4}{3})(\binom{13}{2}) \]

5. We must have exactly one number \( > 8 \) (2 choices) and exactly 3 numbers \( < 8 \) (\( \binom{5}{3} \) possibilities). Thus \( p = 2(\binom{5}{3})(\binom{10}{3}) \).

6. \( (m + 1)/(m + w) \)

7. \( 1 - (\binom{10}{3})(\binom{10}{3}) \)

8. \( [2(\binom{4}{3})(\binom{4}{3}) - (\binom{4}{3})(\binom{4}{3})(\binom{4}{3})]/(\binom{52}{3}) \)

9. Let \( A_i \) be the event that the ticket numbered \( i \) appears at the \( i \)th drawing. The probability of at least one match is \( P(A_1 \cup \cdots \cup A_n) \). Now \( P(A_i) = (n - 1)!/n! = 1/n \) (the first ticket must have number 1, the second may be any one of \( n - 1 \) remaining possibilities, the third one of \( n - 2 \), etc.) By symmetry, \( P(A_i) = 1/n \) for all \( i \). Similarly, \( P(A_i \cap A_j) = (n - 2)!/n! = 1/n(n - 1) \), \( i < j \), \( P(A_i \cap A_j \cap A_k) = (n - 3)!/n! = 1/n(n - 2)(n - 3) \), \( i < j < k \), etc. By the expansion formula (1.4.5) for the probability of a union, \( P(A_1 \cup \cdots \cup \)
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\[ A_n = n(1/n) - (\frac{1}{2})(n-2)!/n! + (\frac{1}{3})(n-3)!/n! - \cdots + (-1)^{n-1}(\frac{n}{n})0!/n! = 1 - 1/2! + 1/3! - \cdots + (-1)^{n-1}/n! \]

11. \((365)^r / 365^r\)

12. The number of arrangements with occupancy numbers \(r_1 = r_2 = 4, r_3 = r_4 = r_5 = 2, r_6 = 0\), is \((\frac{1}{2})^4(\frac{1}{3})^4(\frac{1}{3})^2(\frac{2}{3})^2 = 14! / 4! 4! 2! 2! 1! 1!\). We must select 2 boxes out of 6 to receive 4 balls, then 3 boxes of the remaining 4 to receive 2; the remaining box receives 0. This can be done in \((\frac{1}{2})^4(\frac{1}{3})^4(\frac{1}{3})^2(\frac{2}{3})^2\) ways. Thus the total number is \(14! / (4!)^4(2!)^2\)!

Section 1.5

4. (a) This is a multinomial problem. The probability is

\[ \frac{30!}{10! 10! 10!} \left( \frac{1}{100} \right)^{10} \left( \frac{30}{100} \right)^{10} \left( \frac{20}{100} \right)^{10} \]

(b) This is a binomial problem. The probability is

\[ \binom{30}{12} \left( \frac{30}{100} \right)^{12} \left( \frac{70}{100} \right)^{18} \]

5. The probability that there will be exactly 3 ones and no two (and 3 from \{3, 4, 5, 6\}) is, by the multinomial formula, \((\frac{1}{2})^3(\frac{1}{3})^1(\frac{2}{3})^3\). The probability of exactly 4 ones and 1 two is \((\frac{1}{2})^4(\frac{1}{3})^1(\frac{2}{3})^1\). The sum of these expressions is the desired probability.

Section 1.6

1. \[ (7p^2q^6 + 6p^2q^8 + 5p^2q^4) / \sum_{k=0}^{6} \binom{6}{k} p^k q^{10-k} \]

2. \[ 1 - q^n - npq^{n-1} - \binom{n}{2} p^2 q^{n-2} / 1 - q^n \]

3. We have the following tree diagram:

\[ \begin{array}{c}
\text{n heads} \\
\frac{1}{2} \\
\frac{1}{2} \text{Less than n heads} \\
\frac{1}{2} \text{n heads} \\
\frac{1}{2} \text{Less than n heads} \\
\end{array} \]

PROBLEM 1.6.3
The desired probability is
\[
P\{\text{unbiased coin used and } n \text{ heads obtained} \} \quad \frac{P\{n \text{ heads obtained} \}}{
\frac{(1/2)^{n+1}}{(1/2)^{n+1} + (1/2)^n}}
\]

4. Let \( A = \{\text{heads}\} \). By the theorem of total probability,
\[
P(A) = \sum_{n=1}^{\infty} P(I = n)P(A \mid I = n) = \sum_{n=1}^{\infty} \frac{(1/2)^n e^{-n}}{
(1/2)e^{-1}/(1 - \frac{1}{2}e^{-1})}
\]

**Remark.** To formalize this problem, we may take \( \Omega = \text{all pairs } (n, i), \) \( n = 1, 2, \ldots, i = 0, 1 \) [where \( (n, 1) \) indicates that \( I = n \) and the coin comes up heads, and \( (n, 0) \) indicates that \( I = n \) and the coin comes up tails].

We assign \( p(n, i) = (1/2)^n e^{-n} \) if \( i = 1 \), and \( (1/2)^n(1 - e^{-n}) \) if \( i = 0 \). Alternatively, a tree diagram can be constructed; a typical path is indicated in the diagram.

\[
\begin{align*}
&H \\
\text{e}^{-n} &
\text{e}^{-n} \\
(\frac{1}{2})^n &
1 - e^{-n} \\
&n \\
&\text{T}
\end{align*}
\]

**Problem 1.6.4**

5. (a) \( \left(\frac{5}{9}\right)\left(\frac{20}{9}\right)/\left(\frac{10}{13}\right) = .36 \)
(b) \(\left[\left(\frac{5}{9}\right)\left(\frac{20}{9}\right) + \left(\frac{4}{9}\right)\left(\frac{20}{9}\right)\right]/\left(\frac{10}{13}\right) = 2\left(\frac{5}{9}\right)\left(\frac{20}{9}\right)/\left(\frac{10}{13}\right) = .48 \)
(c) \( 2\left(\frac{4}{9}\right)\left(\frac{19}{9}\right)/\left(\frac{10}{13}\right) = .15 \)
(d) \( 2\left(\frac{4}{13}\right)/\left(\frac{13}{13}\right) = .01 \)

7. By the theorem of total probability, if \( X_n \) is the result of the toss at \( t = n \), then
(see diagram)
\[
y_{n+1} = P\{X_{n+1} = H\} = P(X_n = H)P\{X_{n+1} = H \mid X_n = H\} + P(X_n = T)P\{X_{n+1} = H \mid X_n = T\} = y_n(1/2) + (1 - y_n)(3/4)
\]
Thus \( y_{n+1} + (1/4)y_n = 3/4 \). The solution to the homogeneous equation \( y_{n+1} + (1/4)y_n = 0 \) is \( y_n = A(-1/4)^n \). Since the “forcing function” \( 3/4 \) is constant, we assume as a “particular solution” \( y_n = c. \) Then \((5c)/4 = 3/4\), or \( c = 3/5 \). Thus the general solution is \( y_n = A(-1/4)^n + 3/5 \). Since \( y_0 \) is given as \( 1/2 \), we have \( A = 1/2 - 3/5 = -1/10 \). Thus \( y_n = 3/5 - 1/10(-1/4)^n \).
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Problem 1.6.7

9. We have (see diagram) \( P(D \mid R) = P(D \cap R) / P(R) = .2(9)(.25)/.2(9)(.25) + .8(3)(.25) = 3/7 \)

Problem 1.6.9

CHAPTER 2

Section 2.2

1. (a) \( \{ \omega; R(\omega) \in E^1 \} = \Omega \in \mathcal{F} \), hence \( E^1 \in \mathcal{G} \). If \( B_1, B_2, \ldots \in \mathcal{G} \), then \( \{ \omega; R(\omega) \in \bigcup_{n=1}^{\infty} B_n \} = \bigcup_{n=1}^{\infty} \{ \omega; R(\omega) \in B_n \} \in \mathcal{F} \) since each \( B_n \in \mathcal{G} \); hence \( \bigcup_{n=1}^{\infty} B_n \in \mathcal{G} \).

(b) \( \mathcal{G} \) is a sigma field containing the intervals, hence is at least as large as the smallest sigma-field containing the intervals. Thus all Borel sets belong to \( \mathcal{G} \).

Section 2.3

1. (a) \( F_R(x) = \tfrac{1}{2} e^x, x \leq 0; F_R(x) = 1 - \tfrac{1}{2} e^{-x}, x \geq 0 \)

(b) 1. \( P\{|R| \leq 2\} = P\{-2 \leq R \leq 2\} = \int_{-2}^{2} f_R(x) \, dx = F_R(2) - F_R(-2) = \)}
2. \( P(|R| \leq 2 \text{ or } R \geq 0) = P(R \geq -2) = 1 - F_R(-2) = 1 - \frac{1}{2}e^{-2} \)

3. \( P(|R| \leq 2 \text{ and } R \leq -1) = P(-2 \leq R \leq -1) = \frac{1}{2}(e^{-1} - e^{-2}) \)

4. \( P(|R| + |R - 3| \leq 3) = P(0 \leq R \leq 3) = \frac{1}{2}(1 - e^{-3}) \)

5. \( P(R^3 - R^2 - R - 2 \leq 0) = P((R - 2)(R^2 + R + 1) \leq 0) = P(R \leq 2) \)
    (since \( R^2 + R + 1 > 0 \))
    \( = 1 - \frac{1}{2}e^{-2} \)

6. \( P(e^{\sin R} \geq 1) = P(\sin \pi R \geq 0) = P(0 \leq R \leq 1) + P(2 \leq R \leq 3) \)
    \[ + P(4 \leq R \leq 5) + \cdots \]
    \[ + P(-2 \leq R \leq -1) + P(-4 \leq R \leq -3) + \cdots \]

But \( P(2n - 1 \leq R \leq 2n) = P(-2n \leq R \leq -2n + 1) \) since \( f_R \) is an even function. Thus

\[ P(e^{\sin R} \geq 1) = \sum_{n=0}^{\infty} P(2n \leq R \leq 2n + 1) + \sum_{n=1}^{\infty} P(2n - 1 \leq R \leq 2n) \]
\[ = P(R \geq 0) = \frac{1}{2} \]

7. \( P(R \text{ irrational}) = 1 - P(R \text{ rational}). \) The rationals are a countable set, say \( \{x_1, x_2, \ldots\}. \) Hence, \( P(R \text{ rational}) = \sum_{i=1}^{\infty} P(R = x_i) = 0, \) so \( P(R \text{ irrational}) = 1. \)

2. By direct enumeration,

\[ P(R = 0) = 5 + 5^4q + 5^2q^2 + 5^3q^3 + 5q^4 + q^5 \]
\[ P(R = 1) = 4p^5q + 6p^3q^2 + 6p^2q^3 + 4pq^4 \]
\[ P(R = 2) = 3p^3q^2 + 3p^2q^3 \]

Section 2.4

1. If \( 0 < y < 1, \)

\[ F_2(y) = P(R_1 \leq y) + P\left(R_1 \geq \frac{1}{y}\right) = \int_0^y e^{-x} \, dx + \int_{1/y}^{\infty} e^{-x} \, dx \]
\[ = 1 - e^{-y} + e^{-1/y} \]

\[ F_2(y) = 0, \quad y \leq 0; \quad F_2(y) = 1, \quad y \geq 1 \]

\( F_2 \) is the integral of

\[ f_2(y) = \frac{d}{dy} F_2(y) = e^{-y} + \frac{1}{y^2} e^{-1/y}, \quad 0 < y < 1 \]
\[ = 0 \text{ elsewhere} \]

Hence \( R_2 \) is absolutely continuous.

2. \( f_2(y) = \frac{1}{2y}, \quad e^{-1} < y < e \)
\[ = 0 \text{ elsewhere} \]
3. 
\[ f_2(y) = \begin{cases} 2y^{-2}, & 2 < y < 4 \\ \frac{1}{2}y^{-3/2}, & y > 4 \\ 0, & y < 2 \end{cases} \]

4. If \( 2 \leq y \leq 4 \), \( F_2(y) = P\{R_2 \leq y\} = \int_1^{\sqrt{y/2}} \frac{1}{x^2} \, dx = 1 - \frac{2}{y} \)
   
   If \( 4 \leq y < 5 \), \( F_2(y) = \frac{1}{2} \)
   
   If \( y \geq 5 \), \( F_2(y) = 1 \)
   
   \( P\{R_2 = 5\} = P\{R_1 > 2\} = \frac{1}{3} \), so \( R_2 \) is not absolutely continuous.

7. Consider the graph of \( y = g(x) \). The horizontal line at height \( y \) will intersect

![Graph](image)

**Problem 2.4.7**

the graph at points (say) \( x_{i_1}, \ldots, x_{i_k} \), with \( x_j \in I_j \). If we choose a sufficiently small interval \((a, b)\) about \( y \), we have (except for finitely many \( y \), which lead to intersections at the endpoints of intervals)

\[ F_2(b) - F_2(a) = P\{a < R_2 \leq b\} = \sum_{j=1}^{n} P\{c_j < R_1 \leq d_j\} \]

where \( c_j = h_j(a) \) and \( d_j = h_j(b) \) if \( j \in \{i_1, \ldots, i_k\} \) and \( h_j \) is increasing at \( x_j \)

\[ c_j = h_j(b) \) and \( d_j = h_j(a) \) if \( j \in \{i_1, \ldots, i_k\} \) and \( h_j \) is decreasing at \( x_j \)

\( c_j = \) (say) 1 and \( d_j = 0 \) if \( j \notin \{i_1, \ldots, i_k\} \). Thus

\[ F_2(b) - F_2(a) = \sum_{j=1}^{n} \int_{x_{i_j}}^{x_{i_{j+1}}} f_1(x) \, dx = \sum_{j=1}^{n} [F_1(d_j) - F_1(c_j)] \]

where \( F_1(d_j) - F_1(c_j) \) is interpreted as 0 if \( c_j > d_j \). Differentiate with respect to \( b \) to obtain \( f_2(b) = \sum_{j=1}^{n} f_1[h_j(b)] |h'_j(b)| \), as desired.

**Section 2.5**

1. (a) \( \frac{1}{3} \)
   
   (b) \( \frac{1}{2} \)
   
   (c) \( \frac{1}{3} + F_R(1.5) - F_R(5) = \frac{1}{3} + \frac{1}{3} - \frac{1}{3}(\frac{3}{4}) = \frac{7}{12} \)
   
   (d) \( F_R(3) - F_R(5) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} \)
Section 2.6

4. (a) \( \frac{3}{5} \)
(b) \( \frac{3}{5} \)
(c) \( \frac{5 + \ln 4}{8} \)
(d) \( \frac{5}{8} \)
(e) \( \frac{1}{8} \)
(f) \( \frac{1}{2} \)
(g) \( 1 - \frac{1}{2}e^{-1} \)

In each case, the probability is \( \frac{1}{8} \) (shaded area).

Section 2.7

1. (a) \( f_1(x) = 6x^2 - 4x^3 (0 \leq x \leq 1) \), \( f_2(y) = 2y (0 \leq y \leq 1) \)
(b) \( \frac{3}{8} \)
2. \( f_1(x) = 2e^{-x} - 2e^{-2x}, x \geq 0 \)
   \( f_2(y) = 2e^{-2y}, y \geq 0 \)

3.

\[
\begin{align*}
&f_1(x) \quad f_2(y) \quad f_3(z) \\
&\quad \quad \frac{1}{2} \quad 1 \quad \frac{1}{2}
\end{align*}
\]

\text{Problem 2.7.3}

8. (a) \( P\{R_1^2 + R_2 \leq 1\} = \int_0^1 \int_0^{1-x^2} \frac{1}{3} (x + y) \, dy \, dx = \frac{31}{480} \)

(b) Let \( A = \{R_1 \leq 1, R_2 \leq 1\}, \ B = \{R_1 \leq 1, R_2 > 1\}, \ C = \{R_1 > 1, R_2 \leq 1\} \)

\[
\begin{array}{|c|c|}
\hline
& \text{B} \\
\text{A} & & \text{C} \\
\hline
0 & 1 & 2 \\
\hline
\end{array}
\]

\text{Problem 2.7.8}

The probability that at least one of the random variables is \( \leq 1 \) is

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4}
\]

The probability that exactly one of the random variables is \( \leq 1 \) is \( P(B \cup C) = P(B) + P(C) = \frac{1}{2} \). If \( D = \{\text{exactly one random variable} \leq 1\} \) and \( E = \{\text{at least one random variable} \leq 1\} \), then

\[
P(D \mid E) = \frac{P(D \cap E)}{P(E)} = \frac{P(D)}{P(E)} = \frac{1/2}{5/8} = \frac{4}{5}
\]

(Notice that, for example, \( P(B) = \int_0^1 \int_0^{2} \frac{1}{3} (x + y) \, dy \, dx = \frac{1}{4} \), etc.)

(c) \( P\{R_1 \leq 1, R_2 \leq 1\} = \frac{3}{8} \neq P\{R_1 \leq 1\} P\{R_2 \leq 1\} = (3/8)^2 \), hence the random variables are not independent.
Section 2.8

1. \( f_3(z) = \frac{1}{3}, \; 0 \leq z \leq 1; f_3(z) = \frac{1}{3}z^{-3/2}, \; z > 1; f_3(z) = 0, \; z < 0 \)

2. (a) \( f_3(z) = ze^{-z}, \; z \geq 0; f_3(z) = 0, \; z < 0 \)

(b) \( f_3(z) = \frac{1}{(z+1)^2}, \; z \geq 0, f_3(z) = 0, \; z < 0 \)

3. \( f_3(z) = \frac{1}{\pi(1+z^2)}, \; \text{all } z \)

4. \( f_3(z) = \frac{1}{z^3}, \; z \geq 1, f_3(z) = 0, \; z < 1 \)

5. \( 3\sqrt{3}/8 \)

6. 2/3

7. 8/27

9. 2/33

10. Let \( R_1 = \) arrival time of the man, \( R_2 = \) arrival time of the woman. Then the probability that they will meet is

\[
P(|R_1 - R_2| \leq z) = \frac{\text{shaded area}}{\text{total area}} = 1 - (1 - z)^2 = z(2 - z)
\]

11. \( \left[ 1 - \frac{(n - 1)d}{L} \right]^n \) if \((n - 1)d \leq L\), and 0 if \((n - 1)d > L\)

13. \( f_{R_1}(r) = \frac{1}{b^2} re^{-r^2/2b^2}, \; r > 0; \; f_{\theta_0}(\theta) = \frac{1}{2\pi}, \; 0 < \theta < 2\pi. \)

Section 2.9

2. (a) \( n \geq 4600 \)

(b) \( 1 - 5e^{-2} \)
3. (a) No.
(b) 9

4. \( P(R_1 + R_2 = k) = \sum_{i=0}^{k} P(R_1 = i, R_2 = k - i) \)
   
   \[ = \sum_{i=0}^{k} \binom{n}{i} p^i q^{n-i} \binom{m}{k-i} p^{k-i} q^{m-k+i} \]
   
   \[ = p^k q^{n+m-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}. \]
   But

(a) \[ \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}. \]

[We may select \( k \) positions out of \( n + m \) in \( \binom{n+m}{k} \) ways. The number of selections in which exactly \( i \) positions are chosen from the first \( n \) is \( \binom{n}{i} \binom{m}{k-i} \). Sum over \( i \) to obtain (a).] Thus \( P(R_1 + R_2 = k) = p^k q^{n+m-k} \), \( k = 0, 1, \ldots, n + m \).

(Intuitively, \( R_1 + R_2 \) is the number of successes in \( n + m \) Bernoulli trials, with probability of success \( p \) on a given trial.) Now \( P(R_1 = j, R_1 + R_2 = k) = P(R_1 = j, R_2 = k - j) = \binom{n}{j} p^j q^{n-j} \binom{m}{k-j} p^{k-j} q^{m-k+j} = \binom{n}{j} \binom{m}{k-j} p^k q^{n+m-k} \), \( j = 0, 1, \ldots, n, k = j, j + 1, \ldots, n + m \).

Thus \( P(R_1 = j \mid R_1 + R_2 = k) = \frac{\binom{n}{j} \binom{m}{k-j}}{\binom{n+m}{k}} \), the hypergeometric probability function (see Problem 7 of Section 1.5). Intuitively, given that \( k \) successes have occurred in \( n + m \) trials, the positions for the successes may be chosen in \( \binom{n+m}{k} \) ways. The number of such selections in which \( j \) successes occur in the first \( n \) trials is \( \binom{n}{j} \binom{m}{k-j} \).

### CHAPTER 3

#### Section 3.2

2. (a) \( E([R_1]) = \int_{-\infty}^{\infty} [x] f_1(x) \, dx = \int_{0}^{\infty} [x] e^{-x} \, dx \)

\[ = \int_{1}^{2} e^{-x} \, dx + \int_{2}^{3} 2e^{-x} \, dx + \int_{3}^{4} 3e^{-x} \, dx + \cdots + \int_{n}^{n+1} ne^{-x} \, dx + \cdots \]

\[ = e^{-1} - e^{-2} + 2(e^{-2} - e^{-3}) + 3(e^{-3} - e^{-4}) + \cdots \]

\[ = e^{-1} + e^{-2} + e^{-3} + \cdots = \frac{e^{-1}(1 + e^{-1} + e^{-2} + \cdots)}{1 - e^{-1}} = \frac{e^{-1}}{1 - e^{-1}} \]
(b) \( P\{R_2 = n\} = P\{n \leq R_1 < n + 1\} = \int_n^{n+1} e^{-x} \, dx = e^{-n} - e^{-(n+1)} \)

\[
E(R_2) = \sum_{n=0}^{\infty} nP\{R_2 = n\} = \sum_{n=0}^{\infty} n[e^{-n} - e^{-(n+1)}] \\
= e^{-1} - e^{-2} + 2(e^{-2} - e^{-3}) + 3(e^{-3} - e^{-4}) + \cdots \\
= \frac{e^{-1}}{1 - e^{-1}} \text{ as above.}
\]

3. (a) 1  \hspace{1cm} (b) 0  \hspace{1cm} (c) 1

4. 1/3

5. 2 + 30e^{-3}

8. \( E[R(R - 1) \cdots (R - r + 1)] = \sum_{k=0}^{\infty} \frac{k(k-1) \cdots (k-r+1)}{k!} e^{-r} \lambda^k \\
= \lambda^r \sum_{k=r}^{\infty} \frac{1}{(k-r)!} e^{-r} \lambda^{k-r} = \lambda^{r-e} \lambda^r = \lambda^r
\)

Set \( r = 1 \) to obtain \( E(R) = \lambda \); set \( r = 2 \) to obtain \( E(R^2 - R) = \lambda^2 \), hence \( E(R^2) = \lambda + \lambda^2 \). It follows that \( \text{Var } R = \lambda \).

Section 3.3

1. This is immediate from Theorem 2 of Section 2.7.

2. \( E[(R - m)^n] = 0, \ n \text{ odd} \)

\[
= \sigma^n(n-1)(n-3) \cdots (5)(3)(1) , \ n \text{ even}
\]

Section 3.4

1. Let \( a(R_1 - ER_1) + b(R_2 - ER_2) = 0 \) (with probability 1). If, say, \( b \neq 0 \) then we may write \( R_2 - ER_2 = c(R_1 - ER_1) \). Thus \( \sigma_2^2 = c^2 \sigma_1^2 \) and \( \text{Cov} (R_1, R_2) = c \sigma_1^2 \). Therefore \( \rho(R_1, R_2) = \frac{c \sigma_1^2}{|c| \sigma_1^2} \), hence \( |\rho| = 1 \).

4. In (a), let \( R \) take the values 1, 2, \ldots, \( n \), each with probability 1/n, and set \( R_1 = g(R), R_2 = h(R) \), where \( g(i) = a_i, h(i) = b_i \). Then

\[
E(R_1R_2) = \sum_{i=1}^{n} \frac{1}{n} a_i b_i , \ E(R_1^2) = \sum_{i=1}^{n} \frac{1}{n} a_i^2 , \ E(R_2^2) = \sum_{i=1}^{n} \frac{1}{b} b_i^2
\]

In (b) let \( R \) be uniformly distributed between \( a \) and \( b \), and set \( R_1 = g(R), R_2 = h(R) \). Then

\[
E(R_1R_2) = \int_a^b \frac{g(x)h(x)}{b-a} \, dx , \ E(R_1^2) = \int_a^b \frac{g^2(x)}{b-a} \, dx , \ E(R_2^2) = \int_a^b \frac{h^2(x)}{b-a} \, dx
\]

In each case the result follows from Theorem 2 of Section 3.4.

5. This follows from the argument of property 7, Section 3.3.
Section 3.5

3. \[ P\{R_1 = j, R_2 = k\} = \frac{n!}{j!k!(n-j-k)!} \left(\frac{1}{3}\right)^{j+k} \left(\frac{2}{3}\right)^{n-j-k} \]

\( j, k = 0, 1, \ldots, n, j + k \leq n \) (see Example 1, Section 2.9).

4. \((n - 1)p(1 - p)\).

5. Let \( A_i = \{\text{trial } i \text{ results in success and trial } i + 1 \text{ in failure}\} \). Then \( R_0 = \sum_{i=1}^{n-1} I_{A_i} \) and \( R_0^2 = \sum_{i=1}^{n-1} I_{A_i}^2 + 2 \sum_{i < j} I_{A_i} I_{A_j} \equiv 0 \), and if \( j \geq i + 2 \), \( I_{A_i} \) and \( I_{A_j} \) are independent, with \( E(I_{A_i} I_{A_j}) = P(A_i) P(A_j) = (pq)^2 \). Thus

\[ E(R_0^2) = (n - 1)pq + 2 \sum_{i=1}^{n-3} \sum_{j=i+2}^{n-1} (pq)^2 \]

\[ = (n - 1)pq + 2(pq)^2 \sum_{i=1}^{n-3} (n - 2 - i) \]

\[ = (n - 1)pq + 2(pq)^2[n + 2 + \cdots + (n - 3)] \]

\[ = (n - 1)pq + (pq)^2(n - 3)(n - 2). \]

Therefore, \( \text{Var } R_0 = E(R_0^2) - [E(R_0)]^2 = (n - 1)pq + (n - 2)(n - 3)(pq)^2 - (n - 1)^2(pq)^2, q = 1 - p \) (assuming \( n \geq 2 \)).

6. \( 50(49/50)^{100} \)

Section 3.6

1. (a) \( .532 \)

(b) \( -2.84 \)

Section 3.7

1. \( m = \int_0^\infty xe^{-x} dx = 1 \), \( E(R^2) = \int_0^\infty x^2 e^{-x} dx = 2 \), hence \( \sigma^2 = 1 \). \( P(|R - m| \geq k\sigma) = P(|R - 1| \geq k) \). If \( 0 < k \leq 1 \), this is \( \int_0^{1-k} e^{-x} dx + \int_{1+k}^\infty e^{-x} dx = 1 - e^{-(1-k)} + e^{-(1+k)} \). When \( k > 1 \), it becomes \( \int_{1+k}^\infty e^{-x} dx = e^{-(1+k)} \). Notice that the Chebyshev bound is vacuous when \( k \leq 1 \), and for \( k > 1 \), \( e^{-(1+k)} \) approaches zero much more rapidly than \( 1/k^2 \).

CHAPTER 4

Section 4.2

1. \( \Gamma(r) = \int_0^\infty r^{t-1}e^{-t} dt \) (with \( t = x^2 \)) \( 2 \int_0^\infty x^{2r-1}e^{-x^2} dx \).

Thus \( \Gamma(r) \Gamma(s) = 4 \int_0^\infty \int_0^\infty x^{2r-1}y^{2s-1}e^{-(x^2+y^2)} dx dy \)

\[ = \text{(in polar coordinates)} 4 \int_0^{\pi/2} d\theta \int_0^\infty (\cos \theta)^{2r-1}(\sin \theta)^{2s-1}e^{-\rho^2} \rho^{2r+2s-3} d\rho. \]

Now \( \int_0^\infty \rho^{2r+2s-4}e^{-\rho^2} d\rho = \text{(set } u = \rho^2\text{)} \int_0^\infty u^{r+s-1}e^{-u} du = \frac{1}{2} \Gamma(r + s) \)

Therefore \( \frac{\Gamma(r) \Gamma(s)}{2\Gamma(r + s)} = \int_0^{\pi/2} (\cos \theta)^{2r-1}(\sin \theta)^{2s-1} d\theta \).
Let \( z = \cos^2 \theta \) so that \( 1 - z = \sin^2 \theta \), \( dz = -2 \cos \theta \sin \theta \ d\theta \),

\[
d\theta = -\frac{dz}{2z^{1/2}(1-z)^{1/2}}
\]

Thus

\[
\frac{\Gamma(r)\Gamma(s)}{2\Gamma(r+s)} = -\frac{1}{2} \int_1^0 z^{r-1}(1-z)^{s-1} \, dz = \frac{1}{2} \beta(r, s)
\]

3. \( p_1(e^{-3} - e^{-5}) + p_2(e^{-4} - e^{-8}) + p_3(e^{-3} - e^{-9}) + p_4(1 - e^{-8}) + p_5(1 - e^{-5}) \)

5. \( f_2(y) = \frac{1}{2}, \quad 0 \leq y \leq 1 \)

\[
= \frac{1}{2y^2}, \quad y > 1
\]

Section 4.3

1. \( h(y \mid x) = e^{x-y}, \quad 0 \leq x \leq y, \) and 0 elsewhere; \( P(R_2 \leq y \mid R_1 = x) = 1 - e^{x-y} \), \( y \geq x, \) and 0 elsewhere.

2. \( f_1(x) = \int_{-1}^{x} kx \, dy = kx(x + 1), \quad 0 \leq x \leq 1 \)

\[
= \int_{-1}^{x} -kx \, dy = -kx(x + 1), \quad -1 \leq x \leq 0
\]

\( f_2(y) = \int_{y}^{1} kx \, dx = \frac{1}{2}k(1 - y^2), \quad 0 \leq y \leq 1 \)

\[
= \int_{y}^{1} -kx \, dx + \int_{0}^{1} kx \, dx = \frac{1}{2}k(1 + y^2), \quad -1 \leq y \leq 0
\]

Since \( \int_{-1}^{1} f_1(x) \, dx = \int_{-1}^{1} f_2(y) \, dy = k \), we must have \( k = 1 \).

The conditional density of \( R_2 \) given \( R_1 \) is

\[
h_2(y \mid x) = \frac{f(x, y)}{f_1(x)} = \frac{1}{x + 1}, \quad -1 \leq x \leq 1, \quad -1 \leq y \leq x
\]

The conditional density of \( R_1 \) given \( R_2 \) is

\[
h_1(x \mid y) = \frac{f(x, y)}{f_2(y)} = \frac{2x}{1 - y^2}, \quad 0 \leq y \leq 1, \quad y \leq x \leq 1
\]

\[
= \frac{2x}{1 + y^2}, \quad -1 \leq y \leq 0, \quad 0 \leq x \leq 1
\]

\[
= -\frac{2x}{1 + y^2}, \quad -1 \leq y \leq 0, \quad y \leq x \leq 0
\]

4. The conditional density of \( R_3 \) given \( R_1 = x \) is \( h(z \mid x) = e^{-z}, \) \( z \geq 0, \quad x \geq 0, \)

\[
P\{1 \leq R_3 \leq 2 \mid R_1 = x\} = e^{-1} - e^{-2}, \quad z \geq 0
\]
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5. \( P\{g(R_1, R_2) \leq z \mid R_1 = x\} = P\{g(x, R_2) \leq z \mid R_1 = x\} = \int_{(y: g(x,y) \leq z)} h(y \mid x) \, dy \)

Section 4.4

1. (a) \( h_2(y \mid x) = \frac{f(x, y)}{f_1(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}, \quad 0 \leq y \leq x \leq 1 \)

\[ h_1(x \mid y) = \frac{f(x, y)}{f_2(y)} = \frac{8xy}{4y(1 - y^2)} = \frac{2x}{1 - y^2}, \quad 0 \leq y \leq x \leq 1 \]

Thus \( E(R_2 \mid R_1 = x) = \int_{-\infty}^{\infty} y h_2(y \mid x) \, dy = \int_{0}^{x} y \left(\frac{2y}{x^2}\right) \, dy = \frac{2}{3} x, \quad 0 \leq x \leq 1 \)

\[ E(R_1 \mid R_2 = y) = \int_{-\infty}^{\infty} x h_1(x \mid y) \, dx = \int_{0}^{1} x \left(\frac{2x}{1 - y^2}\right) \, dx = \frac{2}{3} \left(1 - y^2\right) \]

\[ 0 \leq y \leq 1 \]

(b) \( E(R_2^4 \mid R_1 = x) = \int_{-\infty}^{\infty} y^4 h_2(y \mid x) \, dy = \int_{0}^{x} y^4 \left(\frac{2y}{x^2}\right) \, dy = \frac{1}{3} x^4 \)

(c) The conditional density of \( (R_1, R_2) \) given \( A \) is

\[ f(x, y \mid A) = \frac{f(x, y)}{P(A)} = \frac{8xy}{\int_{0}^{1/2} \int_{0}^{x} 8xy \, dy \, dx} = 128xy, \quad 0 \leq y \leq x \leq \frac{1}{2} \]

The conditional density of \( R_2 \) given \( A \) is

\[ f_2(y \mid A) = \int_{-\infty}^{\infty} f(x, y \mid A) \, dx = \int_{0}^{1/2} 128xy \, dx = 16y - 64y^3, \quad 0 \leq y \leq \frac{1}{2} \]

\[ = 0 \text{ elsewhere} \]

\[ E(R_2 \mid A) = \int_{-\infty}^{\infty} y f_2(y \mid A) \, dy = \int_{0}^{1/2} y(16y - 64y^3) \, dy \]

\[ = \frac{2}{3} - \frac{2}{5} = \frac{1}{15} \]

Alternatively, \( E(R_2 \mid A) = \frac{E(R_2^2 \mid A)}{P(A)} = \frac{\int_{0}^{1/2} \int_{0}^{x} y(8xy) \, dy \, dx}{\int_{0}^{1/2} \int_{0}^{x} 8xy \, dy \, dx} = \frac{1/60}{1/16} = 4/15 \)

Alternatively, \( E(R_2 \mid A) = \int_{-\infty}^{\infty} f_1(x \mid A) E(R_2 \mid R_1 = x) \, dx, \) where \( f_1(x \mid A) = f_1(x)/P(A) \) if \( 0 \leq x \leq \frac{1}{2}, \) and 0 elsewhere. Thus

\[ E(R_2 \mid A) = \int_{0}^{1/2} \frac{4x^3}{1/16} \frac{3x}{1/16} \, dx = 4/15 \]

2. \( \frac{(1 + x)^n}{n!} \), \quad \( x = x_1 + \ldots + x_n \)
3. \( E(R_2 | R_1 = x) = \frac{x}{2}, \quad 0 \leq x \leq 1 \)
   \[ = \frac{1}{2}, \quad 1 \leq x \leq 2 \]
   \[ = (x - 1)/2, \quad 2 \leq x \leq 3 \]

4. \( \frac{1}{3}(n - k) \)

5. (a) 3/7
   (b) 19/21

7. \( E(R) = 1P(R = 1) + 2P(R = 2) + 3P(R = 3) \)
   where
   \[
   P(R = 1) = \frac{2}{3} \frac{30!}{10! \cdot 10! \cdot 10!} (.5)^{10}(.3)^{10}(.2)^{10} + \frac{1}{3} \frac{\binom{20}{10} \binom{20}{10} \binom{20}{10}}{\binom{50}{30}}
   
   P(R = 2) = \frac{3}{5}(\frac{20}{12})(.3)^{12}(.7)^{18}
   
   P(R = 3) = 1 - P(R = 1) - P(R = 2)
   
8. (a) \( P(R_3 \leq z) = P(R_3 \leq z, R_1^2 + R_2^2 \leq 1) + P(R_3 \leq z, R_1^2 + R_2^2 > 1) \).
   If \( 0 \leq z \leq 1 \), then \( R_1^2 + R_2^2 > 1 \) implies \( R_3 = 2 > z \), so \( R_3 \leq z, R_1^2 + R_2^2 > 1 \) = 0. Thus
   \[
   F_3(z) = P(R_3 \leq z, R_1^2 + R_2^2 \leq 1)
   = \int_{x^2 + y^2 \leq 1, \frac{x}{z} \leq z} f(x, y) \, dx \, dy
   = \int_0^z \int_0^{(1-x^2)^{1/2}} dy \, dx
   = \frac{1}{2} \left[ z(1 - z^2)^{1/2} + \arcsin z \right], \quad 0 \leq z \leq 1
   
   If \( 1 \leq z < 2 \), \( P(R_3 \leq z, R_1^2 + R_2^2 > 1) \) is still 0, and \( F_3(z) = P(R_3 \leq z, R_1^2 + R_2^2 \leq 1) = \pi/4 \). If \( z \geq 2 \), \( P(R_3 \leq z) = 1 \).

By (4.4.6), \( E(R_3) = 2P(R_3 = 2) + \int_0^1 z(1 - z^2)^{1/2} \, dz \)
   \[ = 2 \left( 1 - \frac{\pi}{4} \right) + \frac{1}{3} \left( \frac{7}{3} - \frac{\pi}{2} \right). \]
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(b) \[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \, dx \, dy = \int_{x^2+y^2 \leq 1} x f_1(x) f_2(y) \, dx \, dy \]
\[ + \int_{x^2+y^2 > 1} 2f_1(x) f_2(y) \, dx \, dy \]
\[ = \int_{0}^{1} x \, dx \int_{0}^{(1-x^2)^{1/2}} dy + 2 \int_{0 \leq x \leq 1, 0 \leq y \leq 1,} \int_{x^2+y^2 > 1} dx \, dy \]
\[ = \int_{0}^{1} x(1-x^2)^{1/2} \, dx + 2 \left(1 - \frac{\pi}{4}\right) \]
\[ = \frac{1}{3} + 2 \left(1 - \frac{\pi}{4}\right) \] as before.

(c) Since \( R_3 = 2 \) when \( R_1^2 + R_2^2 > 1 \), \( E(R_3 \mid R_1^2 + R_2^2 > 1) = 2 \). The conditional density of \((R_1, R_2)\) given \( A = \{R_1^2 + R_2^2 \leq 1\}\) is

\[ f(x, y \mid A) = \frac{f(x, y)}{P(A)} \]
\[ = \frac{4}{\pi}, \quad x^2 + y^2 \leq 1, \quad x, y \geq 0 \]
\[ = 0 \] elsewhere

\[ E(R_3 \mid A) = \frac{E(R_3 1_A)}{P(A)} = \frac{E(R_1 1_A)}{P(A)} \] since \( R_1^2 + R_2^2 \leq 1 \) implies \( R_3 = R_1 \)

\[ = E(R_1 \mid A) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y \mid A) \, dx \, dy = \int_{-\infty}^{\infty} x f_1(x \mid A) \, dx \]
where

\[ f_1(x \mid A) = \int_{-\infty}^{\infty} f(x, y \mid A) \, dy = \frac{4}{\pi} (1-x^2)^{1/2}, \quad 0 \leq x \leq 1 \]
\[ = 0 \] elsewhere

Thus \( E(R_3 \mid A) = \int_{0}^{1} \frac{4}{\pi} (1-x^2)^{1/2} \, dx = \frac{4}{3\pi} \)

Alternatively, \( E(R_1 1_A) = \int_{0}^{1} \int_{0}^{1} x I(x^2+y^2 \leq 1) \, dx \, dy \)
\[ = \int_{0}^{1} x \, dx \int_{0}^{(1-x^2)^{1/2}} \, dy \]
\[ = \frac{4}{3\pi} \]

Now \( E(R_3) = P(A)E(R_3 \mid A) + P(A^c)E(R_3 \mid A^c) \)
\[ = \frac{\pi}{4} \left(\frac{4}{3\pi}\right) + \left(1 - \frac{\pi}{4}\right) \left(\frac{7}{3} - \frac{\pi}{2}\right) \] as before.
10. Let \( R_3 = R_1 + R_2, R_4 = R_1 - R_2 \). The joint density of \( R_3 \) and \( R_4 \) is

\[
f_{34}(u, v) = f_{12}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right|
\]

where \( u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \) (see Section 2.8, problem 12). Thus \( f_{34}(u, v) = \frac{1}{2}, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 \). But \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \) corresponds to \((u, v) \in D\) (see diagram).

\[
\text{Diagram}
\]

PROBLEM 4.4.10

The density of \( R_4 \) is \( f_4(v) = \int_{-\infty}^{\infty} f_{34}(u, v) \, du = 1 - |v|, \ |v| \leq 1 \)

\[= 0 \text{ elsewhere} \]

Therefore \( h_3(x \mid v) = \frac{f_{34}(u, v)}{f_4(v)} = \frac{1}{2(1-v)}, \ 0 \leq v \leq 1, \ v \leq u \leq 2 - v \)

\[= \frac{1}{2(1+v)}, \ -1 \leq v \leq 0, \ -v \leq u \leq 2 + v \]

If \( 0 \leq v \leq 1, E(R_3^2 \mid R_4 = v) = \frac{1}{2(1-v)} \int_{v}^{2-v} u^2 \, du = \frac{4 - 2v + v^2}{3} \]

If \( -1 \leq v \leq 0, E(R_3^2 \mid R_4 = v) = \frac{1}{2(1+v)} \int_{-v}^{2+v} u^2 \, du = \frac{4 + 2v + v^2}{3} \]

Thus \( E((R_3 + R_2)^2 \mid R_4 = v) = \frac{1}{3}[4 - 2|v| + v^2], \ -1 \leq v \leq 1 \).

11. \( x^2 + 7/6 \)

14. \( f_R(x \mid R_1 = x_1, \ldots, R_n = x_n) = \lambda^2(1 - \lambda)^{n-1}/\beta(1 + x, n - x + 1), 0 \leq \lambda \leq 1, \)

where \( x = x_1 + \cdots + x_n \)

\[E(R \mid R_1 = x_1, \ldots, R_n = x_n) = (x + 1)/(n + 2)\]

16. \((np - npq^{n-1})/(1 - q^n - npq^{n-1})\)

17. \((\frac{1}{q} + \frac{1}{p}(8 - 2^{q/p}))((\frac{1}{q} - \frac{1}{p} \sqrt{2})\)


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18. \( f_{12}(x, y | z) = \frac{1}{\pi z^2}, x^2 + y^2 < z^2, \) and 0 elsewhere

(a) \( E(D | R = z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2)^{1/2} f_{12}(x, y | z) \, dx \, dy \)

\[ = \frac{1}{\pi z^2} \int_0^{2\pi} \int_{0}^{\sqrt{z^2 - r^2}} \, dr \, dz \]

\[ = \frac{1}{\pi z^2} \int_0^{2\pi} \left( \int_{0}^{\sqrt{z^2 - r^2}} \, dr \right) \, dz = \frac{2}{3} z. \] Thus \( E(D) = \int_0^{\infty} \frac{2}{3} ze^{-z} \, dz = \frac{2}{3} \)

(b) \( h(z | x, y) = f(x, y, z)/f_{12}(x, y) = f_R(z)f_{12}(x, y)/ \int_{-\infty}^{\infty} f(x, y, z) \, dz \)

\[ = \frac{e^{-z/\pi z^2}}{\int_{(x^2 + y^2)^{1/2}}^{\infty} e^{-z} \, dz}. \]

\( \), \( z > (x^2 + y^2)^{1/2} \)

19. (b)

\( d(x) = -1, \quad -3 \leq x < -1 \)

\[ = 0, \quad -1 \leq x \leq 1 \]

\[ = 1, \quad 1 < x \leq 3 \]

\( E_{\text{min}}[(\theta^* - \theta)^2] = 1/2 \)

20. \( \frac{1}{3}(x + 1) \)

CHAPTER 5

Section 5.2

1. \( N_0(s) = N_1(s)N_2(s) = \frac{1}{2s} \left( e^s - e^{-s} \right)^3 \)

\[ = \frac{1}{8s^3} (e^{2s} - 3e^s + 3e^{-s} - e^{-3s}), \] all \( s \)

\( f_0(x) = \frac{1}{10}[(x + 3)^2u(x + 3) - 3(x + 1)^2u(x + 1) + 3(x - 1)^2u(x - 1) \]

\[ - (x - 3)^2u(x - 3)] \)

Thus

\( f_0(x) = \frac{1}{10}(x + 3)^2, \quad -3 \leq x \leq -1 \)

\[ = \frac{3 - x^2}{8}, \quad -1 \leq x \leq 1 \]

\[ = \frac{1}{10}(x - 3)^2, \quad 1 \leq x \leq 3 \]

\[ = 0 \text{ elsewhere.} \]
2. \( f_0(x) = \frac{1}{2}[(x + 2)u(x + 2) + 2(x + 1)u(x + 1) - 3xu(x) - 4(x - 1)u(x - 1) + 4(x - 2)u(x - 2)] \)

3. \( N_{R_i}(s) = E(e^{-sR_i^2}) = \int_{-\infty}^{\infty} e^{-sx^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = [\text{with } y = (s + \frac{1}{2})^{1/2}x] \)

\[
\int_{-\infty}^{\infty} [2\pi(s + \frac{1}{2})]^{-1/2} e^{-y^2} \, dy = [2(s + \frac{1}{2})]^{-1/2}, \quad \text{Re } s > -\frac{1}{2}
\]

\( N_R(s) = \prod_{i=1}^{n} N_{R_i}(s) = 2^{-n/2}(s + \frac{1}{2})^{-n/2}, \quad \text{Re } s > -\frac{1}{2} \)

and the result follows from Table 5.1.1.

4. \( N_R(s) = \frac{1}{\Gamma(\alpha)\beta^2} \int_{0}^{\infty} x^{\alpha-1} e^{-(s+\beta^{-1})x} \, dx \)

\[
= \frac{1}{\Gamma(\alpha)\beta^2} \int_{0}^{\infty} y^{\alpha-1} e^{-y} \, dy = \frac{1}{(1 + \beta s)^{2\alpha}}, \quad \text{Re } s > -\frac{1}{\beta}
\]

If \( R_0 = R_1 + R_2 \) where \( R_1 \) and \( R_2 \) are independent and \( R_i \) has the gamma distribution with parameters \( \alpha_i \) and \( \beta_i \), then

\( N_0(s) = N_1(s)N_2(s) = \left( \frac{1}{1 + \beta s} \right)^{\alpha_1 + \alpha_2}, \quad \text{Re } s > -1/\beta \)

so \( R_0 \) has the gamma distribution with parameters \( \alpha_1 + \alpha_2 \) and \( \beta_1 \).

5. \( f_0(x) = \lambda^n e^{-\lambda x}u(x)/(n - 1)! \)

9. Integrate \( \frac{e^{-iu\pi}}{\pi(1 + z^2)} \) around contour (a) if \( u \geq 0 \), and around contour (b) if \( u \leq 0 \). (Notice that \( |e^{-iu(x+iy)}| \) is bounded if \( u \geq 0 \), as long as \( y \leq 0 \).) If \( u \geq 0 \),
the integral is 
\[ -2\pi i \left( \text{residue of } \frac{e^{-iu}}{\pi(1 + z^2)} \text{ at } z = -i \right) = -2\pi i \left[ \frac{e^{-i(u-0)}}{\pi(-2i)} \right] = e^u. \]

If \( u \leq 0 \), the integral is 
\[ 2\pi i \left( \text{residue of } \frac{e^{-iu}}{\pi(1 + z^2)} \text{ at } z = i \right) = 2\pi i \left[ \frac{e^{-i(u+i)}}{\pi(2i)} \right] = e^u. \]

The result follows since the integral around the semicircle of radius \( r \) approaches 0 as \( r \to \infty \).

Section 5.3

4. \( M_R(u) = \sum_{n=-\infty}^{\infty} \frac{p_n e^{-i u (a + nd)}}{p_n} \), where \( p_n = P\{ R = a + nd \} \). Thus \( M_R(u) = e^{-ia} M(u) \) where \( M \) is periodic with period \( 2\pi/d \); the numbers \( p_n \) are the coefficients of the Fourier series of \( M \).

5. \( N_R(s) = \exp \{ \lambda (e^{-s} - 1) \}, \) hence 
\[ \frac{dN_R(s)}{ds} = -\lambda e^{-s} \exp \{ \lambda (e^{-s} - 1) \} = -\lambda \] when \( s = 0 \).
\[ \frac{d^2 N_R(s)}{ds^2} = (\lambda e^{-s})^2 \exp \{ \lambda (e^{-s} - 1) \} + \lambda e^{-s} \exp \{ \lambda (e^{-s} - 1) \} = \lambda^2 + \lambda \]
when \( s = 0 \). By (5.3.3), \( E(R) = \lambda, E(R^2) = \lambda^2 + \lambda \), hence \( \text{Var} R = \lambda \).

Section 5.4

1. (a) \( P\{ R_n \leq x \} = P\{ R_n \leq x, R > x + \epsilon \} + P\{ R_n \leq x, R \leq x + \epsilon \} \)

\[ \leq P\{ |R_n - R| \geq \epsilon \} + P\{ R \leq x + \epsilon \} \]
since \( R_n \leq x, R > x + \epsilon \) implies \( |R_n - R| \geq \epsilon \) and
\[ R_n \leq x, R \leq x + \epsilon \) implies \( R \leq x + \epsilon \).

Similarly \( P\{ R \leq x - \epsilon \} = P\{ R \leq x - \epsilon, R_n > x \} + P\{ R \leq x - \epsilon, R_n \leq x \} \)
\[ \leq P\{ |R_n - R| \geq \epsilon \} + P\{ R_n \leq x \} \]

Thus \( F(x - \epsilon) - P\{ |R_n - R| \geq \epsilon \} \leq F_n(x) \leq P\{ |R_n - R| \geq \epsilon \} + F(x + \epsilon) \)

(b) Given \( \delta > 0 \), choose \( \epsilon > 0 \) so small that
\[ F(x + \epsilon) < F(x) + \frac{\delta}{2}, \quad F(x - \epsilon) > F(x) - \frac{\delta}{2}. \]

(This is possible since \( F \) is continuous at \( x \).) For large enough \( n \), \( P\{ |R_n - R| \geq \epsilon \} < \delta/2 \) since \( R_n \xrightarrow{P} R \). By (a), \( F(x) - \delta < F_n(x) < F(x) + \delta \) for large enough \( n \). Thus \( F_n(x) \to F(x) \).

5. (a) \( n \geq 1,690,000 \)
(b) \( n \geq 9604 \)

7. \( P\{|R - \frac{1}{2}n| \geq 0.005n\} = P\left( \left| \frac{R - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}} \right| \geq \frac{0.005n}{\frac{1}{2}\sqrt{n}} \right) \sim P\{|R^*| \geq 0.01 \sqrt{n}\} \)
\[ = 2P\{R^* > 0.01 \sqrt{n}\} = 2 \left( \frac{2}{\sqrt{2\pi}} \right) \int_{0.01\sqrt{n}}^{\infty} e^{-t^2/2} dt \leq \frac{2}{\sqrt{2\pi}} \frac{1}{0.01\sqrt{n}} e^{-0.001n/2} \]
\[ = \frac{200}{\sqrt{2\pi n}} e^{-(1/2)10^{-4}n}. \] For example, if \( n = 10^6 \), this is \( \frac{2}{\sqrt{2\pi}} e^{-50} \)

8. .91
CHAPTER 6

Section 6.2

1. $f_1(x) = \lambda_1^x$ and $f_2(x) = \lambda_2^x$ [or $f_1(x) = \lambda^x, f_2(x) = x\lambda^x$ in the repeated root case] are linearly independent solutions. For if $c_1f_1 + c_2f_2 \equiv 0$ then

$$c_1\lambda_1^x + c_2\lambda_2^x = 0$$

$$c_1\lambda_1^{x+1} + c_2\lambda_2^{x+1} = 0$$

If $c_1$ and $c_2$ are not both 0, then

$$\begin{vmatrix} \lambda_1^x & \lambda_2^x \\ \lambda_1^{x+1} & \lambda_2^{x+1} \end{vmatrix} = 0$$

hence

$$\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = 0,$$

a contradiction.

If $f$ is any solution then for some constants $A$ and $C$,

$$\begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = A \begin{bmatrix} f_1(0) \\ f_1(1) \end{bmatrix} + C \begin{bmatrix} f_2(0) \\ f_2(1) \end{bmatrix}$$

since three vectors in a two-dimensional space are linearly dependent. But then

$$f(2) = d_1 f(0) + d_2 f(1)$$

$$d_1 = \frac{1}{p}, \quad d_2 = \frac{-q}{p}$$

$$= d_1[Af_1(0) + Cf_2(0)] + d_2[Af_1(1) + Cf_2(1)]$$

$$= Af_1(2) + Cf_2(2)$$

Recursively, $f(x) = Af_1(x) + Cf_2(x)$. Thus all solutions are of the form $Af_1(x) + Cf_2(x)$. But we have shown in the text that $A$ and $C$ are uniquely determined by the boundary conditions at 0 and $b$; the result follows.

2. By the theorem of total expectation, if $R$ is the duration of the game and $A = \{\text{win on trial 1}\}$ then

$$E(R) = P(A)E(R \mid A) + P(A^c)E(R \mid A^c)$$

Thus

$$D(x) = p[1 + D(x + 1)] + q[1 + D(x - 1)], \quad x = 1, 2, \ldots, b - 1$$

since if we win on trial 1, the game has already lasted for one trial, and the average number of trials remaining after the first is $D(x + 1)$. [Notice that this argument, just as the one leading to (6.2.1), is intuitive rather than formal.]

3. In standard form, $pD(x + 2) - D(x + 1) + qD(x) = -p - q = -1, \quad D(0) = D(b) = 0$. 
SOLUTIONS TO PROBLEMS

CASE 1. \( p \neq q \).

The homogeneous equation is the same as (6.2.1), with solution \( A + C(q/p)x \).

To find a particular solution, notice that the “forcing function” \(-1\) already satisfies the homogeneous equation, so try \( D(x) = kx \). Then

\[ k[p(x + 2) - (x + 1) + qx] = (2p - 1)k = (p - q)k = -1. \]

Thus

\[ D(x) = A + C(q/p)x + \frac{x}{q - p}. \]

Set \( D(0) = D(b) = 0 \) to solve for \( A \) and \( C \).

CASE 2. \( p = q = 1/2 \).

The homogeneous solution is \( A + Cx \). Since polynomials of degree 0 and 1 already satisfy the homogeneous equation, try as a particular solution \( D(x) = kx^2 \). Then

\[ k[\frac{1}{2}(x + 2)^2 - (x + 1)^2 + \frac{1}{2}x^2] = k = -1 \]

Thus

\[ D(x) = A + Cx - x^2 \]

\[ D(0) = A = 0, \quad D(b) = b(C - b) = 0 \text{ so that } C = b. \]

Therefore \( D(x) = x(b - x) \).

If we let \( b \to \infty \) we obtain

\[ D(x) = \infty \text{ if } p \geq q; \quad D(x) = \frac{x}{q - p} \text{ if } p < q \]

Section 6.3

1. \( P\{S_1 \geq 0, \ldots, S_{2n-1} \geq 0, S_{2n} = 0\} \) is the number of paths from \((0, 0)\) to \((2n, 0)\) lying on or above the axis, times \((pq)^n\). These paths are in one-to-one correspondence with the paths from \((-1, -1)\) to \((2n, 0)\) lying above \(-1\) [connect \((-1, -1)\) to \((0, 0)\) to establish the correspondence]. Thus the number of paths is the same as the number from \((0, 0)\) to \((2n + 1, 1)\) lying above \(0\), namely

\[ \binom{2n + 1}{a} \binom{a - b}{2n + 1} \text{ where } a + b = 2n + 1, \quad a - b = 1, \text{ that is, } a = n + 1, \quad b = n \]

Thus the desired probability is

\[ \binom{2n + 1}{n + 1} \frac{1}{2n + 1} (pq)^n = \frac{(2n)!}{n! \ (n + 1)!} \frac{(pq)^n}{n + 1} = \frac{u_{2n}}{n + 1} \]

5. \( u_{2n} = \binom{2n}{n} (pq)^n = \frac{(2n)!}{n! \ n!} (pq)^n \sim \frac{(2n)^n \sqrt{2\pi 2n}}{(n^n \sqrt{2\pi n})^2} (pq)^n = \frac{(4pq)^n}{\sqrt{n\pi}} = \frac{1}{\sqrt{n\pi}} \text{ if } p = q = \frac{1}{2} \)
By Problem 2,

\[ h_{2n} = \frac{u_{2n-2}}{2n} \sim \frac{1}{2n} \frac{1}{\sqrt{(n-1)\pi}} \sim \frac{1}{2\sqrt{n}n^{3/2}} \]

Let \( T \) be the time required to return to 0. Then

\[ P\{T = 2n\} = h_{2n}, \quad n = 1, 2, \ldots \text{ where } \sum_{n=1}^{\infty} h_{2n} = 1 \]

\[ E(T) = \sum_{n=1}^{\infty} 2nP\{T = 2n\} = \sum_{n=1}^{\infty} 2nh_{2n} \]

But \( 2nh_{2n} \sim K/\sqrt{n} \) and \( \sum 1/\sqrt{n} = \infty \), hence \( E(T) = \infty \).

6. The probability that both players will have \( k \) heads is \([\left(\frac{1}{2}\right)^n]^{\binom{k}{n}}\); sum from \( k = 0 \) to \( n \) to obtain the desired result.

7. The probability that both players will receive the same number of heads = the probability that the number of heads obtained by player 1 = the number of tails obtained by player 2 (since \( p = q = 1/2 \)), and this is the probability of being at 0 after \( 2n \) steps of a simple random walk, namely \( \binom{n}{n/2}(\frac{1}{2})^{2n} \). Comparing this expression with the result of Problem 6, we obtain the desired conclusion. (Alternatively, we may use the formula of Section 2.9, Problem 4, with \( m = k = n \).)

Section 6.4

1. \( (1 - 4pqz^2)^{1/2} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} (-4pqz^2)^n \]

\[ = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} (-4pq)^n z^{2n} \]

Thus

\[ H(z) = 1 - (1 - 4pqz^2)^{1/2} = -\sum_{n=1}^{\infty} \frac{(1/2)^n}{n!} (-4pq)^n z^{2n} \]

Thus

\[ h_{2n} = (-1)^{n+1} \frac{(1/2)^n}{n!} (4pq)^n, \quad n = 1, 2, \ldots \]

But

\[ (-1)^{n+1} \left\{\frac{1/2}{n}\right\} = (-1)^{n+1} \frac{(1/2)(-1/2)(-3/2) \cdots (2n-3)/2}{n!} = \frac{2n-2}{2^n n! 2 \cdot 4 \cdots (2n-2)} \]

\[ = \frac{(2n-2)!}{2^n n! (n-1)! 2^{n-1} (n-1)!} = \frac{2}{n} \left(\frac{1}{2}\right)^{2n} \]

Therefore \( h_{2n} = (2/n)\binom{2n-2}{n-1} (pq)^n \), in agreement with (6.3.5).
Solutions to Problems

2. \[ \sum_{n=0}^{\infty} a_{n+1}z^n - 3 \sum_{n=0}^{\infty} a_n z^n = 4 \sum_{n=0}^{\infty} z^n = \frac{4}{1 - z} \]

If \[ A(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ then } \frac{1}{z} [A(z) - a_0] - 3A(z) = \frac{4}{1 - z} \]

or \[ A(z)(z^{-1} - 3) = \frac{4}{1 - z} + \frac{a_0}{z} \]

Thus \[ A(z) = \frac{4z}{(1 - z)(1 - 3z)} + \frac{a_0}{1 - 3z} \]

\[ = -\frac{2}{1 - z} + \frac{2 + a_0}{1 - 3z} = -2 \sum_{n=0}^{\infty} z^n + (2 + a_0) \sum_{n=0}^{\infty} 3^n z^n \]

Thus \( a_n = (2 + a_0)3^n - 2. \) Notice that \((2 + a_0)3^n\) is the homogeneous solution, 
\(-2\) the particular solution.

6. (a) \( P[N_r = k] = P(\text{the first } k - 1 \text{ trials result in exactly } r - 1 \text{ successes, and trial } k \text{ results in a success}) = \binom{k-1}{r-1} p^{r-1} q^{k-r} p, \quad k = r, r + 1, \ldots. \) Now \( \binom{k-1}{r-1} = \binom{k-1}{k-r} = (k - 1)(k - 2) \cdots (r + 1)r/(k - r)! \)

\[ = (-1)^{k-r}(-r)(-r-1)(-r-2) \cdots [-r - (k - 2 - r)] \]

\[ \times (-r - (k - 1 - r))/(k - r)! \]

\[ = (-1)^{k-r}(k-r), \text{ and the result follows.} \]

Note that if \( j = k - r, \) this computation shows that \((-1)^{j-1}(\cdot)^{-1} = \binom{j-1}{r-1}, \quad r = 1, 2, \ldots, j = 0, 1, \ldots. \)

(b) We show that \( T_1 \) and \( T_2 \) are independent. The argument for \( T_1, \ldots, T_r \) is similar, but the notation becomes cumbersome.

\[ P\{T_1 = j, T_2 = k\} = P\{R_1 = \cdots = R_{j-1} = 0, R_j = 1, \]

\[ R_{j+1} = \cdots = R_{j+k-1} = 0, R_{j+k} = 1\} \]

\[ = p^2qj+k-2, j, k = 1, 2, \ldots. \]

Now \( P\{T_1 = j\} = qj^{-3}p \) by Problem 5, and

\[ P\{T_2 = k\} = \sum_{j=1}^{\infty} P\{T_1 = j, T_2 = k\} = \left( \sum_{j=1}^{\infty} q^{-j+1} \right) p^2q^{k-1} \]

\[ = \frac{p^2q^{k-1}}{1 - q} = pq^{k-1} \]

Hence \( P\{T_1 = j, T_2 = k\} = P\{T_1 = j\}P\{T_2 = k\} \) and the result follows.
(c) \( E(N_r) = r/p \). Var \( N_r = r[(1 - p)/p^2] \) since \( N_r = T_1 + \cdots + T_r \) and the \( T_i \) are independent. The generalized characteristic function of \( N_r \) is

\[
\left( \frac{pe^{-s}}{1 - qe^{-s}} \right)^r, \quad |qe^{-s}| < 1
\]

Set \( s = iu \) to obtain the characteristic function, \( z = e^{-s} \) to obtain the generating function.

7. \( P\{R = k\} = p^kq + q^kp, \ k = 1, 2, \ldots ; E(R) = pq^{-1} + qp^{-1} \)

8. \( 1/\sqrt{2} \)

Section 6.5

2. \( P\{ \) an even number of customers arrives in \( (t, t + \tau) \} = \sum_{k=0,2,4,\ldots}^{\infty} e^{-\lambda \tau}(\lambda \tau)^k/k! \)

\[ = \frac{1}{2} \sum_{k=0}^{\infty} e^{-\lambda \tau}(\lambda \tau)^k/k! + \frac{1}{2} \sum_{k=0}^{\infty} e^{-\lambda \tau}(-\lambda \tau)^k/k! \]

\[ = \frac{1}{2} e^{-\lambda \tau}(e^{\lambda \tau} + e^{-\lambda \tau}) = \frac{1}{2}(1 + e^{-2\lambda \tau}) \]

\( P\{ \) an odd number of customers arrives in \( (t, t + \tau) \} = \sum_{k=1,3,5,\ldots}^{\infty} e^{-\lambda \tau}(\lambda \tau)^k/k! \)

\[ = \frac{1}{2} \sum_{k=0}^{\infty} e^{-\lambda \tau}(\lambda \tau)^k/k! - \frac{1}{2} \sum_{k=0}^{\infty} e^{-\lambda \tau}(-\lambda \tau)^k/k! \]

\[ = \frac{1}{2} e^{-\lambda \tau}(e^{\lambda \tau} - e^{-\lambda \tau}) = \frac{1}{2}(1 - e^{-2\lambda \tau}) \]

[Alternatively, we may note that \( \sum_{k=0,2,4,\ldots}^{\infty} \frac{(\lambda \tau)^k}{k!} = \cosh \lambda \tau \) and \( \sum_{k=1,3,5,\ldots}^{\infty} \frac{(\lambda \tau)^k}{k!} = \sinh \lambda \tau. \)]

3. (a) \( P\{R_t = 1, R_{t+\tau} = 1\} = P\{R_t = -1, R_{t+\tau} = -1\} = \frac{1}{4}(1 + e^{-2\lambda \tau}) \)

\( P\{R_t = 1, R_{t+\tau} = -1\} = P\{R_t = -1, R_{t+\tau} = 1\} = \frac{1}{4}(1 - e^{-2\lambda \tau}) \)

(b) \( K(t, \tau) = e^{-2\lambda \tau} \)

Section 6.6

4. (a) is immediate from Theorem 1.

(b) If \( \omega \in A_n \) for infinitely many \( n \), then \( \omega \in A \), which in turn implies that \( \omega \in A_n \) eventually. Thus \( \lim \sup A_n \leq A \leq \lim \inf A_n \leq \lim \sup A_n \), so all these sets are equal.

(c) Let \( A_n = [1 - 1/n, 2 - 1/n] \); \( \lim \sup A_n = \lim \inf A_n = [1, 2) \) (Another example. If the \( A_n \) are disjoint, \( \lim \sup A_n = \lim \inf A_n = \emptyset \).)

(d) \( \bigcap_{k=1}^{\infty} A_k \subset A_n \subset \bigcup_{k=1}^{\infty} A_k \), and \( B_n = \bigcap_{k=\infty}^{n} A_k \) expands to \( \lim \inf A_n = A \), \( C_n = \bigcup_{k=1}^{\infty} A_k \) contracts to \( \lim \sup A_n = A \). Thus \( P(A_n) \) is boxed between \( P(B_n) \) and \( P(C_n) \), each of which approaches \( P(A) \).

(e) \( \lim \inf A_n \) and \( \lim \sup A_n \) are both closed.

6. \( \lim \inf A_n = \{x, y\}: x^2 + y^2 < 1 \}

\( \lim \sup A_n = \{(x, y): x^2 + y^2 \leq 1\} - \{(0, 1), (0, -1)\}. \)
8. \( P(\lim_n \sup A_n) = \lim_{n \to +\infty} P(\bigcup_{k=n}^\infty A_k) \) by definition of \( \lim \sup \), hence
\[
P(\lim_n \sup A_n) = \lim_{n \to +\infty} \lim_{m \to +\infty} P(\bigcup_{k=n}^m A_k)
\]
Now
\[
P\left( \bigcup_{k=n}^m A_k \right) = P\left( \bigcap_{k=n}^m A_k \right) = \prod_{k=n}^m P(A_k) \text{ by independence}
\]
\[
\leq \prod_{k=n}^m e^{-P(A_k)} \text{ since } P(A_k) = 1 - P(A_k) \leq e^{-P(A_k)}
\]
\[
= e^{-\sum_{k=n}^m P(A_k)} \to 0 \text{ as } m \to \infty \text{ since } \sum_n P(A_n) = \infty
\]
The result follows.

9. If \( \sum_{n=1}^{\infty} P(|R_n - c| \geq \varepsilon) < \infty \) for every \( \varepsilon > 0 \), \( R_n \xrightarrow{a.s.} c \) by Theorem 5. Thus assume that \( \sum_{n=1}^{\infty} P(|R_n - c| \geq \varepsilon) = \infty \) for some \( \varepsilon > 0 \). Then by the second Borel-Cantelli lemma, \( P(|R_n - c| \geq \varepsilon \text{ for infinitely many } n) = 1 \). But \( |R_n - c| \geq \varepsilon \) for infinitely many \( n \) implies that \( R_n \xrightarrow{\text{a.s.}} c \), hence \( P(R_n \xrightarrow{\text{a.s.}} c) = 1 \), that is, \( P(R_n \xrightarrow{} c) = 0 \).

12. Let \( S_n = (R_1 + \cdots + R_n)/n \); then \( E(S_n/n)^2 = (1/n^2) \text{Var} S_n \)
\[
= \frac{1}{n^2} \sum_{k=1}^n \text{Var } R_k \leq \frac{M}{n^2} \sum_{k=1}^n \frac{1}{k}
\]
But
\[
\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \int_1^n \frac{1}{x} \, dx = \ln n \left( \text{notice } \frac{1}{k} \leq \frac{1}{x} \text{ if } k - 1 \leq x \leq k \right)
\]
Hence
\[
E\left( \frac{S_n}{n} \right)^2 \leq \frac{M}{n^2} (1 + \ln n), \text{ so that}
\]
\[
\sum_{n=1}^{\infty} E\left( \frac{S_n}{n} \right)^2 < \infty. \text{ By Theorem 6, } S_n \xrightarrow{a.s.} 0
\]

CHAPTER 7

Section 7.1

1. \( P(R_n = j \mid R_0 = r) = \frac{P(R_n = j, R_0 = r)}{P(R_0 = r)} = \frac{\sum_{i_1, \ldots, i_n=1} P_R r_{i_1 \cdots i_n} \cdots p_{i_{n-2}i_{n-1}} p_{i_{n-1}j}}{P_r} = p_{rj}^{(n)} \)
\[
P(T_n = j) = \sum_{i_0, \ldots, i_{n-1}} P(T_0 = i_0, \ldots, T_{n-1} = i_{n-1}, T_n = j) = \sum_{i_1, \ldots, i_{n-1}} q_r p_{r1} \cdots p_{i_{n-2}i_{n-1}} p_{i_{n-1}j} = p_{rj}^{(n)}
\]
4. \( P(R_{n+1} = i_1, \ldots, R_{n+k} = i_k \mid R_n = i) \)
   \[ = P(R_n = i, R_{n+1} = i_1, \ldots, R_{n+k} = i_k) \]
   \[ = \frac{\sum \sum P_i \cdot P_{i_1} \cdots P_{i_{n-1}} \cdot P_{i_{n-1}} \cdots P_{i_{n-k}}}{\sum \sum P_i \cdot P_{i_1} \cdots P_{i_{n-1}} \cdot P_{i_{n-1}} \cdots P_{i_{n-k}}} = P_{i_1} \cdots P_{i_{n-k}} \]

Section 7.2

1. The desired probability is

\[ \frac{P[D \cap \{R_n = i, R_{n+1} = i_1, \ldots, R_{n+k} = i_k\}]}{P[D \cap \{R_n = i\}]}. \]

\( D \) must be of the form \((R_0, \ldots, R_n) \in B\) for some \( B \subset S^{n+1}\), hence \( D \) is a countable union of sets of the form \( \{R_0 = j_0, \ldots, R_n = j_n\}\). Now

\[ P(R_0 = j_0, \ldots, R_n = j_n, R_n = i, \ldots, R_{n+k} = i_k) = 0 \quad \text{if} \quad j_n \neq i \]

\[ = P(R_0 = j_0, \ldots, R_{n-1} = j_{n-1}, R_n = i)P(R_{n+1} = i_1, \ldots, R_{n+k} = i_k \mid R_n = i) \]

if \( j_n = i \)

Thus the numerator is simply

\[ P[D \cap \{R_n = i\}]P(R_{n+1} = i_1, \ldots, R_{n+k} = i_k \mid R_n = i) \]

and the result follows.

Section 7.3

1. (a) If \( n_1 \in A \), then \( n_1 \) is a multiple of \( d \), say \( n_1 = r_1d \). If \( r_1 = 1 \) then \( \{n_1\} \) is the required set. If not choose \( n_2 \in A \) such that \( n_1 \) does not divide \( n_2 \) (if \( n_2 \) does not exist then \( d = n_1 \), hence \( r_1 = 1 \)). If \( n_2 = r_2d \) then \( \gcd(n_1, n_2) = \gcd(n_1, n_2) = \gcd(r_1d, r_2d) = r_2d \) for some positive integer \( r_2 < r_1 \); (if \( r_2 = r_1 \), then \( n_1 \) divides \( n_2 \)). If \( r_2 = 1 \), \( \{n_1, n_2\} \) is the required set, if not, find \( n_3 \in A \) such that \( r_3d \) does not divide \( n_3 \). If \( n_3 \) does not exist, then \( r_3d \) divides every integer in \( A \) including \( n_1 \) and \( n_2 \). But \( r_3d = \gcd(n_1, n_2) \) so that \( r_3d = d \), hence \( r_3 = 1 \), a contradiction.] If \( n_3 = r_3d \) then \( \gcd(n_1, n_2, n_3) = \gcd(r_3d, r_3d) = r_3d, r_3 < r_3 \). If \( r_3 = 1 \), \( \{n_1, n_2, n_3\} \) is the desired set. If not, find \( n_4 \in A \) such that \( r_4d \) does not divide \( n_4 \), and continue in this fashion. We obtain a decreasing sequence of positive integers \( r_1 > r_2 > \cdots \); the sequence must terminate in a finite number of steps, thus yielding a finite set whose \( \gcd \) is \( d \).

(b) By (a) there are integers \( n_1, \ldots, n_6 \in A \), such that \( \gcd(n_1, \ldots, n_6) = d \). Thus for some integers \( c_1, \ldots, c_6 \) we have \( c_1n_1 + \cdots + c_6n_6 = d \). Collect positive and negative terms to obtain, since \( A \) is closed under addition, \( md, nd \in A \) such that \( md - nd = d \), that is, \( m - n = 1 \). Now let \( q = cd, c \geq n(n - 1) \).
SOLUTIONS TO PROBLEMS

Write \( c = an + b, a \geq n - 1, 0 \leq b \leq n - 1. \) Then \( q = cd = [(a - b)n + (bn + b)]d \in A, \) since \( a - b \geq 0 \) and \( A \) is closed under addition. The result follows.

(c) Let \( A = \{n \geq 1: p_{ii}^{(n)} > 0\}, B = \{n \geq 1: f_{ii}^{(n)} > 0\}. \) Then \( d_i = \gcd(A). \) If \( e_i = \gcd(B), \) then since \( B \subseteq A \) we have \( e_i \geq d_i. \) To show that \( d_i \geq e_i \) we show that \( e_i \) divides all elements of \( A. \) If this is not the case, let \( n \) be the smallest positive integer such that \( p_{ii}^{(n)} > 0 \) and \( e_i \) does not divide \( n. \) Write \( n = ae_i + b, 0 < b < e_i. \) Then

\[
p_{ii}^{(n)} = \sum_{k=1}^{n} f_{ii}^{(k)} p_{ii}^{(n-k)} = \sum_{r=1}^{d_i} f_{ii}^{(r)} p_{ii}^{(a�_{e_i+b-r e_i})}
\]

Now \( e_i \) does not divide \( (a - r)e_i + b, \) so by minimality of \( n, \)

\[
p_{ii}^{(aer_{e_i+b-r e_i})} = 0 \text{ for all } r = 1, 2, \ldots, a.
\]

But then \( p_{ii}^{(n)} = 0, \) a contradiction. The result follows.

4. (a) \( S \) forms a single aperiodic equivalence class. Starting from 1, the probability of returning to 1 is 1 if \( p \leq q, \) and \( q + p(q/p) = 2q \) if \( p > q \) (see 6.2.6). Thus \( S \) is recurrent if \( p \leq q, \) transient if \( p > q. \)

(b) and (c) \( S \) forms a single aperiodic equivalence class. By Theorem 6, \( S \) is recurrent.

(d) The equivalence classes are \( C = \{1\} \) and \( D = \{2, 3\}. \) \( C \) is not closed, hence is transient; \( D \) is closed, hence recurrent. \( C \) and \( D \) are aperiodic.

Section 7.4

1. For each \( n \) so that \( f_n > 0, \) form the following system of states (always originating from a fixed state \( i). \) It is clear from the construction that \( f_{ii}^{(n)} = f_n \) for all \( n. \)

![Diagram]

Problem 7.4.1

Also, \( p_{ii}^{(n)} = u_n \) by induction, using the First Entrance Theorem.

Note. we need not have \( \gcd\{j : f_j > 0\} = 1 \) for this construction.
2. Construct a Markov chain such that for some $i$,

$$f_{i,i}^{(n)} = P(T_1 = n), \quad n = 1, 2, \ldots$$

We claim that $G(n) = p_{i,i}^{(n)}$ for all $n = 1, 2, \ldots$. For $G(1) = P(T_1 = 1) = f_{i,i}^{(1)} = p_{i,i}^{(1)}$, and if $G(r) = p_{i,i}^{(r)}$ for $r = 1, 2, \ldots, n$, then

$$G(n + 1) = \sum_{k=1}^{\infty} P(T_1 + \cdots + T_k = n + 1)$$

$$= \sum_{l=1}^{\infty} \sum_{k=1}^{n+1} P(T_1 = l)P(T_2 + \cdots + T_k = n + 1 - l)$$

$$= \sum_{l=1}^{n+1} f_{i,i}^{(l)} G(n + 1 - l), \text{ if we define } G(0) = 1$$

$$= \sum_{l=1}^{n+1} f_{i,i}^{(l)} p_{i,i}^{(n+1-l)} \text{ by induction hypothesis}$$

$$= p_{i,i}^{(n+1)} \text{ by the First Entrance Theorem}$$

Now state $i$ is recurrent since $\sum_{n=1}^{\infty} P(T_1 = n) = 1$, and has period $d$ by hypothesis. Thus by Theorem 2(c), $\lim_{n \to \infty} G(n d) = d/\mu$.

Since a renewal can only take place at times $n d$, $n = 1, 2, \ldots$, $G(n d)$ is the probability that a renewal takes place in the interval $[n d, (n + 1) d)$. If the average length of time between renewals is $\mu$, for large $n$ it is reasonable that one renewal should take place every $\mu$ seconds, hence there should be, on the average, $d/\mu$ renewals in a time interval of length $d$. Thus we expect intuitively that $G(n d) \to d/\mu$.

4. (a) Let the initial state be $i$. Then $V_{i,i} = \sum_{n=0}^{\infty} I_{(R_n = i)}$, and the result follows.

(b) By (a), $N = \sum_{n=0}^{\infty} Q^n$ so that $QN = \sum_{n=1}^{\infty} Q^n = N - I$. (In particular, $QN$ is finite.)

(c) By (b), $(I - Q)N = I$. But $N = \sum_{n=0}^{\infty} Q^n$ so that $QN = NQ$, hence $N(I - Q) = I$.

Section 7.5

2. (a) $j_0$ is recurrent since $p_{i,j_0}^{(n)} \geq \delta$ for all $n \geq N$ (see Problem 2, Section 7.1), hence

$$\sum_{n=1}^{\infty} p_{i,j_0}^{(n)} = \infty$$

If $i$ is any recurrent state then since $p_{i,j_0}^{(N)} \geq \delta > 0$, $i$ leads to $j_0$. By Theorem 5 of Section 7.3, $j_0$ leads to $i$, so that there can only be one recurrent class. Since $p_{i,j_0}^{(n)} \geq \delta > 0$ for all $n \geq N$, the class is aperiodic, so that $\lim_{n} p_{i,j_0}^{(n)} = 1/\mu_{j_0}$. But then $1/\mu_{j_0} \geq \delta > 0$, hence $\mu_{j_0} < \infty$ and the class is positive.
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(Note also that if \( i \) is any state and \( C \) is the equivalence class of \( j_0 \), then, for \( n \geq N, \ P(R_n \notin C \mid R_0 = i) \leq 1 - \delta \), hence \( P(R_{kn} \notin C \mid R_0 = i) \leq (1 - \delta)^k \to 0 \) as \( k \to \infty \). Thus \( f_{i,0} = 1 \), and it follows that a steady state distribution exists.)

(b) If \( \prod_j \) has a positive column then by (a) there is exactly one recurrent class, which is (positive and) aperiodic, and therefore a steady state distribution exists. Conversely, let \( \{v_j\} \) be a steady state distribution. Pick \( j_0 \) so that

\[ v_{j_0} = \lim_{n \to \infty} p_{i,j_0}^{(n)} > 0 \]

Since the chain is finite, \( p_{i,j_0}^{(n)} > 0 \) for all \( i \) if \( n \) is sufficiently large, say \( n \geq N \). But then \( \prod_j \) has a positive column.

3.1. If \( p \neq q \), the chain is transient, so \( p_{i,j}^{(n)} \to 0 \) for all \( i, j \). If \( p = q \) the chain is recurrent. We have observed (Problem 5, Section 6.3) that the mean recurrence time is infinite, hence the chain is recurrent null, and thus \( p_{i,j}^{(n)} \to 0 \). In either case there is no stationary distribution, hence no steady state distribution. The period is 2.

2. There is one positive recurrent class, namely \( \{0\} \); the remaining states form a transient class. Thus there is a unique stationary distribution, given by \( v_0 = 1, \ v_j = 0, \ j \geq 1 \). Now starting from \( i \geq 1 \), the probability of eventually reaching 0 is \( \lim_{n \to \infty} p_{i,0}^{(n)} \), since the events \( \{R_n = 0\} \) expand to \( \{0\} \). By (6.2.6),

\[
\lim_{n \to \infty} p_{i,0}^{(n)} = (q/p)^i \quad \text{if} \quad p > q \\
= 1 \quad \text{if} \quad p \leq q
\]

(Also \( p_{00}^{(n)} = 1, \ p_{i,j}^{(n)} \to 0, \ j \geq 1 \). If \( p > q \) the limit is not independent of \( i \) so there is no steady state distribution.

3. There are two positive recurrent classes \( \{0\} \) and \( \{b\} \). \( \{1, 2, \ldots, b - 1\} \) is a transient class. Thus, there are uncountably many stationary distributions, given by \( v_0 = p_1, v_0 = p_2, v_i = 0, \ 1 \leq i \leq b - 1, \) where \( p_1, p_2 \geq 0, p_1 + p_2 = 1 \). There is no steady state distribution. By (6.2.3) and (6.2.4),

\[
\lim_{n \to \infty} p_{i,0}^{(n)} = \frac{(q/p)^i - (q/p)^b}{1 - (q/p)^b} \quad \text{if} \quad p \neq q \\
= 1 - \frac{i}{b} \quad \text{if} \quad p = q
\]

\[
\lim_{n \to \infty} p_{i,j}^{(n)} = 1 - \lim_{n \to \infty} p_{i,j}^{(n)} \\
\lim_{n \to \infty} p_{i,j}^{(n)} = 0, \quad 1 \leq j \leq b - 1
\]

4. The chain is aperiodic. If \( p > q \) then \( f_{11} = (q/p)^{i-1} < 1, \ i > 1 \), hence the chain is transient. Therefore \( p_{i,j}^{(n)} \to 0 \) as \( n \to \infty \) for all \( i, j \), and there is no stationary
SOLUTIONS TO PROBLEMS

or steady state distribution. Now if \( p \leq q \) then \( f_{i1} = 1 \) for \( i > 1 \), hence \( f_{i1} = q + pf_{i1} = 1 \), and the chain is recurrent. The equations \( VII = \mathcal{V} \) become

\[
\begin{align*}
    v_1 p + v_2 q &= v_1 \\
    v_1 p + v_3 q &= v_2 \\
    v_2 p + v_4 q &= v_3 \\
    &\vdots
\end{align*}
\]

This may be reduced to \( v_j = (p/q)v_{j-1} \), \( j = 2, 3, \ldots \). If \( p = q \) then all \( v_j \) are equal, hence \( v_j = 0 \) and there is no stationary or steady state distribution. Thus the chain is recurrent null. If \( p < q \), the condition \( \sum_{j=1}^{\infty} v_j = 1 \) yields the unique solution

\[
v_j = \frac{(q - p)}{q} \left( \frac{p}{q} \right)^{j-1}, \quad j = 1, 2, \ldots
\]

Thus there is a unique stationary distribution, so that the chain is recurrent positive; \( \{v_j\} \) is also the steady state distribution.

5. The chain forms a recurrent positive aperiodic class, hence \( p_{j0}^{(n)} \to v_j \) where the \( v_j \) form the unique stationary distribution and the steady state distribution. The equations \( VII = \mathcal{V} \), \( \sum_j v_j = 1 \) yield

\[
v_j = \frac{(p/q)^{j-1}}{\sum_j (p/q)^{j-1}}
\]

6. The chain forms a recurrent positive aperiodic class. Since \( \Pi^a \) has identical rows \( (p^a, pq, qp, q^2) = \mathcal{V} \), there is a steady state distribution (= the unique stationary distribution), namely \( \mathcal{V} \).

7. The chain forms a recurrent positive aperiodic class, hence \( p_{j0}^{(n)} \to v_j \) where the \( v_j \) form the unique stationary distribution and the steady state distribution. The equations \( VII = \mathcal{V} \), \( \sum_j v_j = 1 \) yield

\[
v_1 = \frac{3}{4}, \quad v_2 = \frac{1}{3}, \quad v_3 = \frac{1}{2}, \quad v_4 = \frac{1}{3}
\]

8. There is a single positive recurrent class \( \{2, 3\} \), which is aperiodic, hence \( p_{j0}^{(n)} \to v_j \) where the \( v_j \) form the unique stationary distribution and the steady state distribution. We find that \( v_1 = 0 \), \( v_2 = 3/7 \), \( v_3 = 4/7 \).

9. We may take \( p_{ij} = P\{R_n = j\} \) for all \( i, j \) (with initial distribution \( p_j = P\{R_0 = j\} \) also). The chain forms a recurrent class since from any initial state, \( P\{R_n \neq j\} = \prod_{n=1}^{\infty} P\{R_n = j\} = \prod_{n=1}^{\infty} p_j = 0 \). The class is aperiodic. Clearly \( v_j = p_j \) is a stationary distribution, so that the chain is recurrent positive and the stationary distribution is unique and coincides with the steady state distribution.
10. The chain forms a positive recurrent class of period 3 (see Section 7.3, Example 2). Thus there is a unique stationary distribution given by
\[ v_1 = \frac{1}{6}, \ v_2 = \frac{5}{6}, \ v_3 = \frac{1}{3}, \ v_4 = \frac{2}{3}, \ v_5 = \frac{1}{2}, \ v_6 = \frac{1}{3}, \ v_7 = \frac{2}{3} \]

Now the cyclically moving subclasses are \( C_0 = \{1, 2\}, \ C_1 = \{3, 4\}, \ C_2 = \{5, 6, 7\} \). By Theorem 2(c) of Section 7.4, if \( i \in C_r, j \in C_{r+a} \) then \( p_{ij}^{(3n+a)} \to 3v_j \).

Thus

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & & & & & & & \\
2 & & & & & & & \\
3 & & & & & & & \\
\Pi^{3n} & \to & 4 & & & & & \\
5 & 0 & 0 & 0 & 0 & 1 \frac{1}{3} & 1 \frac{1}{3} & 2 \frac{3}{3} \\
6 & 0 & 0 & 0 & 0 & 1 \frac{1}{3} & 1 \frac{1}{3} & 2 \frac{3}{3} \\
7 & 0 & 0 & 0 & 0 & 1 \frac{1}{3} & 1 \frac{1}{3} & 2 \frac{3}{3} \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & & & & & & & \\
2 & & & & & & & \\
3 & & & & & & & \\
\Pi^{3n+1} & \to & 4 & & & & & \\
5 & \frac{1}{3} & \frac{3}{3} & 0 & 0 & 0 & 0 \\
6 & \frac{1}{3} & \frac{3}{3} & 0 & 0 & 0 & 0 \\
7 & \frac{1}{3} & \frac{3}{3} & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & & & & & & & \\
2 & & & & & & & \\
3 & & & & & & & \\
\Pi^{3n+2} & \to & 4 & & & & & \\
5 & \frac{1}{3} & \frac{3}{3} & 0 & 0 & 0 & 0 \\
6 & \frac{1}{3} & \frac{3}{3} & 0 & 0 & 0 & 0 \\
7 & \frac{1}{3} & \frac{3}{3} & 0 & 0 & 0 & 0 \\
\end{array}
\]
CHAPTER 8

Section 8.2

1. (a) $L(x) = 2e^{-2x}/e^{-x} = 2e^{-x}$, so $L(x) > \lambda$ iff $x < c = -\ln \lambda/2$. Thus

$$
\alpha = \int_0^c e^{-x} \, dx = 1 - e^{-c}
$$

$$
\beta = \int_c^\infty 2e^{-2x} \, dx = e^{-2c} = (1 - \alpha)^2
$$

Hence as in Example 1 of the text, $S_A = \{(\alpha, (1 - \alpha)^2), 0 \leq \alpha \leq 1\}$ and $S = \{(\alpha, \beta): 0 \leq \alpha \leq 1, (1 - \alpha)^2 \leq \beta \leq 1 - \alpha^2\}$.

(b) $e^{-c} = 1 - \alpha = .95$, so that $c = .051$. Thus we reject $H_0$ if $x < .051$, accept $H_0$ if $x > .051$. We have $\beta = (1 - \alpha)^2 = .9025$, which indicates that tests based on a single observation are not very promising here.

(c) Set $\alpha = \beta = (1 - \alpha)^2$; thus $\alpha = (3 - \sqrt{5})/2 = .38 = 1 - e^{-c}$, so that $c = .477$.

3. (a)

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_0(x)$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$p_1(x)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$L(x)$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{3}{3}$</td>
<td>$\frac{3}{3}$</td>
<td>$\frac{3}{3}$</td>
<td>$\frac{3}{3}$</td>
<td>$\frac{3}{3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LRT</th>
<th>Rejection Region</th>
<th>Acceptance Region</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \leq \lambda &lt; \frac{3}{2}$</td>
<td>all $x$</td>
<td>empty</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{3}{2} &lt; \lambda &lt; \frac{3}{2}$</td>
<td>$x = 1, 2$</td>
<td>$x = 3, 4, 5, 6$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\frac{3}{2} &lt; \lambda \leq \infty$</td>
<td>empty</td>
<td>all $x$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The admissible risk points are given in the diagram.
SOLUTIONS TO PROBLEMS

(b) Reject with probability \(a\) if \(x = 1\) or 2, accept if \(x = 3, 4, 5\) or 6, where \(a/3 = .1\), that is, \(a = .3\). We have \(1 - \beta = .3(1/2) = .15\), or \(\beta = .85\).

(c) \(\lambda = pc_1/(1-p)c_2 = 3/2\), so reject with probability \(a\) if \(x = 1\) or 2, accept if \(x = 3, 4, 5, 6\), where \(a\) is any number in \([0, 1]\). Thus there are uncountably many Bayes solutions.

4. \(n \geq 13\).

5. By Problem 2d, the test is of the form

\[
\varphi(x) = \begin{cases} 
1 & \text{if } \sum_{k=1}^{n} x_k^2 > c \\
0 & \text{if } \sum_{k=1}^{n} x_k^2 < c \\
\text{anything} & \text{if } \sum_{k=1}^{n} x_k^2 = c
\end{cases}
\]

Now if the true variance is \(\theta_1\), \(\sum_{k=1}^{n} (R_k^2)/\theta\) has the chi-square density \(h_n\) with \(n\) degrees of freedom (see Problem 3, Section 5.2), hence

\[
P_{\theta}(\{x: \sum_{k=1}^{n} x_k^2 > c\}) = \int_{c/\theta}^{\infty} h_n(x) \, dx = A_n(c/\theta)
\]

(The numbers \(A_n\) are tabulated in most statistics books.) Thus the error probabilities are given by

\[
\alpha = A_n(c/\theta_0), \\
1 - \beta = A_n(c/\theta_1)
\]

For a given value of \(n\), the specification of \(\alpha\) determines \(c\), which in turn determines \(\beta\). In practice, one must keep trying larger values of \(n\) until \(\beta\) is reduced to the desired figure.

6. First consider \(\theta = \theta_0\) versus \(\theta = \theta_1, \theta_1 > \theta_0\).

\[
L(x) = f_{\theta_1}(x)/f_{\theta_0}(x) = (\theta_0/\theta_1)^n \text{ if } 0 \leq t(x) = \max x_i \leq \theta_0 \\
= \infty \text{ if } \theta_0 < t(x) \leq \theta_1
\]

Let \(\lambda = (\theta_0/\theta_1)^n\) and consider the following test

\[
\varphi(x) = \begin{cases} 
1 & \text{if } L(x) > \lambda, \text{ that is, if } \theta_0 < t(x) \leq \theta_1 \\
0 & \text{if } L(x) < \lambda \text{ (this never occurs)} \\
1 & \text{if } L(x) = \lambda \text{ and } t(x) \leq \theta_0 x_1^{1/n}, \text{ that is, if } 0 \leq t(x) \leq \theta_0 x_1^{1/n} \\
0 & \text{if } L(x) = \lambda \text{ and } t(x) > \theta_0 x_1^{1/n}, \text{ that is, if } \theta_0 x_1^{1/n} < t(x) \leq \theta_0
\end{cases}
\]

Since \(t(x)\) can never be \(< 0\) or \(> \theta_1\), \(\varphi\) is exactly the test proposed in the statement of the problem. Its type I error probability is

\[
P_{\theta_0}(x: \max x_i \leq \theta_0 x_1^{1/n}) = (\theta_0 x_1^{1/n}/\theta_0)^n = \alpha
\]
SOLUTIONS TO PROBLEMS

Since $\varphi$ is a LRT, it is most powerful at level $\alpha$. But $\varphi$ does not depend on $\theta_1$, hence is UMP for $\theta = \theta_0$ versus $\theta > \theta_0$.

Now let $\theta_1 < \theta_0$. Then

$$L(x) = (\theta_0/\theta_1)^n \text{ if } 0 \leq t(x) \leq \theta_1;$$

$$= 0 \text{ if } \theta_1 < t(x) \leq \theta_0$$

Let $\lambda = (\theta_0/\theta_1)^n$ and consider the following test.

$$\varphi'(x) = 1 \text{ if } L(x) > \lambda \text{ (this never occurs)}$$

$$= 0 \text{ if } L(x) < \lambda$$

$$= 1 \text{ if } L(x) = \lambda \text{ and } t(x) \leq \theta_0\alpha^{1/n}$$

$$= 0 \text{ if } L(x) = \lambda \text{ and } t(x) > \theta_0\alpha^{1/n}$$

Since $t(x)$ cannot be $> \theta_0$ in this case, $\varphi' \equiv \varphi$. Again, $\varphi$ is UMP for $\theta = \theta_0$ versus $\theta < \theta_0$, and the result follows. The power function is (see diagram)

$$Q(\theta) = E_{\theta} \varphi = 1 \text{ if } 0 < \theta \leq \theta_0\alpha^{1/n}$$

$$= (\theta_0\alpha^{1/n}/\theta)^n = \alpha(\theta_0/\theta)^n, \, \theta_0\alpha^{1/n} \leq \theta \leq \theta_0$$

$$= 1 - P_{\theta} \{x: \theta_0\alpha^{1/n} < t(x) \leq \theta_0\}$$

$$= 1 - [(\theta_0/\theta)^n - (\theta_0\alpha^{1/n}/\theta)^n]$$

$$= 1 - (1 - \alpha)(\theta_0/\theta)^n, \, \theta > \theta_0$$

Problem 8.2.6

7. The risk set is $\{(\alpha, \beta): 0 \leq \alpha \leq 1, (1 - \alpha)2^{-n} \leq \beta \leq 1 - \alpha2^{-n}\}$, and the set of admissible risk points is $\{(\alpha, (1 - \alpha)2^{-n}): 0 \leq \alpha \leq 1\}$.

10. If $\alpha(\varphi) = \alpha < \alpha_0$, let $\varphi' \equiv 1$ and $\varphi_t = (1 - t)\varphi + t\varphi'$. Then

$$\alpha(\varphi_t) = (1 - t)\alpha(\varphi) + t\alpha(\varphi')$$

$$\beta(\varphi_t) = (1 - t)\beta(\varphi) + t\beta(\varphi')$$
Section 8.3

1. (a) $\hat{\theta} = -n / \sum_{i=1}^{n} \ln x_i$
   (b) $\bar{x} = \bar{x}$
   (c) $\hat{\theta} = \max \{x_1, \ldots, x_n\}$

2. $\hat{\theta} = |x|$

3. $\hat{\theta} = r|x|$

4. $\rho(\theta) = \theta(1 - \theta)/n$

5. By (8.3.2) with $g(\theta) = 1$, $0 \leq \theta \leq 1$, we have

$$\psi(x) = \int_0^1 \int_0^1 \left( \frac{\theta x (1 - \theta)^{n-x}}{x} \right) \theta x (1 - \theta)^{n-x} \ d\theta \ d\theta = \frac{\beta(x + 2, n - x + 1)}{\beta(x + 1, n - x + 1)} = \frac{x + 1}{n + 2}$$

6. For each $x$, we wish to minimize $\int_0^1 \psi(\theta) \theta^x (1 - \theta)^{n-x} \{[\theta - \psi(\theta)]^2/\theta(1 - \theta)\} d\theta$ [see (8.3.1)]. In the same way that we derived (8.3.2), we find that

$$\psi(x) = \int_0^1 \frac{\theta x (1 - \theta)^{n-x-1}}{\theta (1 - \theta)^{n-x-1}} \ d\theta = \frac{\beta(x + 1, n - x)}{\beta(x, n - x)} = \frac{x}{n}$$

The risk function is

$$\rho_\psi(\theta) = E_\theta \left[ \frac{(R(n) - \theta)^2}{\theta(1 - \theta)} \right] = \frac{1}{n^2 \theta(1 - \theta)} \text{Var}_\theta R = \frac{1}{n} = \text{constant}$$
Section 8.4

1. \( f_\theta(x_1, \ldots, x_n) = (\theta_2 - \theta_1)^{-n} \prod_{i=1}^n I_{[\theta_1, \theta_2]}(x_i) \), where \( I \) is an indicator function

\[ = (\theta_2 - \theta_1)^{-n} I_{(\theta_1, \infty)}(\min x_i) I_{(-\infty, \theta_2]}(\max x_i) \]

Thus in (a), \( \min R_i, \max R_i \) is sufficient; in (b), \( \max R_i \) is sufficient; in (c) \( \min R_i \) is sufficient.

2.

\[ f_\theta(x_1, \ldots, x_n) = \left( \prod_{i=1}^n x_i \right)^{\theta_1 - 1} e^{-\sum_{i=1}^n x_i/\theta_2} \frac{\theta_1}{[\Gamma(\theta_1) \theta_2^{\theta_1}]^n} \]

hence if \( \theta_1, \theta_2 \) are both unknown, \( \prod_{i=1}^n R_i, \sum_{i=1}^n R_i \) is sufficient; if \( \theta_1 \) is known, \( \sum_{i=1}^n R_i \) is sufficient; if \( \theta_2 \) is known, \( \prod_{i=1}^n R_i \) is sufficient.

3. \( \prod_{i=1}^n R_i, \prod_{i=1}^n (1 - R_i) \) is sufficient if \( \theta_1 \) and \( \theta_2 \) are unknown; if \( \theta_1 \) is known, \( \prod_{i=1}^n (1 - R_i) \) is sufficient, and if \( \theta_2 \) is known, \( \prod_{i=1}^n R_i \) is sufficient.

Section 8.5

1. \((1 - 1/n)T^n\).

2. (a) \( T \) has density \( f_T(y) = ny^{n-1}/\theta^n, 0 \leq y \leq \theta \) (Example 3, Section 2.8), so

\[ E_{\theta} g(T) = \int_0^\theta g(y) f_T(y) \, dy = (n/\theta^n) \int_0^\theta y^{n-1} g(y) \, dy. \]

If \( E_{\theta} g(T) = 0 \) for all \( \theta > 0 \) then \( y^{n-1} g(y) = 0 \), hence \( g(y) = 0 \), for all \( y \) (except on a set of Lebesgue measure 0). Thus

\[ P_\theta \{ g(T) = 0 \} = \int_{\{y; g(y) = 0\}} f_T(y) \, dy = 1 \]

(b) If \( g(T) \) is an unbiased estimate of \( \gamma(\theta) \) then

\[ E_{\theta} g(T) = \frac{n}{\theta^n} \int_0^\theta y^{n-1} g(y) \, dy = \gamma(\theta) \]

Assuming \( g \) continuous we have

\[ \theta^{n-1} g' = \frac{d}{d\theta} \left[ \frac{\theta^n \gamma(\theta)}{n} \right] \]

or \( \gamma(\theta) = \gamma(\theta) + \frac{\theta}{n} \gamma'(\theta) \)

Conversely, if \( g \) satisfies this equation then \( n^{\theta-1} g(\theta) = d/d\theta [n^\theta \gamma(\theta)] \) hence

\[ n \int_0^\theta y^{n-1} g(y) \, dy = \theta^n \gamma(\theta), \]

assuming \( \theta^n \gamma(\theta) \to 0 \) as \( \theta \to 0 \). Thus a UMVUE of \( \gamma(\theta) \) is given by \( \gamma(T) = \gamma(T) + (T/n) \gamma'(T) \). For example, if \( \gamma(\theta) = \theta \) then \( g(T) = T + T/n = [(n+1)/n]T \); if \( \gamma(\theta) = 1/\theta \) then \( g(T) = 1/T + (T/n)(-1/T^2) = (1/T)[1 - (1/n)] \), assuming that \( n > 1 \).
4. We have
\[ P_N(R_1 = x_1, \ldots, R_n = x_n) = \frac{1}{N^n} \prod_{i=1}^{n} I_{i(1,2,\ldots,N)}(x_i) I_{i(1,2,\ldots,N)}(\max x_i) \]
hence \( T = \max R_i \) is sufficient. Now
\[ P_N(T \leq k) = \left( \frac{k}{N} \right)^n, \quad k = 1, 2, \ldots, N; \]
therefore
\[ P_N(T = k) = \frac{k^n - (k - 1)^n}{N^n}, \quad k = 1, 2, \ldots, N \]
Thus
\[ E_N g(T) = \sum_{k=1}^{N} g(k) \left[ \frac{k^n - (k - 1)^n}{N^n} \right] \]
If \( E_N g(T) = 0 \) for all \( N = 1, 2, \ldots \), take \( N = 1 \) to conclude that \( g(1) = 0 \).
If \( g(k) = 0 \) for \( k = 1, \ldots, N - 1 \), then \( E_N g(T) = 0 \) implies that
\[ g(N) \left[ \frac{N^n - (N - 1)^n}{N^n} \right] = 0 \]
hence \( g(N) = 0 \). By induction, \( g \equiv 0 \) and \( T \) is complete. To find a UMVUE of \( \gamma(N) \), we must solve the equation
\[ \sum_{k=1}^{N^2} g(k) [k^n - (k - 1)^n] = N^n \gamma(N), \quad N = 1, 2, \ldots \]
or
\[ g(N) [N^n - (N - 1)^n] = N^n \gamma(N) - (N - 1)^n \gamma(N - 1) \]
Thus
\[ g(N) = \frac{N^n \gamma(N) - (N - 1)^n \gamma(N - 1)}{N^n - (N - 1)^n}, \quad N = 1, 2, \ldots \]

5. \( R \) is clearly sufficient for itself, and
\[ E_{\theta} g(R) = \sum_{k=r}^{\infty} g(k) \left( \frac{k-1}{r-1} \right) (1 - \theta)^{r-\theta} \]
If \( E_{\theta} g(R) = 0 \) for all \( \theta \in [0, 1) \) then \( \sum_{k=r}^{\infty} g(k) (r-1)^{r-k} \theta^{k-r} = 0 \), so that \( g \equiv 0 \).
Thus \( R \) is complete. The above expression for \( E_{\theta} g(R) \) shows that for a UMVUE to exist, \( \gamma(\theta) \) must be expandable in a power series. Conversely, let \( \gamma(\theta) = \sum_{i=0}^{\infty} a_i \theta^i, 0 \leq \theta < 1 \). We must find \( g \) such that
\[ \sum_{k=r}^{\infty} g(k) \left( \frac{k-1}{r-1} \right) \theta^{k-r} = (1 - \theta)^{-r} \gamma(\theta) = \sum_{i=0}^{\infty} b_i \theta^i = \sum_{i=r}^{\infty} b_{i-r} \theta^{i-r} \]
Therefore
\[ g(i) = \frac{b_{i-r}}{\binom{i-1}{r-1}}, \quad i = r, r + 1, \ldots \]

For example, if \( \gamma(\theta) = \theta^k \) then
\[
(1 - \theta)^{-r} \gamma(\theta) = \theta^k \sum_{j=0}^{\infty} (-1)^j \binom{-r}{j} \theta^j = \sum_{j=0}^{\infty} \binom{j + r - 1}{r - 1} \theta^{k+j} \quad \text{(Problem 6a, Section 6.4)}
\]

Thus \( b_i = 0 \) for \( i < k \), and \( b_i = \binom{i+r-1-k}{r-1} \) for \( i \geq k \)
\[
g(i) = \frac{\binom{i-1-k}{r-1}}{\binom{i-1}{r-1}}, \quad i = r + k, r + k + 1, \ldots
\]

= 0 otherwise

In particular, if \( k = 1 \) then
\[
g(i) = \frac{\binom{i-2}{r-1}}{\binom{i-1}{r-1}} = \frac{i-r}{i-1}, \quad i \geq r + 1
\]

Thus a UMVUE of \( \theta = 1 - p \) is \( (R - r)/(R - 1) \); a UMVUE of \( p = 1 - \theta \) is \( 1 - (R - r)/(R - 1) = (r - 1)/(R - 1) \) (The maximum likelihood estimate of \( p \) is \( r/R \), which is biased; see Problem 3, Section 8.3.)

6. (a) \( E_{\theta} \psi(R) = \frac{e^{-\theta}}{1 - e^{-\theta}} \sum_{k=1}^{\infty} \psi(k) \frac{\theta^k}{k!} = e^{-\theta} \)

Thus
\[
\sum_{k=1}^{\infty} \psi(k) \frac{\theta^k}{k!} = 1 - e^{-\theta} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\theta^k}{k!}
\]

The UMVUE is given by
\[
\psi(k) = (-1)^{k-1} = -1 \text{ if } k \text{ is even}
\]
\[
= +1 \text{ if } k \text{ is odd}
\]
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(b) Since $e^{-\theta}$ is always $>0$, the estimate found in part (a) looks rather silly. If $\psi'(k) \equiv 1$ then $E_\theta[\psi'(R) - e^{-\theta}]^2 < E_\theta[\psi(R) - e^{-\theta}]^2$ for all $\theta$, hence $\psi$ is inadmissible.

9.

$$E_\theta \left[ \frac{1}{n} \sum_{i=1}^{n} R_i - \theta \right]^2 = \text{Var}_\theta \left[ \frac{1}{n} \sum_{i=1}^{n} R_i \right] = \frac{4}{n} \text{Var}_\theta R_i = \frac{\theta^2}{3n}$$

$$E_\theta \left[ \frac{(n+1)}{n} T - \theta \right]^2 = \text{Var}_\theta \left[ \frac{(n+1)}{n} T \right] = \frac{(n+1)^2}{n^2} \text{Var}_\theta T$$

$$E_\theta T = \frac{n}{\theta^n} \int_0^\theta y^n \, dy = \frac{n}{n+1} \theta$$ (see Problem 2b),

and

$$E_\theta T^2 = \frac{n}{\theta^n} \int_0^\theta y^{n+1} \, dy = \frac{n\theta^2}{n+2}$$

Thus

$$\text{Var}_\theta T = \frac{\theta^2}{n+2} \left[ \frac{n}{n+2} - \frac{n^2}{(n+1)^2} \right] = \frac{n\theta^2}{(n+1)^2(n+2)}$$

Therefore

$$E_\theta \left[ \frac{(n+1)}{n} T - \theta \right]^2 = \frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n}$$

if $n > 1$

11. In the inequality $[(a + b)/2]^2 \leq (a^2 + b^2)/2$, set $a = \psi_1(R) - \gamma(\theta)$, $b = \psi_2(R) - \gamma(\theta)$, to obtain

$$E_\theta \left[ \frac{\psi_1(R) - \gamma(\theta)}{2} + \frac{\psi_2(R) - \gamma(\theta)}{2} \right]^2 \leq \frac{1}{2} \left[ \rho_{\psi_1}(\theta) + \rho_{\psi_2}(\theta) \right] = \rho_{\psi_1}(\theta)$$

By minimality, we actually have equality. But the left side is

$$\frac{1}{2} \rho_{\psi_1}(\theta) + \rho_{\psi_2}(\theta) + 2E\{[\psi_1(R) - \gamma(\theta)][\psi_2(R) - \gamma(\theta)]\} = \frac{1}{2} \rho_{\psi_1}(\theta) + \frac{1}{2} \text{Cov}_\theta \left[ \psi_1(R), \psi_2(R) \right]$$

Thus

$$\text{Cov}_\theta \left[ \psi_1(R), \psi_2(R) \right] = \rho_{\psi_1}(\theta) = \left[ \rho_{\psi_1}(\theta) \rho_{\psi_2}(\theta) \right]^{1/2}$$

$$= \left[ \text{Var}_\theta \psi_1(R) \right] \left[ \text{Var}_\theta \psi_2(R) \right]^{1/2}$$

We therefore have equality in the Schwarz inequality, and it follows that (with probability 1) one of the two random variables $\psi_1(R) - \gamma(\theta)$, $\psi_2(R) - \gamma(\theta)$ is a multiple of the other (Problem 3, Section 3.4). The multiple is $+1$ or $-1$ since $\psi_1(R)$ and $\psi_2(R)$ have the same variance. If the multiple is $+1$, we are finished, and if it is $-1$, then $(\psi_1(R) + \psi_2(R))/2 = \gamma(\theta)$. The minimum variance is therefore 0, hence $\psi_1(R) = \psi_2(R) = \gamma(\theta)$, as desired.
Section 8.6

1. \( P(\frac{R_1}{R_2} \leq z) = \int_0^\infty \int_0^{xy} f_{12}(x, y) \, dx \, dy \)

\[ = \frac{1}{2^{(m+n)/2} \Gamma(m/2) \Gamma(n/2)} \int_0^\infty \int_0^{xy} y^{(n/2)-1} e^{-y/2} \int_0^{xy} x^{(m/2)-1} e^{-x/2} \, dx \, dy \]

But \( \int_0^{xy} x^{(m/2)-1} e^{-x/2} \, dx = (\text{with } x = uy) \int_0^u (uy)^{(m/2)-1} e^{-uy/2} y \, du \). Thus

\[ P(\frac{R_1}{R_2} \leq z) = \int_0^z h(x) \, dx \]

where

\[ h(x) = \frac{1}{2^{(m+n)/2} \Gamma(m/2) \Gamma(n/2)} \int_0^\infty y^{(m+n)/2-1} e^{-(y/2)(1+x)} \, dy \]

\[ = \frac{\Gamma((m+n)/2)x^{(m/2)-1}}{2^{(m+n)/2} \Gamma(m/2) \Gamma(n/2)} \frac{2^{(m+n)/2}}{(1 + x)^{(m+n)/2}} \]

\[ = \frac{x^{(m/2)-1}}{\beta(m/2, n/2)} (1 + x)^{(m+n)/2}, \quad x \geq 0 \]

If

\[ W = \frac{R_1/m}{R_2/n} = \frac{n R_1}{m R_2} \quad \text{then} \quad f_W(x) = h\left(\frac{m}{n} x\right)^{m/n} \]

\[ = f_{mn}(x), \quad \text{as desired} \]

2. If \( R \) is chi-square with \( n \) degrees of freedom then \( R = R_1^2 + \cdots + R_n^2 \) where the \( R_i \) are independent and normal \((0, 1)\). Thus \( E(R) = n \), and \( \text{Var} R = n \text{Var} R_i^2 = n[E(R_i^4) - (E(R_i^2))^2] = n(3 - 1) = 2n \). If \( R \) has the \( t \) distribution with \( n \) degrees of freedom, then \( E(R) = 0 \) by symmetry, unless \( n = 1 \), in which case \( R \) has a Cauchy density and \( E(R) \) does not exist. Now in the integral

\[ \int_0^\infty \frac{x^2}{[1 + (x^2/n)]^{(n+1)/2}} \, dx, \quad \text{let } y = \frac{1}{1 + (x^2/n)} \]

so that

\[ dy = \frac{-2x/n}{[1 + (x^2/n)]^2} \, dx = \frac{-2xy^2}{n} \, dx \]

But

\[ \frac{x^2}{n} = \frac{1}{y} - 1, \quad \text{hence} \quad dy = -\frac{2}{\sqrt{n}} \frac{\sqrt{1 - y} y^2 \, dx}{y} \]

The integral becomes

\[ \frac{1}{2} \int_0^1 n \left(\frac{1 - y}{y}\right)^{y(n+1)/2} \sqrt{n} \sqrt{\frac{y}{1 - y} \frac{1}{y^2}} \, dy \]

\[ = \frac{n^{3/2}}{2} \int_0^1 y^{(n/2)-2}(1 - y)^{1/2} \, dy = \frac{1}{2} n^{3/2} \beta\left(\frac{n}{2} - 1, \frac{3}{2}\right) \]
Thus
\[ \text{Var } R = E(R^2) = \frac{\Gamma[(n + 1)/2]}{\sqrt{n\pi} \Gamma(n/2)} \frac{\Gamma[(n/2) - 1] \Gamma(3/2)}{\Gamma[(n + 1)/2]} n^{n/2} \]
\[ = \frac{n/2}{(n/2) - 1} = \frac{n}{n - 2}, \quad n > 2 \]

If \( n = 2 \), the same calculation gives \( \text{Var } R = \infty \).

A similar calculation shows that if \( R \) has the \( F(m, n) \) distribution then \( E(R) = n/(n - 2) \) if \( n > 2 \), \( E(R) = \infty \) if \( n = 1 \) or \( 2 \), \( \text{Var } R = [2n^2(m + n - 2)]/\left[m(n - 2)^2(n - 4)\right] \) if \( n > 4 \), \( \text{Var } R = \infty \) if \( n = 3 \) or \( 4 \).