Definition 1.1. Newton’s method is a technique to approximate a root of $f(x)$. A first guess $x_1$ is made, and then it proceeds recursively according to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 1.2. Estimate the $x$-value for the point of intersection on the graphs of $y = x^3 + 2x$ and $y = 2x + 4$ using Newton’s Method with an initial estimate of $x_1 = 1$. You should use this method 2 times in order to obtain estimates $x_2$ and $x_3$.

$$2x + 4 = x^3 + 2x$$

\[0 = x^3 - 4 = f(x)\]

\[3x^2 = f'(x)\]

$$x_2 = 1 + \frac{3}{3} = \sqrt{2} = x_2$$

\[x_3 = 2 - \frac{1}{2} = \frac{5}{3} = x_3\]
Math 221 AD2: Test 3 Review—Riemann Sums and Definite Integrals  
November 9, 2018

Theorem 2.1. If $f$ is integrable on $[a, b]$ or if $f$ has only a finite number of jump discontinuities on $[a, b]$, then $f$ is integrable on $[a, b]$.

Theorem 2.2. If $f$ is integrable on $[a, b]$, then the integral of $f$ from $a$ to $b$ can be computed using a limit of right Riemann sums. That is,

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{f(a + k \Delta x)}{\Delta x} \cdot \Delta x
$$

Example 2.3. Compute the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using a limit of right Riemann sums. Sketch a picture illustrating this.

1. $\Delta x = \frac{2 - (-1)}{n} = \frac{3}{n}$

2. $x_k = -1 + k \left( \frac{3}{n} \right)$

3. $R_n = \sum_{k=1}^{n} = \left[ 1 + \left( -1 + k \left( \frac{3}{n} \right) \right)^2 \right] \left( \frac{3}{n} \right)$

4. $R_n = \sum_{k=1}^{n} = \left[ 1 + \left( -1 + k \left( \frac{3}{n} \right) \right) \cdot \left( -1 + \left( \frac{3}{n} \right) \right) \right] \left( \frac{3}{n} \right)$

5. $R_n = \sum_{k=1}^{n} = \left[ 1 + \left( 1 - \left( \frac{3}{n} \right)^2 \right) \right] \left( \frac{3}{n} \right)$

6. $R_n = \sum_{k=1}^{n} = \frac{3}{n} + \left( \frac{3}{n} - \frac{18}{n^2} + \frac{27}{n^3} \right)$

7. $R_n = \sum_{k=1}^{n} = \frac{6}{n} + \frac{-18}{n^2} + \frac{27}{n^3}$

8. $R_n = \sum_{k=1}^{n} = \frac{6}{n} + \frac{-18(n+1)}{2n^2} + \frac{27(n(n+1)(2n+1))}{6n^3}$

9. $\lim_{n \to \infty} 6 + \lim_{n \to \infty} \frac{-18(n+1)}{2n^2} + \lim_{n \to \infty} \frac{27(n(n+1)(2n+1))}{6n^3}$

10. $\lim_{n \to \infty} 6 + \lim_{n \to \infty} 0 + \lim_{n \to \infty} 0 = 6$
Math 221 AD2: Test 3 Review—Fundamental Theorem of Calculus, part I
November 9, 2018

Theorem 3.1 (Fundamental Theorem of Calculus, part I).
If $f$ is continuous on $[a, b]$ then
\[ g(x) = \int_{a}^{x} f(t) \, dt \]
is continuous on $[a, b]$ and differentiable on $(a, b)$ with $g'(x) = f(x)$.

Example 3.2. Find the derivative of the following function.
\[ g(x) = \int_{2}^{3} e^{t^2 + 3t^6} \, dt = -\int_{3}^{2} e^{t^2 + 3t^6} \, dt \]
\[ g'(x) = -(e^{x^2 + 3x^6}) \]
e $e^{x^2 + 3x^6}$ is continuous and differentiable for all numbers.

Example 3.3. Find the derivative of the following function.
\[ \int_{0}^{\sqrt{x^2 + \cos(x)}} \frac{1}{1 + u^2} + 2u - 5\sqrt{u+1} \, du \]
is continuous and differentiable for all numbers
\[ g'(x) = \left(\frac{1}{1 + (x^2 + \cos(x))^2} + 2(x^2 + \cos(x)) - 5\sqrt{x^2 + \cos(x) + 1}\right)(2x - \sin x) \]
Math 221 AD2: Test 3 Review—Fundamental Theorem of Calculus, part II
November 9, 2018

**Theorem 4.1** (Fundamental Theorem of Calculus, part II).

\[
\int_a^b f(x) \, dx = F(b) - F(a) \text{ if } F \text{ is any antiderivative of } f \text{ that is } \left( F'(x) = f(x) \right)
\]

**Example 4.2.** Compute the integral.

\[
\int_0^1 \frac{1}{x^2 + 1} + 2x^3 - \sqrt{x} + \sec^2(x) + e^x \, dx
\]

\[
\int_0^\frac{\pi}{6} \frac{1}{x^2 + 1} \, dx + \int_0^1 2x^3 \, dx - \int_0^1 \sqrt{x} \, dx + \int_0^1 \sec^2(x) \, dx + \int_0^1 e^x \, dx
\]

\[
\arctan(x) + \frac{x}{2} - \frac{2}{3}x^{\frac{3}{2}} + \tan(x) + e^x \bigg|_0^1
\]

\[
\left( \arctan(1) - \arctan(0) \right) + \left( \frac{1}{2} - \frac{0}{2} \right) - \left( \frac{2}{3} \cdot \frac{3}{2} - \frac{2}{3} \cdot 0 \right) + \left( \tan(1) - \tan(0) \right) + (e^1 - e^0)
\]

\[
\left( \frac{\pi}{4} - 0 \right) + \left( \frac{1}{2} - 0 \right) - \left( \frac{2}{3} - 0 \right) + \left( \tan(1) - 0 \right) + (e - 1)
\]

**Example 4.3.** At time \( t \) hours, a population of bacteria is growing at a rate of \( 3t^2 + 2t + 50 \) bacteria per hour. If the population is 2000 at time \( t = 2 \) hours, then what is the population at time \( t = 4 \) hours?

\[
\int_2^4 3t^2 + 2t + 50 \, dt
\]

\[
2000 + \int_2^4 3t^2 + 2t + 50 \, dt
\]

\[
\left. t^3 + t^2 + 50t \right|_2^4
\]

\[
\left( 4^3 + 4^2 + 50 \cdot 4 \right) - \left( 2^3 + 2^2 + 50 \cdot 2 \right)
\]

\[
(280) - (112)
\]

\[
= 168
\]

\[
2000 + 168 = 2168
\]
Math 221 AD2: Test 3 Review—Integration by Substitution
November 9, 2018

**Theorem 5.1** (The Substitution Rule). If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x))g'(x)\,dx = \int f(u)\,du
\]

**Example 5.2.** Find the indefinite integral.

\[
\int \frac{\ln \sqrt{x}}{x} \, dx = \int \left( \frac{u}{x} \right) 2x \, du = \int u \, 2 \, du = u^2 + C
\]

\[u = \ln \sqrt{x}\]
\[du = \frac{1}{2x} \, dx\]
\[dx = 2x \, du\]

\[(\ln \sqrt{x})^2 + C\]

**Example 5.3.** Find the indefinite integral.

\[
\int \frac{3x + 2x^3}{x^4 + 16} \, dx
\]

\[
= \int \frac{3x}{x^4 + 16} \, dx + \int \frac{2x^3}{x^4 + 16} \, dx
\]

\[
= \int \frac{3x}{x^4 + 16} \, dx
\]

\[
= \frac{1}{2} \ln (x^4 + 16)
\]

\[
= \int \frac{2x^3}{x^4 + 16} \, dx
\]

\[
= \frac{1}{2} \cdot \arctan \left( \frac{\sqrt{x^2}}{4} \right)
\]

\[
= \frac{1}{2} \cdot \ln \left| x^4 + 16 \right|
\]

\[
= \frac{3 \arctan \left( \frac{\sqrt{x^2}}{4} \right)}{32} + \frac{\ln |x^4 + 16|}{2} + C
\]